Stochastic sub-Riemannian geodesics on the Grushin distribution

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Abstract. Recent years have seen intensive scientific activities of describing diffusion processes with Brownian covariance given by a Riemannian metric on a manifold. In our paper the dynamics is specified through a stochastic variational principle for a generalization of the classical action, with a given kinetic Riemannian metric.

In short, we introduce the concept of stochastic sub-Riemannian geodesics and find their equations in the case of Grushin distribution. We also discuss the number of stochastic geodesics between any two given points and calculate their energies.

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1 Geodesics on Grushin distribution

The pair of linear differential operators (vector fields) $X_1 = \partial_{x_1}$ and $X_2 = x_1\partial_{x_2}$, on $\mathbb{R}^2$, generate what is known as a *Grushin distribution.* It is not really a distribution in the classical sense because the span of X_1 and X_2 drops rank along the x^2-axis, $\{x^1 = 0\}$. Nevertheless, the vector fields X_1 and X_2 are bracket generating in the sense that taking sufficiently many Lie brackets among them generates all vector fields on $\mathbb{R}^2$.

This paper continues our ideas from [9]. The study of these problems has connections to many fields, including geometric, functional, and stochastic analysis; and differential geometry together optimal control.

We shall start with a review of a few results regarding sub-Riemann- ian geodesics on the Grushin distribution. These statements focus on the existence of geodesics between the origin and any other given point in the plane, and provide an explicit parameterization of their number. These results are well known in the literature, the reader being reffered to Calin et al. [3], [8], and Gaveau and Greiner [11], [12]. Some theorems will be extended to the stochastic environment in the next Sections.

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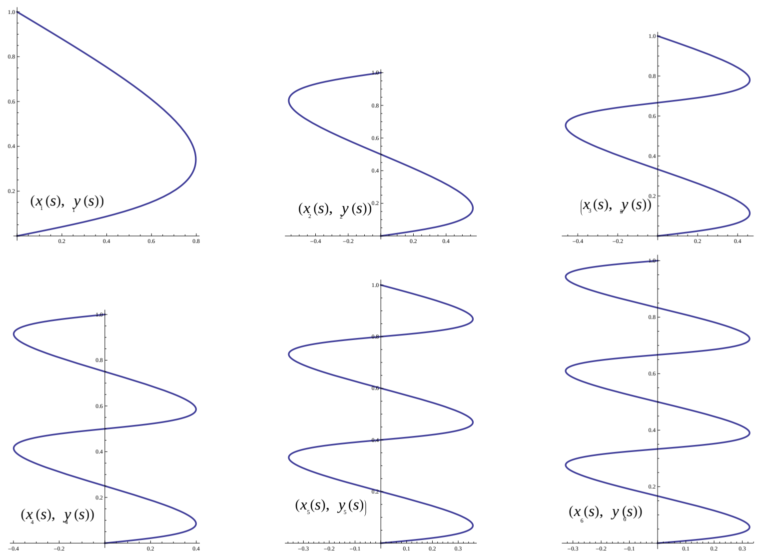


Figure 1: *Geodesics* $(x_m(s), y_m(s))$ *joining the origin with the point* $(0, 1)$ *for* $m = 1, \dots, 6$.

In order to discuss the geodesics on Grushin distribution, we shall consider a metric g on $\mathbb{R}^2$ with respect to which the vector fields $X_1 = \partial_{x_1}$ and $X_2 = x_1\partial_{x_2}$ are orthonormal. The velocity of a smooth curve tangent to the distribution is given by

$$\frac{dx}{dt}(t) = u_1(t)X_1 + u_2(t)X_2,$$

and hence the energy of the curve $x(t)$, with respect to the metric g, is defined by

$$J(u(\cdot)) = \frac{1}{2}\int_0^T \big(u_1^2(t) + u_2^2(t)\big)\, dt.$$

A sub-Riemannian geodesic between two points A and B is a curve which minimizes the functional $J(u(\cdot))$ over all smooth curves $x(t)$ satisfying $x(0) = A$ and $x(T) = B$. These geodesics are completely characterized in Calin and Chang [3], p. 275 and in the paper [8]. The following three Theorems are taken from the aforementioned references.

Theorem 1.1. *Let* $y_1 > 0$. *There are infinitely many geodesics connecting the points* $(0, 0)$ *and* $(0, y_1)$. *The equations of the geodesics are given by*

$$x_m(s) = \sqrt{\frac{2y_1}{m\pi}}\sin(m\pi s), \qquad y_m(s) = y_1\Big(s - \frac{\sin(2m\pi s)}{2m\pi}\Big),\ m = 1, 2, 3, \cdots$$

The length of the mth geodesic is $\sqrt{2m\pi y_1}$. *For each* $m \geq 1$, *there are exactly two geodesics of the same length connecting the preceding points, see Fig. 1.*

Theorem 1.2. *Given a point* $Q(x_1, y_1)$ *with* $x_1 \neq 0$, *there are only finitely many geodesics joining the point* $P(0, y_0)$ *and the point* Q. *Let* $\theta_1, \theta_2, \cdots, \theta_N$ *be the solutions*

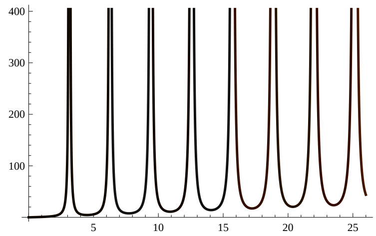


Figure 2: *The graph of function* $\mu(\theta) = \frac{\theta}{\sin^2(\theta)} - \cot\theta$*, for* $\theta \in [0, \infty) \setminus \{k\pi, k \in \mathbb{Z}\}$.

of the equation

$$\frac{2(y_1 - y_0)}{x_1^2} = \mu(\theta), \tag{1.1}$$

where

$$\mu(\theta) = \frac{\theta}{\sin^2\theta} - \cot\theta.$$

Then the equations of the geodesics are

$$x_m(s) = \frac{\sin(\theta_m s)}{\sin(\theta_m)} x_1$$

$$y_m(s) = y_0 + \frac{x_1^2}{2\sin^2\theta_m}\Big(\theta_m s - \frac{1}{2}\sin(2\theta_m s)\Big), \quad m = 1, 2, \cdots, N.$$

The length of these geodesics are given by

$$\ell_m^2 = \nu(\theta_m)[(y_1 - y_0) + x_1^2],$$

where

$$\nu(z) = \frac{2z^2}{z - \sin z \cos z + \sin^2 z}.$$

The graph of the "brush function" $\mu(\theta)$ is given in Fig. 2. This function was introduced, analyzed and used for the first time by Beals, Gaveau and Greiner in the study of geodesics on the Heisenberg group, see [2]. We shall see in the next section that this function plays an important role for counting the number of stochastic geodesics between two points in the Grushin plane.

Theorem 1.3. *Assume that* $x_0 x_1 \neq 0$. *The number of geodesics connecting the points* $P(x_0, y_0)$ *and* $Q(x_1, y_1)$ *is finite.*

In conclusion, in the deterministic case, there is always at least one geodesic between any two points, and at most infinitely many. The stochastic formulation of this problem will be discussed in detail in the next section.

2 Stochastic geodesics on Grushin distribution

In order to introduce stochastic geodesics, we consider the stochastically perturbed Grushin distribution

$$dx_t = (u_1(t)X_1(x_t) + u_2(t)X_2(x_t))dt + \sigma dW_t, \tag{2.1}$$

with $u_i(t)$ stochastic processes, $\sigma = (\sigma_1, \sigma_2)^T$, a constant vector, and $W_t = \big(W_1(t), W_2(t)\big)^T$, a 2-dimensional Brownian motion. This writes also as

$$dx_1(t) = u_1(t)dt + \sigma_1\, dW_1(t) \tag{2.2}$$
$$dx_2(t) = u_2(t)x_1(t)dt + \sigma_2\, dW_2(t). \tag{2.3}$$

We shall consider in the following that the *energy of a stochastic process* X_t is the energy of the curve $t \to E(X_t)$ with respect to a given Riemannian metric, where E means the expectation of the process. In the stochastic Grushin plane $\mathbb{R}^2 \setminus \{(0, x_2); x_2 \in \mathbb{R}\}$, endowed with the metric

$$g = (g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{x_1^2} \end{pmatrix},$$

the energy of the process $(x_1(t), x_2(t))$ is $\frac{1}{2}\big(u_1^2(t) + u_2^2(t)\big)$. Since this energy does not depend on the point (x_1, x_2), the connectivity by stochastic geodesics problem extends to $\mathbb{R}^2$.

The following definition introduces the concept of stochastic geodesic on the Grushin distribution.

Definition 2.1. A *stochastic geodesic* from the point A towards point B, in the stochastic Grushin plane, is a continuous process $x_t = \big(x_1(t), x_2(t)\big)$ satisfying the stochastic ODEs (2.2-2.3) with initial condition $x_0 = A$, the expectation at the final configuration $\mathbb{E}(x_T) = B$, and for which the negative energy functional

$$J(u(\cdot)) = -\frac{1}{2}\mathbb{E}\Big[\int_0^T \big(u_1^2(t) + u_2^2(t)\big)\, dt\Big]$$

is maximum.

It is worth noting the possible non-symmetry of connectivity by stochastic geodesics. This means that if there is a stochastic geodesic starting at A and aiming to B, it is not necessarily true that there is a stochastic geodesic starting at B and aiming to A. This is unlike in the case of connectivity by deterministic geodesics, which is obviously symmetric.

In order to find the equations satisfied by stochastic geodesics, we shall employ the method of stochastic Hamiltonian, applying Theorem 4.1 of Udriste and Damian [19] (see also [20], [21], [22]). The associated Hamiltonian 1-form in this case is

$$H(t, x_t, u_t, p_t) = -\frac{1}{2}\big(u_1^2(t) + u_2^2(t)\big)dt + \big(p_1 u_1(t) + p_2 u_2(t) x_1(t)\big)dt. \tag{2.4}$$

We have

$$\begin{aligned} H_{x_1} &= p_2 u_2 dt, & H_{x_2} &= 0 \\ H_{p_1} &= u_1 dt, & H_{p_2} &= u_2 x_1 dt \\ H_{u_1} &= (-u_1 + p_1)dt, & H_{u_2} &= (-u_2 + p_2 x_1)dt. \end{aligned}$$

The critical point conditions

$$H_{u_1}(t, x_t, u_t^*, p_t) = 0, \; H_{u_2}(t, x_t, u_t^*, p_t) = 0$$

yield the optimal controls $u_1^* = p_1$ and $u_2^* = p_2 x_1$. The momenta p_i satisfy the adjoint linear stochastic differential system

$$\begin{aligned} dp_1(t) &= -H_{x_1}(t, x_t, u_t^*, p_t) = -p_2 u_2^* dt = -p_2(t)^2 x_1(t) dt \\ dp_2(t) &= -H_{x_2}(t, x_t, u_t^*, p_t) = 0 \end{aligned} \tag{2.5}$$

and the initial stochastic equations describing the stochastic process x_t are

$$\begin{aligned} dx_1(t) &= H_{p_1}(t, x_t, u_t^*, p_t) + \sigma_1 dW_1(t) = p_1(t)dt + \sigma_1 dW_1(t) \\ dx_2(t) &= H_{p_2}(t, x_t, u_t^*, p_t) + \sigma_2 dW_2(t) = p_2(t)(x_1(t))^2 dt + \sigma_2 dW_2(t), \end{aligned}$$

$$(x_1(0), x_2(0)) = (x_1^A, x_2^A), \; \mathbb{E}[(x_1(T), x_2(T))] = (x_1^B, x_2^B).$$

We shall find the general solution of the aforementioned stochastic ODEs. Since the equation (2.5) implies $p_2(t) = c_2$, the system takes the more simple form

$$\begin{aligned} dx_1(t) &= p_1(t)dt + \sigma_1 dW_1(t) \\ dx_2(t) &= c_2 x_1(t)^2 dt + \sigma_2 dW_2(t) \\ dp_1(t) &= -c_2^2 x_1(t) dt \\ p_2 &= c_2. \end{aligned}$$

Notice that the associated ODEs containing only the "drift parts" (see the first and third equations),

$$dp_1(t) = -c_2^2 x_1(t) dt, \qquad dx_1(t) = p_1(t) dt,$$

has the general solution

$$\begin{aligned} x_1(t) &= a_1 \cos(c_2 t) + a_2 \sin(c_2 t) \\ p_1(t) &= -c_2 \left[a_1 \sin(c_2 t) - a_2 \cos(c_2 t)\right]. \end{aligned}$$

In order to solve the stochastic system, we try something similar to the variation of parameters method, i.e., we are looking for a solution of the following form

$$x_1(t) = a_1(t) \cos(c_2 t) + a_2(t) \sin(c_2 t) + \sigma_1 W_1(t), \tag{2.6}$$

$$p_1(t) = -c_2 \left[a_1(t) \sin(c_2 t) - a_2(t) \cos(c_2 t)\right]. \tag{2.7}$$

It follows that the coefficients satisfy the stochastic system

$$\begin{aligned} \cos(c_2 t) da_1(t) + \sin(c_2 t) da_2(t) &= 0, \\ \sin(c_2 t) da_1(t) - \cos(c_2 t) da_2(t) &= c_2 \sigma_1 W_1(t) dt \end{aligned}$$

or equivalently,

$$da_1(t) = c_2\sigma_1 W_1(t)\sin(c_2 t)dt$$
$$da_2(t) = -c_2\sigma_1 W_1(t)\cos(c_2 t)dt.$$

Integrating and using Ito's formula, we obtain

$$a_1(t) = a_1(0) - \sigma_1\cos(c_2 t)W_1(t) + \sigma_1\int_0^t \cos(c_2 s)\,dW_1(s)$$
$$a_2(t) = a_2(0) - \sigma_1\sin(c_2 t)W_1(t) + \sigma_1\int_0^t \sin(c_2 s)\,dW_1(s).$$

Using properties of Wiener integrals, we have

$$E(a_1(t)) = a_1(0), \qquad E(a_2(t)) = a_2(0).$$

Substituting the boundary conditions

$$x_1(0) = x_1^A, \quad \mathbb{E}[x_1(T)] = x_1^B$$

into (2.6-2.7) leads to $x_1(0) = a_1(0) = x_1^A$ and

$$x_1^A\cos(c_2 T) + a_2(0)\sin(c_2 T) = x_1^B. \tag{2.8}$$

This provides the relation

$$a_2(0) = \frac{x_1^B - x_1^A\cos\theta}{\sin\theta}, \tag{2.9}$$

where $\theta = c_2 T$.

Substituting the expressions for $a_1(t)$ and $a_2(t)$ into (2.6-2.7) yields

$$x_1(t) = \Big(x_1^A + \sigma_1\int_0^t \cos(c_2 s)\,dW_1(s)\Big)\cos(c_2 t) + \Big(a_2(0) + \sigma_1\int_0^t \sin(c_2 s)\,dW_1(s)\Big)\sin(c_2 t),$$

$$p_1(t) = -c_2\Big(x_1^A + \sigma_1\int_0^t \cos(c_2 s)\,dW_1(s)\Big)\sin(c_2 t) + c_2\Big(a_2(0) + \sigma_1\int_0^t \sin(c_2 s)\,dW_1(s)\Big)\cos(c_2 t),$$

with $a_2(0)$ and c_2 satisfying (2.8).

The second component of the stochastic geodesic, $x_2(t)$, depends on its first component, $x_1(t)$, as follows

$$x_2(t) = x_2^A + c_2\int_0^t x_1^2(s)\,ds + \sigma_2 W_2(t).$$

The integral in the middle is not solvable in closed form. However, the boundary condition $\mathbb{E}[x_2(T)] = x_2^B$ provides the "compatibility condition"

$$x_2^A + c_2 \int_0^T E[x_1^2(s)]\, ds = x_2^B. \tag{2.10}$$

The integral in the middle can now be computed. We first note that

$$\mathbb{E}\left[\left(\int_0^t \cos(c_2 s)\, dW_1(s)\right)^2\right] = \int_0^t \cos^2(c_2 s)\, ds = \frac{t}{2} + \frac{\sin(2c_2 t)}{4c_2},$$
$$\mathbb{E}\left[\left(\int_0^t \sin(c_2 s)\, dW_1(s)\right)^2\right] = \int_0^t \sin^2(c_2 s)\, ds = \frac{t}{2} - \frac{\sin(2c_2 t)}{4c_2},$$
$$\mathbb{E}\left[\left(\int_0^t \sin(c_2 s)\, dW_1(s)\right)\left(\int_0^t \cos(c_2 s)\, dW_1(s)\right)\right] = \int_0^t \sin(c_2 s)\cos(c_2 s)\, ds = \frac{\sin^2(c_2 t)}{2c_2}.$$

Then

$$\begin{aligned}\mathbb{E}[x_1^2(s)] &= \left((x_1^A)^2 + \sigma_1^2\Big(\frac{s}{2} + \frac{\sin(2c_2 s}{4c_2}\Big)\right)\cos^2(c_2 s) \\ &\quad + \left(a_2(0)^2 + \sigma_1^2\Big(\frac{s}{2} - \frac{\sin(2c_2 s}{4c_2}\Big)\right)\sin^2(c_2 s) \\ &= \frac{1}{2}\sigma_1^2 s + (x_1^A)^2\cos^2(c_2 s) + a_2(0)^2\sin^2(c_2 s) + \frac{\sigma_1^2}{8c_2}\sin(4c_2 s).\end{aligned}$$

Integrating yields

$$\begin{aligned}\int_0^T \mathbb{E}[x_1^2(s)]\, ds &= \frac{1}{4}\sigma_1^2 T^2 + (x_1^A)^2\Big(\frac{T}{2} + \frac{\sin(2c_2 T)}{4c_2}\Big) \\ &\quad + a_2(0)^2\Big(\frac{T}{2} - \frac{\sin(2c_2 T)}{4c_2}\Big) + \frac{\sigma_1^2}{8c_2}\frac{\sin^2(2c_2 T)}{2c_2}.\end{aligned}$$

Substituting $\theta = c_2 T$, using relation (2.9), the previous identity becomes

$$\begin{aligned}c_2\int_0^T \mathbb{E}[x_1^2(s)]\, ds &= \frac{1}{4}\sigma_1^2 T\theta + \frac{(x_1^A)^2}{2}\Big(\theta + \frac{\sin(2\theta)}{2}\Big) \\ &\quad + \frac{1}{2}(x_1^B - x_1^A\cos\theta)^2\Big(\frac{\theta}{\sin^2\theta} - \cot\theta\Big) + \frac{\sigma_1^2 T}{16}\frac{\sin^2(2\theta)}{\theta}.\end{aligned}$$

Then the condition (2.10) is equivalent to

$$\begin{aligned}x_2^B - x_2^A &= \frac{1}{4}\sigma_1^2 T\theta + \frac{(x_1^A)^2}{2}\Big(\theta + \frac{\sin(2\theta)}{2}\Big) \\ &\quad + \frac{1}{2}(x_1^B - x_1^A\cos\theta)^2\mu(\theta) + \frac{\sigma_1^2 T}{16}\frac{\sin^2(2\theta)}{\theta},\end{aligned} \tag{2.11}$$

where

$$\mu(\theta) = \frac{\theta}{\sin^2\theta} - \cot\theta. \tag{2.12}$$

Each solution θ of equation (2.11) determine a unique pair $(c_2, a_2(0))$, with $c_2 = \theta/T$ and and $a_2(0)$ given by (2.9). The pair $(c_2, a_2(0))$ define the stochastic geodesic $\big(x_1(t), x_2(t)\big)$ starting at A and aiming to B.

3 Geodesics starting at the origin

In the case when the stochastic geodesic starts at the origin, i.e., when $x_1^A = x_2^A = 0$, the equation (2.11) takes the following more simple form

$$x_2^B = \frac{1}{4}\sigma_1^2 T\theta + \frac{1}{2}(x_1^B)^2\mu(\theta) + \frac{\sigma_1^2 T}{16}\frac{\sin^2(2\theta)}{\theta}. \tag{3.1}$$

Let $x_1^B \neq 0$ and set

$$g(\theta) = g(\theta; x_1^B, x_2^B) := \frac{2}{(x_1^B)^2}\Big[x_2^B - \frac{1}{4}\sigma_1^2 T\theta - \frac{\sigma_1^2 T}{16}\frac{\sin^2(2\theta)}{\theta}\Big].$$

Then the number of stochastic geodesics starting at the origin and aiming to B is given by the number of solutions θ of the equation

$$g(\theta) \quad = \quad \mu(\theta). \tag{3.2}$$

It is worthy to know the behavior of both sides in order to predict the number of solutions. The function on the left side has the following properties

(i) $g(0^+) = \dfrac{2x_2^B}{(x_1^B)^2}$.

(ii) $g(\theta)$ oscillates with the amplitude decreasing to 0 as $\theta \to \infty$. In fact

$$g(\theta) \sim \frac{2}{(x_1^B)^2}\Big[x_2^B - \frac{1}{4}\sigma_1^2 T\theta\Big], \quad \theta \sim \pm\infty,$$

i.e. the function $g(\theta)$ approaches a linear function of negative slope for $|\theta|$ large.

The function $\mu(\theta)$ is characterized by the following result, see Beals et al. [2].

Lemma 3.1. *The function μ defined by* (2.12) *is a monotone-increasing diffeomorphism of the interval $(-\pi,\pi)$ onto $\mathbb{R}$. On each interval $\big(m\pi,(m+1)\pi\big)$, $m = 1, 2, \ldots$, the function μ has a unique critical point c_m. On this interval μ decreases strictly from $+\infty$ to $\mu(c_m)$ and then increases strictly from $\mu(c_m)$ to $+\infty$. Moreover, the minima values are increasing*

$$\mu(c_m) + \pi < \mu(c_{m+1}), \qquad m = 1, 2, \ldots.$$

If $g(0) > 0$, then the solutions θ of equation (3.2) are positive, see Fig. 3. Since there is $M > 0$ such that

$$g(\theta) < 0, \quad \mu(\theta) > 0, \qquad \forall\theta > M,$$

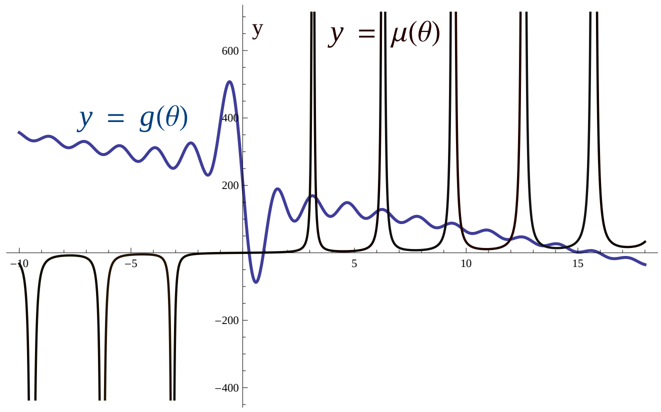


Figure 3: *The graphs of the functions $g(\theta)$, $\mu(\theta)$ and their intersections.*

by continuity reasons it follows that the equation (3.2) has at least one solution. Since $g(\theta)$ tends to a line with negative slope, the number of solutions must be finite. The case $g(0) < 0$ is similar, but the solutions are negative.

It can also be inferred from Fig. 3 that when $x_1^B \to \infty$, i.e., when the slope of the line is small, then the number of solutions increases unbounded. They are given by $\theta_k = k\pi$, or $c_2^k = k\pi/T$.

When $x_2^B = 0$ and $x_2^B \neq 0$, the only solution of the equation (3.2) is $\theta = 0$. This implies $c_2 = 0$, and hence $x_1(t) = \sigma_2 W_t$ and $x_2(t) = x_2^A = 0$.

To conclude, we state the following result.

Theorem 3.2. (i) *Given a point B in the Grushin plane, with $x_1^B \neq 0$, there is at least one stochastic geodesic starting at the origin and aiming to B. The number of stochastic geodesics with this property is finite and is given by the number of solutions θ of the equation* (3.2).

(ii) *Given a point B on the x_2-axis, there are infinitely many stochastic geodesics starting at the origin and aiming to B.*

(iii) *If B belongs to the x_1-axis, then there is a unique stochastic geodesic from the origin towards B; this is given by a Brownian motion along the x_1-axis*

$$x_t = \big(\sigma_1 W_1(t), 0\big).$$

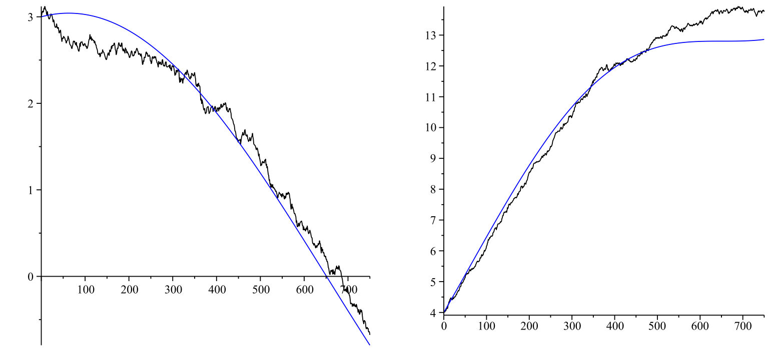
Remark 3.1. (a) If set $\sigma_1 = 0$, then the equation (3.2) becomes

$$\frac{2x_2^B}{(x_1^B)^2} = \mu(\theta),$$

which is exactly the equation (1.1).

(b) It is worth noting that the number of stochastic geodesics is independent of σ_2.

(c) A MAPLE simulation of the graphs of geodesic components $x_1(t)$ and $x_2(t)$ (deterministic versus stochastic) are given in Fig. 4 (a) and (b).



(a) (b)

Figure 4: *Graphs of geodesic components: deterministic versus stochastic;* (*a*) $(x_1(t), X_1(t))$*;* (*b*) $(x_2(t), X_2(t))$*.*

4 The energy along stochastic geodesics

In this section we shall compute the energy along stochastic geodesics starting at the origin and aiming towards a point B of coordinates (x_1^B, x_2^B). This is given by

$$I = \frac{1}{2}\mathbb{E}\Big[\int_0^T \Big(u_1^*(t)^2 + u_2^*(t)^2\Big)\,dt\Big], \tag{4.1}$$

and we shall express it in terms of x_1^B and x_2^B. Using the stochastic Ito relations $(dW_t)^2 = dt$, $dtdW_t = 0$, and $(dt)^2 = 0$, we have

$$\begin{aligned} dx_1(t) &= p_1(t)dt + \sigma_1 dW_1(t) \\ dp_1(t) &= -c_2^2 x_1(t)dt = 0. \end{aligned}$$

or

$$\begin{aligned} (dx_1(t))^2 &= \sigma_1^2 dt \\ (dp_1(t))^2 &= 0. \end{aligned}$$

Since the optimal controls are

$$u_1^*(t) = p_1(t),\ \ u_2^*(t) = p_2(t)x_1(t) = c_2 x_1(t),$$

with c_2 constant, differentiating using Ito's formula we have

$$\begin{aligned} d\Big(u_1^*(t)^2 + u_2^*(t)^2\Big) &= d\Big(p_1(t)^2 + c_2^2 x_1(t)^2\Big) \\ &= 2p_1(t)dp_1(t) + (dp_1(t))^2 + 2c_2^2 x_1(t)dx_1(t) + c_2^2(dx_1(t))^2 \\ &= c_2^2\sigma_1^2 dt + 2\sigma_1 c_2^2 x_1(t)dW_1(t). \end{aligned}$$

Integrating, we get

$$u_1^*(t)^2 + u_2^*(t)^2 = u_1^*(0)^2 + u_2^*(0)^2 + (c_2\sigma_1)^2 t + 2\sigma_1 c_2^2 \int_0^t x_1(s)\, dW_1(s).$$

Since the geodesic starts at the origin, $x_1^A = 0$, we find

$$u_1^*(0) = p_1(0) = c_2 a_2(0) = \frac{c_2 x_1^B}{\sin(c_2 T)}$$
$$u_2^*(0) = c_2 x_1(0) = c_2 x_1^A = 0.$$

Consequently, we have $u_1^*(0)^2 + u_2^*(0)^2 = \dfrac{(c_2 x_1^B)^2}{\sin(c_2T)^2}$, and hence we obtain

$$u_1^*(t)^2 + u_2^*(t)^2 = \frac{(c_2 x_1^B)^2}{\sin(c_2T)^2} + (c_2\sigma_1)^2 t + 2\sigma_1 c_2^2 \int_0^t x_1(s)\, dW_1(s).$$

Integrating between 0 and T yields

$$\frac{1}{2}\int_0^T \Big(u_1^*(t)^2 + u_2^*(t)^2\Big)\, dt = \frac{T(c_2 x_1^B)^2}{2\sin(c_2T)^2} + \Big(\frac{\sigma_1 c_2 T}{2}\Big)^2 + \sigma_1 c_2^2 \int_0^T \int_0^t x_1(s)\, dW_1(s)\, dt.$$

Since the expectation of an Ito integral is zero, taking the expectation yields

$$I = \frac{1}{2}E\Big[\int_0^T \Big(u_1^*(t)^2 + u_2^*(t)^2\Big)\, dt\Big] = \frac{T(c_2 x_1^B)^2}{2\sin(c_2T)^2} + \Big(\frac{\sigma_1 c_2 T}{2}\Big)^2.$$

This can be written in terms of the new variable $\theta = c_2 T$ as

$$I = \frac{(\theta x_1^B)^2}{2T(\sin\theta)^2} + \Big(\frac{\sigma_1\theta}{2}\Big)^2,$$

where θ satisfies equation (3.2). If $x_1^B \neq 0$ there are finitely many solutions θ, and hence finitely many energies, one for each stochastic geodesic.

In the particular case, when $x_1^B = 0$, i.e. B belongs to the x_2-axis, then $\theta = \theta_k = k\pi$, $k = 1, 2, 3, \cdots$, and the energies are given by

$$I_k = \Big(\frac{\sigma_1 k\pi}{2}\Big)^2, k = 1, 2, 3, \cdots.$$

It is worth noting that the energies do not depend on σ_2.

5 Conclusions

The paper starts with few notions and results of sub-Riemannian geodesics on the Grushin distribution. It is worth noting that in this case there are no abnormal or singular geodesics.

Then the stochastic geodesics are defined as energy minimizing stochastic processes starting at a given point and having a given end point expectation. The number of stochastic geodesics between the origin and any other point in the plane is found and explicit formulas for the energy along stochastic geodesics are found.

Therefore, the paper will have an impact on the future approach of sub-Riemannian geometry treated from the stochastic point of view. More precisely, in mathematical literature there are papers based on Brownian metric, and and here one used the kinetic metric. Our paper opens the eyes for further research: geometric problems which depends on both metrics, the Brownian one and the kinetic one.

The novelty of this article is based on elements of differential geometry placed over stochastic processes. There could be considered also problems in which the stochastic theory is placed on a background of differential geometry. The mixture between probability theory, differential geometry and control theory is now in progress and requires new footprint differential geometry ideas.

We believe that the mixture can be continued, but stochastic-deterministic balance cannot be understood without new points of view.

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