A stochastic variant of Chow-Rashewski Theorem on the Grushin distribution

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Abstract. This paper introduces and proves a stochastic variant of Chow-Rashewski global connectivity theorem for the Grushin distribution. Given two points A and B, in the Grushin distribution of step 2, then for any ball centered at B the probability that an admissible stochastic process ever reaches a disk centered at B of arbitrary small radius r can be made as large as possible by choosing proper controls $u_1(t)$, $u_2(t)$. The details refer to admissible curves, stochastic admissibility and stochastic controllability.

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1 Introduction

Let M be a finite dimensional, connected differentiable manifold. Any subbundle $\mathcal{H}$ of the tangent bundle TM will be called a distribution on M. The distribution $\mathcal{H}$ is supposed to be non-integrable, because otherwise we obtain a foliation on M. Most usually, the distribution $\mathcal{H}$ is given locally as the linear hull of a set $\{X_1, X_2, \cdots, X_m\}$ of smooth vector fields on M, where $m \leq dimM = n$ is the rank of $\mathcal{H}$. The codimension of $\mathcal{H}$, which is equal to $n - m$, provides the number of missing directions of the sub-Riemannian geometry. We can always construct locally a metric g defined on $\mathcal{H} \times \mathcal{H}$, with respect to which the vector fields X_j are assumed orthonormal.

A *sub-Riemannian manifold* is a triplet $(M, \mathcal{H}, g)$, where M is a finite dimensional connected manifold, $\mathcal{H}$ is a horizontal distribution, and g is a sub-Riemannian metric, i.e. a positive definite, non-degenerate metric $g : \mathcal{H} \times \mathcal{H} \to \mathcal{F}(M)$.

The set of smooth functions on M is denoted by $\mathcal{F}(M)$ and the horizontal vector fields, which are linear combinations of X_j, are considered sections of the horizontal subbundle, i.e. elements of $\Gamma(\mathcal{H})$.

The first approach of this geometry from the dual equivalent point of view of constraints was done by Vranceanu [21] in 1926, under the name of *non-holonomic geom-*

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etry. Sub-Riemannian geometry is also known under the synonym name of Carnot-Carathéodory geometry.

The distribution $\mathcal{H}$ satisfies the *bracket generating condition* if the vector fields X_j, together with finitely many of their iterated brackets span the tangent space of the space M at each point. This means that for each $x \in M$, there is an $r > 1$ such that

$$X_i, \cdots, [X_i, X_j], \cdots, [X_i, [X_j, X_k]], \cdots, [\cdots [X_{i_1}, \cdots, [X_{i_r}, X_{i_{r+1}}]] \cdots]$$

span $T_x M$.

The aforementioned condition has been used independently by Chow and Rashevskii ([10], [18]) to prove global connectedness of M by horizontal curves; it was also used by Hörmander [14] as a sufficient, but not necessary condition for the differential operator $\sum_j X_j^2$ to be hypoelliptic. It is worth noting that there are sub-Riemannian manifolds on which the global connectedness holds but the bracket-generating condition fails, see for instance [6].

More precisely, Chow's theorem states the following:

If the distribution $\mathcal{H}$ is bracket generating at each point on a connected manifold M, then any two points of M can be joined by a piecewise curve tangent to the distribution.

The horizontal curves are called *admissible curves.* If we perform a stochastic perturbation of the distribution $\mathcal{H}$, the admissible curves will be replaced by *admissible stochastic processes*, see Section 3. Hopping that any two points can be joined by an admissible stochastic process is not too realistic. Take for instance the 2-dimensional difussion process $X_t = (W_t^1, W_t^2)$, where W_t^j is a 1-dimensional Brownian motion. Then X_t starts at the origin and the probability that it reaches a given point (different than the origin) is zero. Therefore, we cannot expect that an exact analog of Chow's theorem would hold. However, we can modify the connectivity property, replacing it by a *stochastic connectivity.* In this case, one stochastic variant of Chow's theorem reads:

Let (M, H, g) be a connected sub-Riemannian manifold with the distribution $\mathcal{H}$ bracket generating. Given any two points A, B on M, then for any $r, \epsilon > 0$, there is an admissible process x_t, with $x_0 = A$ and

$$P\Big(x_t \in D(B, r) \text{ for some } t > 0\Big) \geq 1 - \epsilon, \tag{1.1}$$

where $D(B, r) = \{y \in M; \|y - B\|_g < r\}$ is the ball centered at B of radius r.

This states that given any two points A and B, we can choose proper control functions $u_1(t)$, $u_2(t)$ such that the corresponding admissible stochastic process reaches any disk centered at B with a probability sufficiently close to 1 (see Fig. 1).

The plan of the paper is as in the following. Section 2 shows the connectivity by admissible curves on the Grushin plane. Then Section 3 defines admissible stochastic processes and proves the aforementioned stochastic connectivity property for the case of Grushin distribution.

The stochastic version of the Grushin problem may have some applications in finding heat kernels and fundamental solutions. It is known that the heat kernel of a sub-elliptic operator depends on the geometry of the underlying horizontal distribution $\mathcal{H}$. It is expected that the heat kernel $K(x, x_0; t)$ is given by the path integral

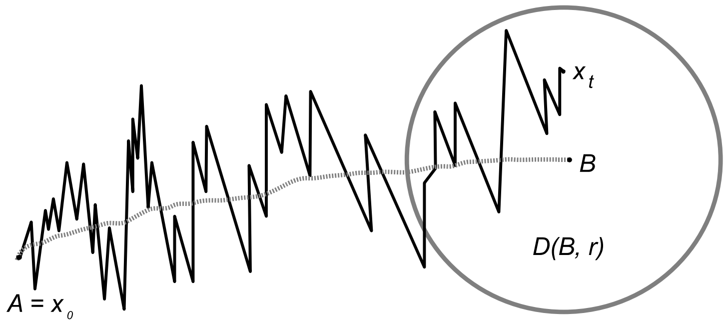

Figure 1: *The admissible stochastic process x_t starting at $x_0 = A$ and reaching the disk $D(B, r)$.*

with respect to all horizontal curves joining the points x_0 and x in time t. We believe that the aforementioned path integral can be replaced by a probabilistic argument, where the probability that an admissible process starting at x_0 reaching a volume element centered at x plays a central role in defining a measure on the space of admissible stochastic processes starting at a given point. A similar well known result states that the fundamental solution, $F(x_0, x)$, of a second order differential operator, which is associated with the diffusion X_t, can be expressed probabilistically by

$$F(x_0, x)dx = P^{x_0}\Big(X_t \in dx \text{ for some } t > 0\Big),$$

where dx is an infinitesimal volume element centered at x, and X_t starts at x_0.

In a second paper (see [9]) we define the stochastic geodesic processes as minimizers of a stochastic energy action. The exact number of stochastic geodesics from the origin to any other point is computed and the corresponding energies are calculated. It is also shown that the number of stochastic geodesics increases unbounded when the second point approaches the vertical axis (stochastic cut locus).

2 Grushin Distribution

The linear differential operators $X_1 = \partial_{x_1}$, $X_2 = x_1\partial_{x_2}$ in $\mathbb{R}^2$ are called Grushin vector fields. The rank of the distribution generated by $\{X_1, X_2\}$ is 1 along the vertical axis $\{x_1 = 0\}$ and 2 elsewhere. Since their bracket is $[X_1, X_2] = \partial_{x_2}$, then Chow's bracket generating condition is satisfied; this means that $X_1, X_2, [X_1, X_2]$ span all tangent space at each point. By Chow-Rashevski theorem, any two points A and B in the plane can be joined by a continuous, piecewise differentiable curve which is tangent to the distribution generated by $\{X_1, X_2\}$. This means that there are two controls $u_1(t), u_2(t)$ such that there is a solution $x : [0, T] \to \mathbb{R}^2$ to the following boundary problem

$$\frac{dx}{dt}(t) = u_1(t)X_1\big(x(t)\big) + u_2(t)X_2\big(x(t)\big) \tag{2.1}$$
$$x(0) = A,\ x(T) = B.$$

The system can be written equivalently as

$$dx_1(t) = u_1(t)\,dt \tag{2.2}$$
$$dx_2(t) = u_2(t)x_1(t)\,dt \tag{2.3}$$

$$x_j(0) = x_j^A,\ x_j(T) = x_j^B,\ j = 1, 2.$$

An *admissible* curve between A and B is a solution $x(t) = (x_1(t), x_2(t))$ of the system (2.1), or, equivalently, the system $(2.2-2.3)$.

In fact, the admissible curves between A and B can be constructed explicitly. Assume for the sake of simplicity that $A = (0, 0)$ and $B = (x_1^B, x_2^B) \neq A$ (otherwise we can join separately A and B with the origin and concatenate the resulting admissible curves). There are two cases to investigate, which come from the rank-varying property of the Grushin distribution: $x_1^B \neq 0$ and $x_1^B = 0$.

1. We assume that B is outside of the x_2-axis. If we consider the constant controls $u_1(t) = a$, $u_2(t) = b$, then the system can be easily solved, obtaining $x_1(t) = at$, $x_2(t) = \frac{1}{2}abt^2$. The boundary condition yields

$$a = \frac{x_1^B}{T}, \qquad b = \frac{2x_2^B}{x_1^B T}, \tag{2.4}$$

which corresponds to the admissible curve

$$x_1(t) = \frac{x_1^B}{T}t, \qquad x_2(t) = \frac{x_2^B}{T^2}t^2,\ 0 \le t \le T,$$

joining the origin and the point B. Since

$$x_2 = \frac{x_2^B}{(x_1^B)^2}x_1^2,$$

the curve is just an arc of parabola.

2. We assume that B is on the x_2-axis. In this case we look for controls of type $u_1(t) = a\cos(at)$, $u_2(t) = b$, with a and b constants. Solving the system and using the boundary conditions yields $a = \frac{\pi}{T}$, $b = \frac{\pi x_2^B}{2T}$. The admissible curve joining the origin and $B(0, x_2^B)$ is given by

$$x_1(t) = \sin(\pi t/T)$$
$$x_2(t) = \frac{1}{2}x_2^B(1 - \cos(\pi t/T)), \quad 0 \le t \le T,$$

i.e., the semi-ellipse

$$(x_1)^2 + \left(1 - \frac{2}{x_2^B}\,x_2\right)^2 = 1.$$

Much more it is known about the Grushin distribution. The reader can find more information, for instance, in [3] p.271 and [8].

3 Stochastic Admissibility

We shall stochastically perturb the system $(2.2-2.3)$ by adding some "noise". The traditional way of doing this is to add normal errors given by increments of a Brownian motion process. For the definition and properties of Brownian motions the reader can consult Kuo [15].

Let $W_1(t), W_2(t)$ be two independent Brownian motion processes and σ_i two non-negative constants used in controlling the amplitudes of the error term, so $\sigma_i dW_i(t) \sim N(0, \sigma_i dt)$, i=1,2. Consider the following stochastic perturbation of the system (2.2–2.3)

$$dx_1(t) = u_1(t)dt + \sigma_1 dW_1(t) \tag{3.1}$$
$$dx_2(t) = u_2(t)x_1(t)dt + \sigma_2 dW_2(t) \tag{3.2}$$
$$x_1(0) = x_2(0) = 0.$$

An *admissible process* is a stochastic process $x_t = \big(x_1(t), x_2(t)\big)$, which starts at the origin and satisfies the system (3.1-3.2). It is worth noting that the end point $x_T = x(T)$ is left free.

In this section we shall prove the connectivity property from the origin by admissible stochastic processes.

Theorem 3.1. *Let B be any point in the plane and consider the origin $A = (0,0)$. Then for any $r, \epsilon > 0$, there is an admissible stochastic process x_t such that $x_0 = A$ and*

$$P\Big(x_t \in D(B,r) \text{ for some } t > 0\Big) \geq 1 - \epsilon. \tag{3.3}$$

This can be restated equivalently as: the probability that an admissible stochastic process, which starts at the origin, ever reaches a given disk centered at B of arbitrary radius r can be made as large as possible by choosing proper controls $u_1(t)$, $u_2(t)$.

Proof. We need to address the following two cases:

1. If B is outside of the x_2-axis, we look for a solution x_t with constant controls $u_1(t) = a$ and $u_2(t) = b$, with a and b subject to be found later on. Solving the system (3.1-3.2) yields

$$x_1(t) = at + \sigma_1 W_1(t) \tag{3.4}$$
$$x_2(t) = \frac{ab}{2}t^2 + b\sigma_1 Z_t + \sigma_2 W_2(t), \tag{3.5}$$

where $Z_t = \int_0^t W_1(s)\,ds$. Since $Z_t \sim N(0, t^3/3)$ and $W_2(t) \sim N(0,t)$, using the independence of $W_1(t)$ and $W_2(t)$, yields

$$x_1(t) \sim N(at, \sigma_1^2 t)$$
$$x_2(t) \sim N\Big(\frac{ab}{2}t^2, b^2\sigma_1^2 t^3/3 + \sigma_2^2 t\Big).$$

Since the pair $(x_1(t), x_2(t))$ does not have an obvious joint distribution, we shall use the following approach.

Let $A_t = \{\omega \in \Omega; \|x_t(\omega) - B\| \geq r\}$ be the complementary event, which consists of all states ω for which x_t does not belong to the disk $D(B,r)$. Then evaluating the expectation of the square of the Euclidean distance $\|x_t - B\|^2$ with respect to the natural probability measure dP we have the following estimation

$$\begin{aligned}\mathbb{E}[\|x_t - B\|^2] &= \int_\Omega \|x_t(\omega) - B\|^2 \, dP(\omega) \geq \int_{A_t} \|x_t(\omega) - B\|^2 \, dP(\omega) \\ &\geq r^2 \int_{A_t} dP(\omega) = r^2 P(A_t),\end{aligned}$$

so $P(A_t) \leq \dfrac{1}{r^2} E[\|x_t - B\|^2]$, or

$$P(\|x_t - B\| \geq r) \leq \frac{1}{r^2}\mathbb{E}[\|x_t - B\|^2].$$

This is equivalent to

$$P\big(x_t \in D(B,r)\big) = P(\|x_t - B\| \leq r) \geq 1 - \frac{1}{r^2}\mathbb{E}[\|x_t - B\|^2].$$

Comparing to the inequality (3.3), we set

$$\epsilon = \frac{1}{r^2}\mathbb{E}[\|x_t - B\|^2].$$

We need to show now that for any $r, \epsilon > 0$, there is a $t > 0$ such that

$$\mathbb{E}[\|x_t - B\|^2] = \epsilon \, r^2. \tag{3.6}$$

The left side of (3.6) can be evaluated directly using formulas (3.4-3.5). We have

$$\begin{aligned}\|x_t - B\|^2 &= (x_1(t) - x_1^B)^2 + (x_2(t) - x_2^B)^2 \\ &= (at - x_1^B + \sigma_1 W_1(t))^2 + \Big(\frac{ab}{2}t^2 - x_2^B + b\sigma_1 Z_t + \sigma_2 W_2(t)\Big)^2 \\ &= \big(at - x_1^B\big)^2 + 2\sigma_1(at - x_1^B)W_1(t) + \sigma_1^2 W_1(t)^2 \\ &\quad + \Big(\frac{ab}{2}t^2 - x_2^B\Big)^2 + b^2\sigma_1^2 Z_t^2 + \sigma_2^2 W_2(t)^2 \\ &\quad + 2\Big(\frac{ab}{2}t^2 - x_2^B\Big)b\sigma_1 Z_t + 2\Big(\frac{ab}{2}t^2 - x_2^B\Big)\sigma_2 W_2(t) + 2b\sigma_1\sigma_2 Z_t W_2(t).\end{aligned}$$

Using

$$\begin{aligned}&\mathbb{E}[W_1(t)] = \mathbb{E}[W_2(t)] = \mathbb{E}[Z_t] = 0 \\ &\mathbb{E}[W_1(t)^2] = \mathbb{E}[W_2(t)^2] = t \\ &\mathbb{E}[Z_t^2] = \frac{t^3}{3} \\ &\mathbb{E}[Z_t W_2(t)] = \mathbb{E}[Z_t]\mathbb{E}[W_2(t)] = 0,\end{aligned}$$

we have

$$\mathbb{E}[\|x_t - B\|^2] = \big(at - x_1^B\big)^2 + (\sigma_1^2 + \sigma_2^2)t + \Big(\frac{ab}{2}t^2 - x_2^B\Big)^2 + b^2\sigma_1^2\frac{t^3}{3}.$$

Then condition (3.6) becomes the polynomial equation

$$\big(at - x_1^B\big)^2 + (\sigma_1^2 + \sigma_2^2)t + \Big(\frac{ab}{2}t^2 - x_2^B\Big)^2 + b^2\sigma_1^2\frac{t^3}{3} = \epsilon r^2. \tag{3.7}$$

We need to show that we can choose the constants a and b such that the previous equation has a solution $t > 0$. First we ask the expectaion of the random variable x_t to be the center of the disk, i.e., $\mathbb{E}[x_1(t)] = x_1^B$ and $\mathbb{E}[x_2(t)] = x_2^B$. This implies

$$at - x_1^B = 0 \tag{3.8}$$

$$\frac{ab}{2}t^2 - x_2^B = 0. \tag{3.9}$$

Then the first and the third terms of equation (3.7) vanishes, leading to the more simple equation

$$(\sigma_1^2 + \sigma_2^2)t + b^2\sigma_1^2\frac{t^3}{3} = \epsilon r^2. \tag{3.10}$$

Solving for a and b in terms of t in (3.8-3.9) yields

$$a = \frac{x_1^B}{t}, \qquad b = \frac{2x_2^B}{x_1^B t}. \tag{3.11}$$

It is worth noting the similarity with relations (2.4). Substituting in (3.10) leads to the following linear equation in t

$$t\Big[\sigma_1^2 + \sigma_2^2 + \frac{4}{3}\sigma_1^2\Big(\frac{x_2^B}{x_1^B}\Big)^2\Big] = \epsilon r^2,$$

which obviously has a positive solution. Substituting back in (3.11) determines the following values of controls

$$u_1(t) = a = \frac{x_1^B}{\epsilon r^2}\Big[\sigma_1^2 + \sigma_2^2 + \frac{4}{3}\sigma_1^2\Big(\frac{x_2^B}{x_1^B}\Big)^2\Big]$$
$$u_2(t) = b = \frac{2x_2^B}{x_1^B\epsilon r^2}\Big[\sigma_1^2 + \sigma_2^2 + \frac{4}{3}\sigma_1^2\Big(\frac{x_2^B}{x_1^B}\Big)^2\Big].$$

We make the remark that the previous computation assumed that $\sigma_1^2 + \sigma_2^2 \neq 0$, i.e., at least one constant σ_i is nonzero; we also have $\epsilon > 0$.

If $\sigma_1 = \sigma_2 = 0$, then the second and the fourth terms of (3.7) disapear, leading to

$$\big(at - x_1^B\big)^2 + \Big(\frac{ab}{2}t^2 - x_2^B\Big)^2 = \epsilon r^2. \tag{3.12}$$

Substituting relations (3.11) into (3.12) yields $\epsilon = 0$. This can be restated by saying that in the non-stochastic case the probability that an admisible curve reaches a given disk is always equal to 1.

2. If B belongs to the x_2-axis, then $x_1^B = 0$. This case is similar to case **1** and consists in a straightforward calculation. We shall provide it in the following for the sake of completeness. In this case we look for controls $u_1(t) = a\cos(at)$ and $u_2(t) = b$, with a and b constants. The system (3.1-3.2) becomes

$$\begin{aligned} dx_1(t) &= a\cos(at)dt + \sigma_1\, dW_1(t) \\ dx_2(t) &= bx_1(t)dt + \sigma_2\, dW_2(t) \\ x_1(0) &= x_2(0) = 0. \end{aligned}$$

The solution is

$$x_1(t) = \sin(at) + \sigma_1 W_1(t) \tag{3.13}$$

$$x_2(t) = \frac{b}{a}(1-\cos(at)) + b\sigma_1 Z(t) + \sigma_2 W_2(t), \tag{3.14}$$

with $Z_t = \int_0^T W_1(t)\,dt$.

As before, we need to prove that there are constants a and b such that for any $\epsilon, r > 0$, the equation

$$\mathbb{E}[\|x_t - B\|^2] = \epsilon r^2 \tag{3.15}$$

has a positive solution t. Evaluating the left side we get

$$\begin{aligned} \mathbb{E}[\|x_t - B\|^2] &= \mathbb{E}[(x_1(t) - x_1^B)^2] + \mathbb{E}[(x_2(t) - x_2^B)^2] \\ &= \mathbb{E}\Big[\big(\sin(at) + \sigma_1 W_1(t)\big)^2\Big] \\ &\quad + \mathbb{E}\Big[\Big(\frac{b}{a}\big(1-\cos(at)\big) - x_2^B\Big) + b\sigma_1 Z_t + \sigma_2 W_2(t)\Big)^2\Big] \\ &= (\sigma_1^2 + \sigma_2^2)t + \sin^2(at) + \frac{b^2\sigma_1^2}{3}t^3 + \Big(\frac{b}{a}\big(1-\cos(at)\big) - x_2^B\Big)^2. \end{aligned}$$

The following conditions are sufficient for (3.15) to hold

$$\sin^2(at) = 0 \tag{3.16}$$

$$\frac{b}{a}\big(1-\cos(at)\big) - x_2^B = 0 \tag{3.17}$$

$$(\sigma_1^2 + \sigma_2^2)t + \frac{b^2\sigma_1^2}{3}t^3 = \epsilon r^2. \tag{3.18}$$

From (3.16) we obtain $a = \dfrac{\pi}{t}$, and then (3.17) yields $b = \dfrac{\pi x_2^B}{2t}$. Since the equation (3.18) has the positive solution

$$t = \frac{\epsilon r^2}{\sigma_1^2 + \sigma_2^2 + \frac{\pi^2}{4}(x_2^B)^2\sigma_1^2},$$

substituting in the formulas of a and b yields

$$a = \frac{\pi(\sigma_1^2 + \sigma_2^2 + \frac{\pi^2}{4}(x_2^B)^2\sigma_1^2)}{\epsilon r^2} \tag{3.19}$$

$$b = \frac{\pi x_2^B(\sigma_1^2 + \sigma_2^2 + \frac{\pi^2}{4}(x_2^B)^2\sigma_1^2)}{2\epsilon r^2}. \tag{3.20}$$

Hence, in the case $x_1^B = 0$, the solution (3.13-3.14) with a and b given by (3.19-3.20) satisfies condition (3.3). □

A general stochastic connectivity result for any sub-Riemannian manifold is hard to address at this incipient stage. However, the stochastic connectivity proved in this section shows the transience property of the stochastic admissible processes in the case of Grushin distribution; it shows, actually, an "almost recurrence" property. It is worth mentioning that in the case of Brownian motion on surfaces, there are two main factors which influence its recurrence: the dimension and the curvature. The lower the dimension and the higher the curvature, the more recurrent the diffusion tends to be. When the problem is studied on distributions, it is expected that the rank and the step of the distribution play a similar role. An answer to this problem requires future endeavors.

4 Stochastic Controllability by Bang-Bang Controls

This section deals with the controllability of a stochastic ODE system induced by the Grushin distribution using bang-bang controls. The concept of controllability is given by the following definition.

Definition 4.1. A stochastic dynamical system on $\mathbb{R}^n$ is said to be controllable if for every initial condition $x(0)$ and every vector $x_1 \in \mathbb{R}^n$, there is a finite random time t_1 and a control $u(t) \in \mathbb{R}^m$, $t \in [0, t_1]$, such that $x(t_1; x(0), u)$ steers to x_1 with probability 1.

In the following we shall solve the *random time minimum problem*, i.e. to find the optimal admissible process that steers an initial point x_0 to the origin in the shortest possible time.

Let $U = [-1, 1]^2 \subset \mathbb{R}^2$ be the control set and τ be a random time. Giving the starting point $x_0 \in \mathbb{R}^2$, we shall find an optimal control $u^*(\cdot)$ such that

$$I(u^*(\cdot)) = \min_{u(\cdot)} E\left(\int_0^\tau dt\right),$$

where $x(t)$ satisfies the stochastic evolution ODEs induced by the perturbed Grushin distribution

$$dx_1(t) = u_1(t)\,dt + \sigma_1 dW_1(t),\; dx_2(t) = u_2(t)x_1(t)\,dt + \sigma_2 dW_2(t).$$

Since $\tau^* = I(u^*(\cdot))$, the optimal point τ^* ensures the minimum random time to steer to the origin with probability 1. This random time optimum problem is equivalent to a stochastic controllability problem.

In order to prove the existence of a bang-bang control, we use the stochastic maximum principle as in Theorem 4.1 of Udriste and Damian [19]. The Hamiltonian 1-form

$$H(x,p,u) := -[1 + u_1(t)p_1(t) + u_2(t)p_2(t)x_1(t)]dt$$

gives the adjoint stochastic system

$$dp_j(t) = -H_{x^j}(x(t), p(t), u(t), W_t) - p_i(t)\sigma^i_{ax^j} dW^a_t$$

or

$$dp_1(t) = p_2(t)u_2(t)dt, \; dp_2(t) = 0.$$

The maximum of the linear 1-form $u \to H$ exists since each control variable belongs to the interval $[-1,1]$; for optimum, the control must be at a vertex of ∂U (see linear optimization, simplex method). The optimal controls u^*_a must be the functions

$$u_1(t) = \operatorname{sgn} p_1(t), \; u_2(t) = \operatorname{sgn}[p_2(t)x_1(t)].$$

Consequently

$$H^* = -[1 + |p_1(t)| + |p_2(t)x_1(t)|]dt.$$

Next we shall look at possible bang-bang trajectories. For $u_2(t) = \pm 1$, we obtain the general solution

$$p_1(t) = \pm\, at + b, \; p_2(t) = a.$$

We have the following optimal stochastic evolutions:

1) For $u_1(t) = 1$, $u_2(t) = 1$, we find

$$x_1(t) = t + c_1 + \sigma_1 W_1(t), x_2(t) = \frac{1}{2}t^2 + c_1 t + c_2 + \sigma_2 W_2(t) + \sigma_1 \int_0^t W_1(s)ds.$$

2) For $u_1(t) = 1$, $u_2(t) = -1$, we obtain

$$x_1(t) = t + c_1 + \sigma_1 W_1(t), x_2(t) = -\left(\frac{1}{2}t^2 + c_1 t + c_2\right) + \sigma_2 W_2(t) - \sigma_1 \int_0^t W_1(s)ds.$$

3) For $u_1(t) = -1$, $u_2(t) = 1$, we find

$$x_1(t) = -t + c_1 + \sigma_1 W_1(t), x_2(t) = -\frac{1}{2}t^2 + c_1 t + c_2 + \sigma_2 W_2(t) + \sigma_1 \int_0^t W_1(s)ds.$$

4) For $u_1(t) = -1$, $u_2(t) = -1$, we get

$$x_1(t) = -t + c_1 + \sigma_1 W_1(t), x_2(t) = \frac{1}{2}t^2 - c_1 t + c_2 + \sigma_2 W_2(t) - \sigma_1 \int_0^t W_1(s)ds.$$

It is worthy to note that all optimum solutions given above are Gaussian processes. For instance, in the case of 1), using the independence of $W_1(t)$ and $W_2(t)$ we have

$$x_1(t) \sim N\Big(t + c_1, \sigma_1 t\Big), \; x_2(t) \sim N\Big(\frac{1}{2}t^2 + c_1 t + c_2, \sigma_2^2 t + \sigma_1^2 \frac{t^3}{3}\Big).$$

The Lebesgue measure of each set $\{t \in [0,\tau] : p_1(t) = 0\}$, $\{t \in [0,\tau] : p_2(t)x_1(t) = 0\}$ vanishes. Then, the singular control is ruled out and the remaining possibilities are bang-bang controls. This optimal control is discontinuous since each component jumps from a minimum to a maximum and vice versa, in response to each change in the sign of each function $p_1(t)$, respectively $p_2(t)x_1(t)$. The functions $p_1(t)$, $p_2(t)x_1(t)$ are called *switching functions*.

5 Conclusions

We recall that Chow's theorem states that on a sub- Riemannian manifold which satisfies the bracket generating condition any two points can be joined by an admissible piecewise smooth curve (a curve tangent to the horizontal distribution). In the case when the horizontal distribution is stochastically perturbed, the admissible curves are replaced by admissible stochastic processes. Our ideea was to introduce and prove a stochastic variant of Chow's theorem of connectedness by admissible curves in the case of Grushin distribution.

Our research interests lie at the boundary of probability theory, differential geometry and control theory, mainly in the area of admissibility and controllability. This field is concerned with the properties of solutions of suitable stochastic equations and the stochastic processes to which they correspond.

The novelty of this article is based on a mixture between probability theory, differential geometry and control theory. We believe that the mixture can be continued, but stochastic-deterministic balance cannot be understood without new points of view. Obviously it all depends on which side of the fence you stand : (1) differential geometry over a stochastic theory plus control theory or (2) a stochastic theory over geometry plus control theory.

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