Galloway's compactness theorem on Finsler manifolds

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Abstract. The compactness theorem of Galloway is a stronger version of the Bonnet-Myers theorem allowing the Ricci scalar to take also negative values from a set of real numbers which is bounded below. In this paper we allow any negative value for the Ricci scalar, and adding a condition on its average, we find again that the manifold is compact and provide an upper bound of its diameter. Also, with no condition on Ricci scalar itself, but with a condition on its average, we find again the compactness of the manifold. All considerations are done in the category of Finsler manifolds.

M.S.C. 2010: 53C60.
Key words: Finsler manifold; Ricci scalar; compactness theorems.

1 Introduction

The classical results from the Riemannian geometry as Hopf-Rinow theorem, Bonnet-Myers Theorem, Synge Theorem and others have been extended to Finsler manifolds due to the efforts of many geometers. These results have been summarized in the well-known textbook by D. Bao, S.S. Chern and Z. Shen [4], in a coherent and clear theory of geodesics on such manifolds. The quoted text-book was followed by a lot of papers aiming to extend to the Finslerian framework and other important results from Riemannian geometry. See [2], [3], [5], [8], [9], [10] etc.

Recall that the Bonnet-Myers Theorem states that if the Ricci scalar Ric of a Finsler manifold M satisfies $Ric \geq (n-1)a > 0$ then every geodesic with length $\pi/\sqrt{a}$ or longer must contain conjugate points, the diameter of M is at most $\pi/\sqrt{a}$ and in fact M is compact.

The later two assertions are direct consequences of the former when it is written as follows: Let $\sigma(t)$, $0 \leq t \leq L$ be a unit speed geodesic with velocity field T and $Ric(t) := Ric_{(\sigma(t),T)}$. If $Ric(t) \geq (n-1)a > 0$ for every $t \in [0, L]$ and if $L \geq \dfrac{\pi}{\sqrt{a}}$, then σ must contain conjugate points to $\sigma(0)$.

Balkan Journal of Geometry and Its Applications, Vol.20, No.2, 2015, pp. 1-8.

In [2] we extended to Finsler manifolds the compactness theorem of Galloway (see [7]). The essential step was to prove that if

$$Ric(t) \geq (n-1)a + \frac{df}{dt}$$

for some function f with $|f(t)| \leq \dfrac{\Lambda}{\pi}$, $\Lambda \geq 0$ and if

$$L \geq \frac{\Lambda}{a(n-1)} + \sqrt{\frac{\pi^2}{a} + \frac{\Lambda^2}{a^2(n-1)^2}},$$

then σ must contain conjugate points to $\sigma(0)$. Then when the Finsler manifold M is forward(backward) geodesically complete, it follows that it is compact with $\operatorname{diam}(M) \leq \dfrac{\Lambda}{a(n-1)} + \sqrt{\dfrac{\pi^2}{a} + \dfrac{\Lambda^2}{a^2(n-1)^2}}$.

In this paper we prove

Theorem 1.1. *Let (M, F) be a forward geodesically complete connected Finsler manifold of dimension n. Suppose that*

a) The Ricci scalar Ric has the following uniform positive upper bound

$$Ric < (n-1)a$$

for a constant $a > 0$,

b) For every geodesic σ parameterized by the arc-length $t \in [0, L]$ we have

$$\int_0^L Ric(t)dt \geq a(n-1)L + \varepsilon\Lambda,$$

for $\varepsilon = \pm 1$ and a constant $\Lambda > 0$.

Then:

(1) *Along every geodesic the distance between any two successive conjugate points is at most $-\varepsilon\frac{\Lambda}{a(n-1)} + \sqrt{\frac{\pi^2}{a} + \frac{\Lambda^2}{a^2(n-1)^2}}$.*

(2) *The diameter of M is at most $-\varepsilon\frac{\Lambda}{a(n-1)} + \sqrt{\frac{\pi^2}{a} + \frac{\Lambda^2}{a^2(n-1)^2}}$.*

(3) *M is compact.*

Theorem 1.2. *Let (M, F) be a forward geodesically complete connected Finsler manifold of dimension n. If there exists a point $p \in M$ such that along each geodesic $\sigma : [0, \infty) \to M$ emanating from p and parameterized by arc length t the condition*

$$\int_0^\infty Ric(t)dt = \infty,$$

holds, then M is compact.

The above theorems have possible applications in a Finslerian theory of Relativity. Their Riemannian versions were already used in standard theory of Relativity. Thus in his Thesis submitted at the University of California, San Diego, T. Frankel has used Myers Theorem to obtain a bound on the size of a fluid mass in stationary space-time universe. Later, in [7] G. Galloway made use of Frankel's method to obtain a closure theorem which has as its conclusion the "finiteness" of the "spatial part" of a space-time obeying certain cosmological assumptions for cosmological models more general than the classical Friedmann models. He continued to discuss the implications of his theorem in Physics in a joint paper with T. Frankel, [6]. In 2013, G. Galloway and E. Woolgar [8] extended some of Galloway's results to the so-called Bakry-Émery Ricci tensor.

The structure of the paper is as follows. In Section 2 we recall, mainly following the text-book [4], the results from Finsler geometry to be used. The next two sections are devoted to the proofs of the Theorem 1.1 and Theorem 1.2, respectively.

2 Preliminaries

We shall use the notations, the terminology and results from [4] without comments.

Finsler manifolds. Index form.

Let (M, F) be a Finsler manifold. The Finsler structure F is a function $F : TM \to [0, \infty)$, $(x, y) \to F(x, y)$ which is C^∞ on the slit tangent bundle $TM \backslash 0$, positively homogeneous of degree 1 in y, and whose Hessian matrix $g_{ij} := \dfrac{1}{2}\dfrac{\partial^2 F^2}{\partial y^i \partial y^i}$ is positive-definite at every point of $TM \backslash 0$.

The Chern connection of local coefficients $\Gamma^i_{jk}(x, y)$ is a linear connection in the pull-back bundle $\pi^* TM$ over $TM \backslash 0$, where $\pi : TM \to M$ is the natural projection. It is only h-metrical and it has two curvatures $R_j{}^i{}_{kh}$, $P_j{}^i{}_{kh}$.

Let y be a non zero element of $T_x M$. Then, $g_y(x) := g(x, y) = g_{ij}(x, y) dx^i \otimes dx^j$ is an inner product, which is used to measure lengths and angles in $T_x M$.

For a vector field $W(t) := W^i(t)\dfrac{\partial}{\partial x^i}$ along a curve σ, whose tangent vector field is T, the expression,

$$D_T W = \left[\frac{dW^i}{dt} + W^j T^k (\Gamma^i_{jk}(\sigma, T))\right] \frac{\partial}{\partial x^i} \tag{2.1}$$

is called the covariant derivative with reference vector T.

One says that W is parallel along σ if $D_T W = 0$, with reference vector T. One defines parallel transport (with reference vector T) on the standard way. The parallel transport preserves g_T-lengths and angles.

The constant speed geodesics are solutions of $D_T T = 0$, with reference vector T.

Let $\sigma(t) = exp_x(tT), x \in M, 0 \le t \le L$ be a geodesic of constant speed 1. One abbreviates $g_{(\sigma,T)}$ by g_T.

For two continuous and piecewise C^∞ vector fields V and W along σ the index form is

$$I(V, W) = \int_0^L [g_T(D_T V, D_T W) - g_T(R(V, T)T, W)]dt. \tag{2.2}$$

Here D_T is calculated with reference vector T of length 1 and

$$R(V,T)T := (T^j R^i_{jkh} T^h) V^k \frac{\partial}{\partial x^i}$$

is evaluated at the point (σ, T).

The index form is bilinear and symmetric.

Let $T \wedge V$ be the flag (a plane in $T_x M$) spanned by the flagpole T and by a unit vector V which is orthogonal to the flagpole. The flag curvature in the point $(\sigma(t), T)$ and for the said flag is then given by

$$K(T \wedge V) = g_T(R(V,T)T, V) = V^i (T^j R_{jikh} T^h) V^k =: V^i R_{ik} V^k. \tag{2.3}$$

If W is a continuous piecewise C^∞ vector field such that it is g_T-orthogonal to σ we have

$$I(W,W) = \int_0^L [g_T(D_T W, D_T W) - K(T \wedge W) g_T(W,W)] dt, \tag{2.2'}$$

where $K(T \wedge W)$ is the flag curvature of the flag with flagpole T and transverse edge W.

Let $0 =: t_0 < t_1 < ... < t_h := L$ be a partition of $[0, L]$ such that V and W are both C^∞ on each closed subinterval $[t_{s-1}, t_s]$. Using integration by parts, one can rewrite the index form as

$$\begin{aligned} I(V,W) = g_T(D_T V, W)\Big|_0^L - \textstyle\sum_{s=1}^{h-1} g_T(D_T V, W)\big|_{t_s^-}^{t_s^+} - \int_0^L g_T(D_T D_T V + \\ + R(V,T)T, W) dt. \end{aligned} \tag{2.4}$$

The second term in the right side of the above equality disappears if V is of the class C^1 along σ. And the first term vanishes if $W(0) = W(r) = 0$. A vector field J along σ is said to be a *Jacobi field* if it satisfies the equation

$$D_T D_T J + R(J,T)T = 0. \tag{2.5}$$

One says that $q = \sigma(L)$ is conjugate with $p = \sigma(0)$ along σ if there exists a nonzero Jacobi field J along σ which vanishes at p and q, i.e. $J(0) = J(L) = 0$.

We recall from [4] p.182 the following result

Proposition 2.1. *Let $\sigma(t), 0 \leq t \leq r$ be a geodesic in a Finsler manifold (M, F). Suppose no point $\sigma(t), 0 < t \leq r$ is conjugate to $p := \sigma(0)$. Let W be any piecewise C^∞ vector field along σ and let J denote the unique Jacobi field along σ that has the same boundary values as W. That is, $J(0) = W(0)$ and $J(r) = W(r)$. Then*

$$I(W,W) \geq I(J,J). \tag{2.6}$$

Equality holds if and only if W is actually a Jacobi field, in which case the said J coincides with W.

As an application of this result we obtain the following corollary

Corollary 2.2. *Let $\sigma(t)$, $0 \leq t \leq r$ be a geodesic in a Finsler manifold (M, F). Let W be a piecewise C^∞ vector field along σ, which is nowhere 0 on $(0, r)$, satisfies $W(0) = W(r) = 0$ and $I(W, W) \leq 0$ on $[0, r]$. Then, the geodesic $\sigma(t)$ must contain conjugate points with $\sigma(0)$.*

Proof. We proceed by contradiction. Suppose that no point $\sigma(t), 0 < t \leq r$ is conjugate to $\sigma(0)$. By the definition of the conjugate points, the unique Jacobi field which vanishes at the endpoints of $\sigma(t), 0 \leq t \leq r$ is identically zero. The vector field W satisfies $W(0) = W(r) = 0$ and it can not be a Jacobi field since is nowhere zero on $(0, r)$. By the Proposition 2.1 we have $0 = I(J, J) < I(W, W) \leq 0$ which is a contradiction. Thus $\sigma(r)$ or an $\sigma(t)$ for $t < r$ should be conjugate with $\sigma(0)$. □

Ricci scalar

Let $\{l = \frac{y}{F(x,y)}, e_\alpha, \alpha = 1, \ldots, n-1\}$ be a g_y-orthonormal basis for the fiber of π^*TM over the point $(x, y) \in TM\backslash 0$. With respect to it one has $K_{(x,y)}(l \wedge e_\alpha) = g_y(R(e_\alpha, l)l, e_\alpha) = R_{\alpha\alpha}$.

The Ricci scalar denoted by $Ric_{(x,y)}$ is

$$Ric_{(x,y)} := \sum_{\alpha=1}^{n-1} K(x, y, l \wedge e_\alpha) = \sum_{\alpha=1}^{n-1} R_{\alpha\alpha}.$$

If (M, F) has constant flag curvature c, then $Ric_{(x,y)} = (n-1)c$.

3 Proof of Theorem 1.1

It suffices to prove that if along every unit speed geodesic $\sigma(t)$, $0 \leq t \leq L$ the Ricci scalar satisfies the hypothesis a) and b) of the Theorem 1.1 and if

$$L \geq -\varepsilon \frac{\Lambda}{a(n-1)} + \sqrt{\frac{\pi^2}{a} + \frac{\Lambda^2}{a^2(n-1)^2}},$$

then σ must contain conjugate points to $\sigma(0)$.

Using the parallel transport with reference vector T we construct a moving frame $\{e_i(t)\}$ along σ such that

(i) Each e_i is parallel along σ, that is $D_T e_i = 0$,

(ii) $\{e_i(t)\}$ is a g_T-orthonormal frame,

(iii) $e_n = T$.

Define $W_\alpha(t) = f(t)e_\alpha(t)$ for some smooth function f, $\alpha = 1, 2, ..., n-1$.

Fix a positive $r \geq L$ and consider the index from I for $\sigma(t), 0 \leq t \leq r$. By (2.2') we have

$$I(W_\alpha, W_\alpha) = \int_0^r [g(D_T W_\alpha, D_T W_\alpha) - g(W_\alpha, W_\alpha)K(T, W_\alpha)]dt,$$

where $K(T \wedge W_\alpha)$ is the flag curvature evaluated at the point $(\sigma(t), T) \in TM\backslash 0$.

We have $D_T W_\alpha = \frac{df_\alpha}{dt} e_\alpha$ and since the flag curvature does not depend on vectors spanning the flag, the equality $K(T, W_\alpha) = K(T, e_\alpha)$ holds.

Using these facts, $I(W_\alpha, W_\alpha)$ takes the form

$$I(W_\alpha, W_\alpha) = \int_0^r \left[\left(\frac{df}{dt} \right)^2 - f^2 K(T, e_\alpha) \right] dt. \tag{3.1}$$

We take $f(t) = \sin \dfrac{\pi t}{r}$ and we get

$$I(W_\alpha, W_\alpha) = \frac{\pi^2}{2r} - \int_0^r \sin^2 \frac{\pi t}{r} K(T, e_\alpha) dt. \tag{3.2}$$

Summing over α one obtains

$$\sum_\alpha I(W_\alpha, W_\alpha) = (n-1)\frac{\pi^2}{2r} - \int_0^r Ric(t)dt + \int_0^r Ric(t)\cos^2 \frac{\pi t}{r} dt. \tag{3.3}$$

By the assumptions a) and b) one gets

$$\sum_\alpha I(W_\alpha, W_\alpha) \leq (n-1)\frac{\pi^2}{2r} - (n-1)ar - \varepsilon\Lambda + (n-1)a \int_0^r cos^2 \frac{\pi t}{r} dt. \tag{3.4}$$

Computing the indicated integral one yields

$$\sum_\alpha I(W_\alpha, W_\alpha) \leq \frac{(n-1)}{2r}(\pi^2 - 2\varepsilon \frac{\Lambda}{n-1} r - ar^2) \tag{3.5}$$

and we have $\sum_\alpha I(W_\alpha, W_\alpha) \leq 0$ if $r \geq L = -\varepsilon \frac{\Lambda}{a(n-1)} + \sqrt{\frac{\pi^2}{a} + \frac{\Lambda^2}{a^2(n-1)^2}}$. It follows that some $I(W_\alpha, W_\alpha)$ must be non-positive and let denote that W_α by W.

This W satisfies the hypothesis of the Corollary 2.2. Applying it the desired conclusion follows.

In order to prove the statements 2)-3) of the Theorem 1.1, the same arguments as those from [4], p. 196-198, are used. We outline them in the following.

Since M is forward geodesically complete, by the Hopf-Rinow theorem any pair of points in M can be joined by a minimal geodesic. It is known that the cut point of $\sigma(0)$ appears before or coincide with the first conjugate point to $\sigma(0)$. As we have just proved, such a geodesic must have the length less than or equal with $-\varepsilon \frac{\Lambda}{a(n-1)} + \sqrt{\frac{\pi^2}{a} + \frac{\Lambda^2}{a^2(n-1)^2}}$. Thus $\operatorname{diam}(M) \leq -\varepsilon \frac{\Lambda}{a(n-1)} + \sqrt{\frac{\pi^2}{a} + \frac{\Lambda^2}{a^2(n-1)^2}}$, hence 2) holds. By the statement 2) the manifold M is forwardly bounded from the above. As it is always closed in its own topology, using again the Hopf-Rinow theorem one concludes that M is compact, that is, the statement 3) holds. Thus the Theorem 1.1 is completely proved. □

Remark 3.1. If in the main theorem (Theorem 1.2) from [10] the function *max* is explicitly written, two statements are obtained. The one covers the first three items of the Bonnet-Myers Theorem. The other one is similar with the case $\varepsilon = -1$ from Theorem 1.1 except that the bound of the diameter of M is $\frac{\pi}{\sqrt{a}} + \frac{\Lambda}{a(n-1)}$. This is clearly less than our bound in the case $\varepsilon = -1$. But our bound in the case $\varepsilon = +1$

is strictly lesser then the bound $\frac{\pi}{\sqrt{a}} + \frac{\Lambda}{a(n-1)}$. The latter was found in [10] by using a Ricatti inequation satisfied by the trace of the Hessian of the Finslerian distance function on M. Thus we have three different bounds for the diameter of M, all depending on Λ. If Λ increases to $+\infty$ two of them monotonically increase also to $+\infty$ and one monotonically decreases to zero. For $\Lambda = 0$ all three reduce to the bound given by the Bonnet-Myers Theorem.

4 Proof of Theorem 1.2

Before going on we notice that in the proof of Theorem 1.1 a main fact was that for given a point $p \in M$ every unit speed geodesic emanating from p contains a first point conjugates to p. Then using the Morse index form a evaluation of length of the geodesic from p to this first conjugate point was performed. Based on it a bound of the diameter of M was found and from here the conclusion that M is compact. But the same conclusion can be derived directly from the just mentioned main fact. In the Riemannian case the remark is due to W. Ambrose [1]. In our framework it can be formulated as follows.

Lemma 4.1. *Let (M, F) be a forward geodesically complete connected Finsler manifold of dimension n. If there exists a point $p \in M$ such that every geodesic ray emanating from p has a point conjugate to p along that ray, then M is compact.*

Proof. Let S_p be the indicatrix in the point $p \in M$. For each $p \in M$ and $y \in S_p$ we consider the unit speed geodesic from p with the initial velocity y. Each such geodesic is defined for any $t \in [0, \infty)$. Let c_y be the value of t in the first conjugate point of p and i_y the value of t in the cut point of p. By the hypothesis, the set of c_y is forwardly bounded from above (if $c_y = \infty$ one says that p has no conjugate points along that geodesic) and since one has $i_y \leq c_y$ it follows that $\sup_{y \in S_p} i_y \leq \sup_{y \in S_p} c_y$ and because the diameter of M is less or equal with $\sup_{y \in S_p} i_y$ it comes out that M is forwardly bounded from the above. Since M is closed in its own topology, by the Hopf-Rinow theorem it is compact. $\square$

Thus in order to prove the Theorem 1.2 it suffices to prove that there exists a point $p \in$ such that every unit speed geodesic $\sigma : [0, \infty) \to M$ issuing from p has a point conjugate to p along $\sigma : [0, \infty) \to M$. The Morse index lemma will be used again. We repeat the construction leading to the formula (3.1) from Section 3 and replace the function f by the following one:

$$f(t) = \begin{cases} t, & t \in [0, 1) \\ 1, & t \in [a, b] \\ \dfrac{r - t}{r - b}, & t \in [b, r] \end{cases}$$

Then summing over α, instead of (3.3) one gets

$$\begin{aligned} \textstyle\sum_\alpha I(W_\alpha, W_\alpha) &= \textstyle\int_0^1 ((n-1) - t^2 Ric(t))dt - \int_1^b Ric(t)dt + \\ &+ \textstyle\int_b^r \left(\dfrac{n-1}{(r-b)^2} - \dfrac{(r-t)^2}{(r-b)^2} Ric(t) \right) dt. \end{aligned} \tag{4.1}$$

In the right hand of this equality, the first integral is finite, by the hypothesis of the Theorem 1.2, the second integral in (4.1) diverges to $-\infty$ and by an integration by parts it comes out that the third integral tends to 0 when r tends to ∞.

Thus $\sum_\alpha I(W_\alpha, W_\alpha) \leq 0$ and hence there exists a W as in Corollary 2.1 such that $I(W, W) \leq 0$. The Corollary 2.1 implies that p has a conjugate point along the geodesic $\sigma : [0, \infty) \to M$. □

Acknowledgements. The author was supported by a grant of the Romanian National Authority for Scientific Research, CNSS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0256 as well as by the bilateral Romanian-Hungarian grant RO-HU 672/2013.

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