Global pinching theorem for spacelike submanifolds in semi-Riemannian space forms

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Abstract. In this paper, we deal with the compact spacelike submanifolds with parallel normalized mean curvature vector in an indefinite space form and obtain a global pinching result.

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1 Introduction

Let $Q_p^{n+p}(c)$ be an $(n+p)$-dimensional connected semi-Riemannian manifold of index p and of constant curvature c, which is called an indefinite space form of index p. If $c > 0$, we call it the de Sitter space of index p and denote it by $S_p^{n+p}(c)$. If $c < 0$, we call it the semi-Hyperbolic space of index p and denote it by $H_p^{n+p}(c)$. A smooth immersion $\varphi : M^n \to Q_p^{n+p}(c)$ of an n dimensional connected manifold M^n is said to be a *spacelike* if the induced metric via φ is a Riemannian metric on M^n. As is usual, the spacelike submanifold is said to be complete if the Riemannian induced metric is a complete metric on M^n.

The study of spacelike hypersurfaces in an indefinite space form of index p has been recently the the focus of substantial interest from both physics and mathematical community. It was pointed by Marsdenand Tipler [21] and Stumbles [27] that spacelike hypersurfaces with constant mean curvature in arbitrary spacetime are interesting in the relativity theory. The interest in the study of spacelike hypersurfaces immersed in the de Sitter space is motivated by their nice Bernstein-type properties. It was proved by E. Calabi [5] (for $n \leq 4$) and by S.Y. Cheng and S.T. Yau [16] (for all n) that a complete maximal spacelike hypersurface in L^{n+2} is totally geodesic. In [24], S. Nishikawa obtained similar results for others Lorentzian manifolds. In particular, he proved that a complete maximal spacelike hypersurface in $S_1^{n+1}(1)$ is totally geodesic.

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Goddard [17] conjectured that a complete spacelike hypersurface with constant mean curvature in de Sitter space S_1^{n+1} should be umbilical. Although the conjecture turned out to be false in its original statement, it motivated a great deal of work of several authors trying to find a positive answer to the conjecture under appropriate additional hypotheses ([3, 11, 22, 23]). There are also many works about the Goddard's problem for spacelike hypersurface with constant scalar curvature in de Sitter space ([6, 8, 10, 13, 19, 29]).

In higher codimension, the condition on the mean curvature is replaced by a condition on the mean curvature vector. Let $Q_p^{n+p}(c)$ be the complete connected semi-Riemannian manifolds of index p with constant curvature c and M^n be a spacelike submanifold of $Q_p^{n+p}(c)$ with parallel mean curvature vector h. When M^n is maximal, i.e., $h \equiv 0$, T. Ishihara [18] established a inequality for the squared norm S of the second fundamental form of M^n: $\frac{1}{2}\triangle S \geq S(nc + S/2)$. As an important application, Ishihara proved that maximal complete spacelike submanifolds in $Q_p^{n+p}(c), c \geq 0$, are totally umbilical and, if $c < 0$, then $0 \leq S \leq -npc$. Moreover, he determined all the complete spacelike maximal submanifolds M^n of $Q_p^{n+p}(c), c < 0$, satisfying $S = -npc$. R. Aiyama [2] studied compact spacelike submanifolds in $S_p^{n+p}(c)$ with parallel mean curvature vector and proved that if the normal connection of M^n is flat, then M^n is totally umbilical. She also proved that compact spacelike submanifolds in $S_p^{n+p}(c)$ with parallel mean curvature vector and non-negative sectional curvatures are also totally umbilical. Q. M. Cheng [12] showed that Akutagawa's result [3] is valid for complete spacelike submanifolds in $S_p^{n+p}(c)$ with parallel mean curvature vector.

In [14] and [15], Chaves-Sousa obtained a Simon type formula for the squared norm of the traceless tensor $\phi = B - Hg$, where g stands for the induced metric on a spacelike submanifold in $Q_p^{n+p}(c)$ with parallel mean curvature vector. As an application of this formula, Brasil-Chaves-Mariano [4] obtained an other limitation for the supremum of the mean curvature $\sup H^2 < \frac{4(n-1)c}{(n-2)^2p+4(n-1)}$ as an extension of results of [3] and [12]. Camargo-Chaves-Sousa [7] considered complete spacelike submanifold in $Q_p^{n+p}(c)$ with parallel normalized mean curvature vector (which is much weaker than the condition to have parallel mean curvature vector) and constant normalized scalar curvature r satisfying $r \leq c$. They proved that if the mean curvature satisfies $\sup H^2 < \frac{4(n-1)c}{(n-2)^2p+4(n-1)}$, then M^n is totally umbilical. In [9], the author improved this result and proved a rigidity theorem under the hypothesis of the mean curvature and the normalized scalar curvature being linearly related.

However, all these works have pointwise conditions on the squared norm of the second fundamental form S or on the mean curvature H. There are some works that consider L_p-pinching conditions instead of pointwise one. Shen [26] proved that if M^n be an oriented closed embedded minimal hypersurface in $S^{n+1}(1)$ with nonnegative Ricci curvature and $\int_M S^{\frac{n}{2}}\, dv < C(n)$, where $C(n)$ is a positive universal constant, then M^n is a totally geodesic hypersurface. Lin and Xia [20] proved that if M^{2n} be an even dimensional oriented closed minimal submanifold in $S^{2n+p}(1)$ with Euler characteristic not greater than two and $\int_M S^{\frac{n}{2}}\, dv < C(n,p)$, where $C(n,p)$ is a positive universal constant depending on n and p, then M is totally geodesic. Xu [28] proved that if M^n be an oriented closed submanifold with parallel mean curvature in $S^{n+p}(1)$ with $\int_M (S - nH^2)^{\frac{n}{2}}\, dv < C(n,p)$, then M is totally umbilical. Recently Araujo and Barbosa [1] considered the case of compact spacelike submanifolds with parallel mean curvature vector in an indefinite space form and proved the following result.

Theorem 1.1 ([1]). *Let M^n be a compact spacelike submanifold in $Q_p^{n+p}(c)$, with mean curvature $H \neq 0$ such that the mean curvature vector is parallel. Then there exists a positive constant $C = C(n, H)$ such that if $\|S\|_{\frac{n}{2}} < C$, where S is the squared norm of the second fundamental form of M^n and*

$$\|S\|_{\frac{n}{2}} = \left(\int_M S^{\frac{n}{2}} dv \right)^{\frac{2}{n}},$$

then M^n is totally umbilical.

In this paper, we deal with the case of compact spacelike submanifolds with parallel normalized mean curvature vector (which is much weaker than the condition to have parallel mean curvature vector as stated above) in an indefinite space form and obtain the following global pinching result.

Theorem 1.2. *Let M^n be a compact spacelike submanifold in $Q_p^{n+p}(c)$ $(c \geq 0)$ with mean curvature H bounded away from zero and parallel normalized mean curvature vector. If the normalized scalar curvature $r = aH + b, a, b \in \mathbb{R}$ and $b < c$, then there exists positive constant $C(n)$ such that if $\|S\|_{\frac{n}{2}} < C(n)$, then M^n is totally umbilical.*

As a corollary, taking $a = 0$ in Theorem 1.2, we have a global pinching result on compact spacelike submanifold in $Q_p^{n+p}(c)$ with parallel normalized mean curvature vector and constant scalar curvature.

Corollary 1.3. *Let M^n be a compact spacelike submanifold in $Q_p^{n+p}(c)$ $(c \geq 0)$ with parallel normalized mean curvature vector and constant scalar curvature $r < c$. If mean curvature H is bounded away from zero, then there exists positive constant $C(n)$ such that if $\|S\|_{\frac{n}{2}} < C(n)$, then M^n is totally umbilical.*

2 Preliminaries

Let M^n be an n-dimensional Riemannian manifold immersed in $Q_p^{n+p}(c)$. For any $q \in M$, we choose a local orthonormal frame $e_1, \cdots, e_{n+p}$ in $Q_p^{n+p}(c)$ around q such that $e_1, \cdots, e_n$ are tangent to M^n. Take the corresponding dual coframe $\omega_1, \cdots, \omega_{n+p}$. We use the following standard convention for indices:

$$1 \leq A, B, C, \cdots \leq n+p, \quad 1 \leq i, j, k, \cdots \leq n, \quad n+1 \leq \alpha, \beta, \gamma \cdots \leq n+p.$$

Let $\varepsilon_i = 1, \varepsilon_\alpha = -1$, then the structure equations of $Q_p^{n+p}(c)$ are given by

$$d\omega_A = \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \varepsilon_C \varepsilon_D R_{ABCD} \omega_C \wedge \omega_D, \tag{2.2}$$

$$R_{ABCD} = c\varepsilon_A \varepsilon_B (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2.3}$$

Restricting those forms to M^n, we have

$$\omega_\alpha = 0, \quad n+1 \leq \alpha \leq n+p. \tag{2.4}$$

So the Riemannian metric of M^n is written as $ds^2 = \sum_i \omega_i^2$. Since $0 = d\omega_\alpha = \sum_i \omega_{\alpha i} \wedge \omega_i$, from Cartan lemma, we can write

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{2.5}$$

Let $B = \sum_{\alpha,i,j} h_{ij}^\alpha \omega_i \omega_j e_\alpha$ be the second fundamental form. We will denote by $h = \frac{1}{n}\sum_\alpha (\sum_i h_{ii}^\alpha) e_\alpha$ and by $H = |h| = \frac{1}{n}\sqrt{\sum_\alpha (\sum_i h_{ii}^\alpha)^2}$ the mean curvature vector and the mean curvature of M^n, respectively.

The structure equations of M^n are

$$d\omega_i = \sum_{j=1}^n \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.6}$$

$$d\omega_{ij} = \sum_{k=1}^n \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l=1}^n R_{ijkl}\omega_k \wedge \omega_l. \tag{2.7}$$

The Gauss equations are

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{2.8}$$

$$n(n-1)r = n(n-1)c - n^2H^2 + S, \tag{2.9}$$

where r is the normalized scalar curvature of M^n and $S = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2$ is the norm square of the second fundamental form of M^n.

The Codazzi equations are

$$h_{ijk}^\alpha = h_{ikj}^\alpha = h_{jik}^\alpha, \tag{2.10}$$

where the covariant derivative of h_{ij}^α is defined by

$$\sum_k h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha + \sum_k h_{kj}^\alpha \omega_{ki} + \sum_k h_{ik}^\alpha \omega_{kj} - \sum_\beta h_{ij}^\beta \omega_{\beta\alpha}. \tag{2.11}$$

Similarly, the components h_{ijkl}^α of the second derivative $\nabla^2 h$ are given by

$$\sum_l h_{ijkl}^\alpha \omega_l = dh_{ijk}^\alpha + \sum_l h_{ljk}^\alpha \omega_{li} + \sum_l h_{ilk}^\alpha \omega_{lj} + \sum_l h_{ijl}^\alpha \omega_{lk} - \sum_\beta h_{ijk}^\beta \omega_{\beta\alpha}. \tag{2.12}$$

By exterior differentiation of (2.11), we can get the following *Ricci formula*

$$h_{ijkl}^\alpha - h_{ijlk}^\alpha = \sum_m h_{im}^\alpha R_{mjkl} + \sum_m h_{jm}^\alpha R_{mikl} + \sum_\beta h_{ij}^\beta R_{\alpha\beta kl}. \tag{2.13}$$

The Laplacian $\triangle h_{ij}^\alpha$ of h_{ij}^α is defined by $\triangle h_{ij}^\alpha = \sum_k h_{ijkk}^\alpha$, from the Codazzi equation and Ricci formula, we have

$$\triangle h_{ij}^\alpha = \sum_k h_{kkij}^\alpha + \sum_{m,k} h_{km}^\alpha R_{mijk} + \sum_{m,k} h_{im}^\alpha R_{mkjk} + \sum_{k,\beta} h_{ik}^\beta R_{\alpha\beta jk}. \tag{2.14}$$

If $H \neq 0$, we choose $e_{n+1} = \frac{h}{H}$, then it follows that

$$H^{n+1} := \frac{1}{n} tr(h^{n+1}) = H; \quad H^{\alpha} := \frac{1}{n} tr(h^{\alpha}) = -H\omega_{n+1\alpha}, \forall \alpha \geq n+2, \tag{2.15}$$

where h^{α} denotes the matrix (h^{α}_{ij}). From (2.11) and (2.15), we can see that

$$\sum_k H^{n+1}_k \omega_k = dH; \quad \sum_k H^{\alpha}_k \omega_k = -H\omega_{n+1\alpha}, \forall \alpha \geq n+2. \tag{2.16}$$

From (2.12),(2.15) and (2.16) we have

$$H^{n+1}_{kl} = H_{kl} - \frac{1}{H} \sum_{\beta > n+1} H^{\beta}_k H^{\beta}_l, \tag{2.17}$$

where

$$dH = \sum_i H_i \omega_i, \quad \nabla H_k = \sum_l H_{kl} \omega_l = dH_k + \sum_l H_l \omega_{lk}.$$

If M^n has parallel normalized mean curvature vector, we have

$$\omega_{n+1\alpha} = 0, \quad h^{n+1} h^{\alpha} = h^{\alpha} h^{n+1}, \forall \alpha. \tag{2.18}$$

Then (2.16) and (2.17) yield

$$H^{\alpha}_k = 0, \forall k, \alpha \geq n+2; \quad H^{n+1}_{kl} = H_{kl}. \tag{2.19}$$

From (2.12) and (2.19) we obtain

$$H^{\alpha}_{kl} = 0, \alpha \geq n+2. \tag{2.20}$$

From (2.24) of [7] we have

$$\begin{aligned}
\frac{1}{2}\triangle S =& \frac{1}{2} \sum_{\alpha,i,j} \triangle (h^{\alpha}_{ij})^2 = \sum_{\alpha,i,j,k} (h^{\alpha}_{ijk})^2 + \sum_{\alpha,i,j} h^{\alpha}_{ij} \triangle h^{\alpha}_{ij} \\
=& \sum_{\alpha,i,j,k} (h^{\alpha}_{ijk})^2 + n \sum_{\alpha,i,j} h^{\alpha}_{ij} H^{\alpha}_{ij} + nc(S - nH^2) \\
& - nH \sum_{\alpha} tr(h^{n+1}(h^{\alpha})^2) + \sum_{\alpha,\beta} (tr(h^{\alpha} h^{\beta}))^2 \\
& + \sum_{\alpha,\beta} N(h^{\alpha} h^{\beta} - h^{\beta} h^{\alpha}),
\end{aligned} \tag{2.21}$$

where $N(A) = tr(AA^t)$, for all matrix $A = (a_{ij})$.

Set $\phi^{\alpha}_{ij} = h^{\alpha}_{ij} - H^{\alpha}\delta_{ij}$, it is easy to check that ϕ^{α} is traceless and

$$\begin{aligned}
&|\phi|^2 = \sum_{\alpha,i,j} (\phi^{\alpha}_{ij})^2 = S - nH^2 \\
&N(\phi^{\alpha}) = N(h^{\alpha}) - n(H^{\alpha})^2, \ \ n+1 \leq \alpha \leq n+p,
\end{aligned} \tag{2.22}$$

where ϕ^α denotes the matrix (ϕ^α_{ij}). Following the operator $\Box$ in [16], as in [9], we introduce a Cheng-Yau's modified operator L

$$L = \Box + \frac{n-1}{2} a\triangle. \tag{2.23}$$

Here the operator $\Box$ acting on any smooth function f is defined by

$$\Box(f) = \sum_{i,j} (nH\delta_{ij} - h^{n+1}_{ij}) f_{ij}$$

and f_{ij} is given by the following

$$\sum_j f_{ij}\omega_j = df_i + f_j\omega_{ij}.$$

Lemma 2.1. *Let $M^n \hookrightarrow Q^{n+p}(c)$ be an oriented isometrically immersed submanifold with $R = aH + b$, where a, b are real constants and $b < c$. Then L is elliptic.*

Proof. Combining Gauss equation (2.9) and the assumption $r = aH + b$, we have

$$S = n^2H^2 + n(n-1)(aH + b - c). \tag{2.24}$$

Together with the assumption $b < c$ gives

$$nH[nH + (n-1)a] = n(n-1)(c-b) + S > 0.$$

Thus the connectedness of M implies that H does not change sign if $b < c$. So we can choose the orientation of M such that $H > 0$ on Σ. Choose a local orthonormal frame field $e_1, \ldots, e_n$ at $q \in M$ such that $h^{n+1}_{ij} = \lambda^{n+1}_i \delta_{ij}$. Let μ_i be the eigenvalue of $P = (nH + \frac{n-1}{2}a)I - h^{n+1}$ at every point $q \in M$, then $\mu_i = nH + \frac{n-1}{2}a - \lambda^{n+1}_i$ $(i = 1, 2, \cdots, n)$. Since $H \neq 0$, we can obtain from (2.9) that

$$\frac{n-1}{2}a = \frac{1}{2nH}\big(S - n^2H^2 + n(n-1)(c-b)\big).$$

Therefore, for every i,

$$\begin{aligned} \mu_i &= nH + \frac{n-1}{2}a - \lambda^{n+1}_i \\ &= nH - \lambda^{n+1}_i + \frac{1}{2nH}\big(S - n^2H^2 + n(n-1)(c-b)\big) \\ &= \frac{n-1}{2H}(c-b) + \frac{1}{2nH}\big(S + n^2H^2 - 2nH\lambda^{n+1}_i\big). \end{aligned} \tag{2.25}$$

Observe now that

$$\begin{aligned} S + n^2H^2 - 2nH\lambda^{n+1}_i &= \sum_{j=1}^n (\lambda^{n+1}_j)^2 + \Big(\sum_{j=1}^n \lambda^{n+1}_j\Big)^2 - 2\Big(\sum_{j=1}^n \lambda^{n+1}_j\Big)\lambda^{n+1}_i \\ &= \sum_{j=1, j\neq i}^n (\lambda^{n+1}_j)^2 + \Big(\sum_{j=1, j\neq i}^n \lambda^{n+1}_j\Big)^2 \geq 0. \end{aligned} \tag{2.26}$$

So (2.25), (2.26) and $b < c$ imply that $\mu_i > 0$ for each i and L is elliptic. $\Box$

Lemma 2.2 ([25]). *Let $A, B : \mathbb{R}^n \to \mathbb{R}^n$ be symmetric linear maps such that $AB - BA = 0$ and $tr(A) = tr(B) = 0$. Then*

$$|trA^2B| \leq \frac{n-2}{\sqrt{n(n-1)}} N(A)\sqrt{N(B)}.$$

Proposition 2.3. *Let M^n be a spacelike submanifold in $Q_p^{n+p}(c)$ with parallel normalized mean curvature vector. If $r = aH + b, a, b \in \mathbb{R}$, then the following inequality holds*

$$L(nH) \geq \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 - n^2|\nabla H|^2 + |\phi|^2\Big(\frac{|\phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi| + n(c - H^2)\Big). \tag{2.27}$$

Proof. From (2.23) we have

$$\begin{aligned} L(nH) &= \sum_{i,j}((nH + \frac{1}{2}(n-1)a)\delta_{ij} - h_{ij}^{n+1})(nH)_{ij} \\ &= (nH + \frac{1}{2}(n-1)a)\triangle(nH) - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} \\ &= (nH + \frac{1}{2}(n-1)a)\triangle(nH + \frac{1}{2}(n-1)a) - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} \\ &= \frac{1}{2}\triangle(nH + \frac{1}{2}(n-1)a)^2 - |\nabla(nH + \frac{1}{2}(n-1)a)|^2 - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} \\ &= \frac{1}{2}\triangle(nH + \frac{1}{2}(n-1)a)^2 - n^2|\nabla H|^2 - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij}. \end{aligned} \tag{2.28}$$

On the other side, from Gauss equation and $r = aH + b$, we have

$$\begin{aligned} \triangle S &= \triangle(n^2H^2 + n(n-1)(r-c)) \\ &= \triangle(n^2H^2 + n(n-1)(aH + b - c)) \\ &= \triangle(n^2H^2 + (n-1)anH) = \triangle(nH + \frac{1}{2}(n-1)a)^2. \end{aligned} \tag{2.29}$$

From (2.21), (2.28) and (2.29) we get

$$\begin{aligned} L(nH) =& \frac{1}{2}\triangle S - n^2|\nabla H|^2 - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} \\ =& \sum_{\alpha,i,j,k}(h_{ijk}^\alpha)^2 - n^2|\nabla H|^2 + n\sum_{\alpha,i,j} h_{ij}^\alpha H_{ij}^\alpha - n\sum_{i,j} h_{ij}^{n+1}H_{ij} \\ &+ nc(S - nH^2) - nH\sum_\alpha tr(h^{n+1}(h^\alpha)^2) + \sum_{\alpha,\beta}(tr(h^\alpha h^\beta))^2 \\ &+ \sum_{\alpha,\beta} N(h^\alpha h^\beta - h^\beta h^\alpha). \end{aligned} \tag{2.30}$$

Since M^n has parallel normalized mean curvature vector, (2.19), (2.20) and (2.30) yield

$$
\begin{aligned}
L(nH) = &\sum_{\alpha,i,j,k} (h^{\alpha}_{ijk})^2 - n^2|\nabla H|^2 + nc(S - nH^2)\\
&- nH\sum_{\alpha} tr(h^{n+1}(h^{\alpha})^2) + \sum_{\alpha,\beta}(tr(h^{\alpha}h^{\beta}))^2 + \sum_{\alpha,\beta} N(h^{\alpha}h^{\beta} - h^{\beta}h^{\alpha}).
\end{aligned}
\tag{2.31}
$$

From (2.15) and (2.22), we have

$$
\begin{aligned}
&\phi^{n+1}_{ij} = h^{n+1}_{ij} - H\delta_{ij},\\
&N(\phi^{n+1}) = tr(\phi^{n+1})^2 = tr(h^{n+1})^2 - nH^2 = N(h^{n+1}) - nH^2,\\
&tr(h^{n+1})^3 = tr(\phi^{n+1})^3 + 3HN(\phi^{n+1}) + nH^3,\\
&\phi^{\alpha}_{ij} = h^{\alpha}_{ij}, \ \ N(\phi^{\alpha}) = N(h^{\alpha}), \ \ \alpha \geq n+2.
\end{aligned}
\tag{2.32}
$$

By (2.31) and (2.32), we see that

$$
\begin{aligned}
L(nH) \geq &\sum_{\alpha,i,j,k} (h^{\alpha}_{ijk})^2 - n^2|\nabla H|^2 + n|\phi|^2(c - H^2) - nH\sum_{\alpha} tr(\phi^{n+1}(\phi^{\alpha})^2)\\
&+ \sum_{\alpha,\beta}(tr(h^{\alpha}h^{\beta}))^2 + \sum_{\alpha,\beta} N(h^{\alpha}h^{\beta} - h^{\beta}h^{\alpha}).
\end{aligned}
\tag{2.33}
$$

By (2.18) we know that the traceless matrix ϕ^{n+1} commutes with the traceless matrices ϕ^{α}, for all α. Hence we can apply Lemma 2.2 in order to obtain

$$
\sum_{\alpha} tr(\phi^{n+1}(\phi^{\alpha})^2) \leq \frac{n-2}{\sqrt{n(n-1)}}\sqrt{N(\phi^{n+1})}|\phi|^2 \leq \frac{n-2}{\sqrt{n(n-1)}}|\phi|^3.
\tag{2.34}
$$

Moreover, Cauchy-Schwarz inequality implies that

$$
|\phi|^4 \leq p\sum_{\alpha}(N(\phi^{\alpha}))^2 \leq p\sum_{\alpha,\beta}(tr(h^{\alpha}h^{\beta}))^2.
\tag{2.35}
$$

Inserting (2.34) and (2.35) into (2.33), we arrive to (2.27). □

Following the idea of Lemma 1 in [28], we obtain the next key lemma for the proof of Theorem 1.2.

Lemma 2.4. *Let M^n be a spacelike submanifold in $Q^{n+p}_p(c)$ with parallel normalized mean curvature vector. Setting*

$$
f_{\varepsilon} = \Big(\sum_{i,j}(h^{n+1}_{ij})^2 - nH^2 + n\varepsilon^2\Big)^{1/2} \ \ \textit{and} \ \ g_{\varepsilon} = \Big(\sum_{i,j,\beta\neq n+1}(h^{\beta}_{ij})^2 + n(p-1)\varepsilon^2\Big)^{1/2},
$$

we have

$$
\begin{aligned}
&\sum_{i,j,k}(h^{n+1}_{ijk})^2 - n|\nabla H|^2 \geq \frac{n+2}{n}|\nabla f_{\varepsilon}|^2,\\
&\sum_{i,j,k,\beta\neq n+1}(h^{\beta}_{ijk})^2 \geq \frac{n+2}{n}|\nabla g_{\varepsilon}|^2, \quad \textit{for} \quad p \neq 1.
\end{aligned}
$$

Proof. Set $x_{ij}^{n+1} = h_{ij}^{n+1} - H\delta_{ij} + \varepsilon\delta_{ij}$. Hence

$$\sum_{i,j}(x_{ij}^{n+1})^2 = f_\varepsilon^2,$$

and

$$\sum_{i,j,k}(x_{ijk}^{n+1})^2 = \sum_{i,j,k}(h_{ijk}^{n+1})^2 - n|\nabla H|^2. \tag{2.36}$$

Let e_i be a frame diagonalizing the matrix (h_{ij}^{n+1}) such that $h_{ij}^{n+1} = \lambda_i\delta_{ij}, 1 \le i,j \le n$. Then,

$$x_{ij}^{n+1} = (\lambda_i - H + \varepsilon)\delta_{ij}, \quad \sum_{i,j}(x_{ij}^{n+1})^2 = f_\varepsilon^2$$

and

$$\begin{aligned}(2f_\varepsilon|\nabla f_\varepsilon|)^2 &= 4\sum_k \big(\sum_i x_{ii}^{n+1}x_{iik}^{n+1}\big)^2\\ &\le 4\big(\sum_i(x_{ii}^{n+1})^2\big)\big(\sum_{i,k}(x_{iik}^{n+1})^2\big) = 4f_\varepsilon^2\sum_{i,k}(x_{iik}^{n+1})^2.\end{aligned} \tag{2.37}$$

Also,

$$\sum_{i,j,k}(x_{ijk}^{n+1})^2 \ge 2\sum_{i\ne k}(x_{iik}^{n+1})^2 + \sum_{i,k}(x_{iik}^{n+1})^2. \tag{2.38}$$

Now, for each fixed k, we have

$$\begin{aligned}\sum_i(x_{iik}^{n+1})^2 &= \sum_{i\ne k}(x_{iik}^{n+1})^2 + (x_{kkk}^{n+1})^2\\ &= \sum_{i\ne k}(x_{iik}^{n+1})^2 + \big(\sum_i x_{iik}^{n+1} - \sum_{i\ne k}x_{iik}^{n+1}\big)^2\\ &= \sum_{i\ne k}(x_{iik}^{n+1})^2 + \big(\sum_{i\ne k}x_{iik}^{n+1}\big)^2\\ &\le \sum_{i\ne k}(x_{iik}^{n+1})^2 + (n-1)\sum_{i\ne k}(x_{iik}^{n+1})^2 = n\sum_{i\ne k}(x_{iik}^{n+1})^2.\end{aligned} \tag{2.39}$$

Combining (2.36),(2.37),(2.38) and (2.39), we obtain

$$\sum_{i,j,k}(h_{ijk}^{n+1})^2 - n|\nabla H|^2 \ge \frac{n+2}{n}\sum_{i,k}(x_{iik}^{n+1})^2 \ge \frac{n+2}{n}|\nabla f_\varepsilon|^2. \tag{2.40}$$

If $p \ge 2$, we put $x_{ij}^\beta = h_{ij}^\beta + \varepsilon\delta_{ij}, n+2 \le \beta \le n+p$. By using the argument above, we obtain

$$|\nabla(g_\varepsilon^\beta)^2|^2 \le \frac{4n}{n+2}(g_\varepsilon^\beta)^2\sum_{i,j,k}(h_{ijk}^\beta)^2, \tag{2.41}$$

where $g_\varepsilon^\beta = \Big(\sum\limits_{i,j}(x_{ij}^\beta)^2\Big)^{\frac{1}{2}}$. From (2.41) we have

$$
\begin{aligned}
|\nabla g_\varepsilon^2| &\le \sum_{\beta\neq n+1}|\nabla (g_\varepsilon^\beta)^2| \le 2\sqrt{\frac{n}{n+2}}\sum_{\beta\neq n+1}\Big((g_\varepsilon^\beta)^2\sum_{i,j,k}(h_{ijk}^\beta)^2\Big)^{\frac{1}{2}}\\
&\le 2\sqrt{\frac{n}{n+2}}\sum_{\beta\neq n+1}(g_\varepsilon^2)^{\frac{1}{2}}\Big(\sum_{i,j,k,\beta\neq n+1}(h_{ijk}^\beta)^2\Big)^{\frac{1}{2}}.
\end{aligned}
$$

It follow that

$$\sum_{i,j,k,\beta\neq n+1}(h_{ijk}^\beta)^2 \ge \frac{n+2}{n}|\nabla g_\varepsilon|^2.$$

□

Lemma 2.5 ([1]). *Let M^n $(n\ge 3)$ be a compact connected Riemannian manifold. Then, for every $f\in C^\infty(M)$ and $t\in\mathbb{R}_+$, we have*

$$\int_M |\nabla f|^2 dv \ge \frac{k_1}{1+t}\|f\|_{2^*}^2 - \frac{k_2}{t}\|f\|_2^2, \tag{2.42}$$

where

$$k_1 = 2^{-3-\frac{2}{n}}\Big(\frac{n-2}{n-1}\Big)^2 C_1^{\frac{2}{n}}, \quad k_2 = 2^{E(n)+\frac{2}{n}-2}\Big(\frac{n-2}{n-1}\Big)^2 C_1^{\frac{2}{n}} vol(M)^{-\frac{2}{n}}$$

and C_1 is the isoperimetric contant of M,

$$E(n) = \begin{cases} \frac{(n-4)(n-2)}{2}, & n>3\\ 1, & n=3.\end{cases}$$

3 Proof of Theorem 1.2

Let M^n be a compact spacelike submanifold in $Q_p^{n+p}(c)$ with parallel normalized mean curvature vector. From proposition 2.3 and Lemma 2.4, we have

$$
\begin{aligned}
L(nH) \ge& \sum_{i,j,k}(h_{ijk}^{n+1})^2 - n^2|\nabla H|^2 + \sum_{i,j,k,\beta\neq n+1}(h_{ijk}^\beta)^2\\
&+|\phi|^2\Big(\frac{|\phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi| + n(c-H^2)\Big)\\
\ge& \frac{n+2}{n}|\nabla f_\varepsilon|^2 + \frac{n+2}{n}|\nabla g_\varepsilon|^2\\
&+|\phi|^2\Big(\frac{|\phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi| + n(c-H^2)\Big).
\end{aligned}
\tag{3.1}
$$

Since, for any $s\in\mathbb{R}_+$,

$$-\frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi| \ge -\frac{s}{2}|\phi|^2 - \frac{n(n-2)^2}{2(n-1)s}H^2. \tag{3.2}$$

Therefore, from (3.1) and (3.2), we get

$$
\begin{aligned}
L(nH) \geq & \frac{n+2}{n}|\nabla f_\varepsilon|^2 + \frac{n+2}{n}|\nabla g_\varepsilon|^2 \\
& + |\phi|^2\Big(\frac{|\phi|^2}{p} - \frac{s}{2}|\phi|^2 - \frac{n(n-2)^2}{2(n-1)s}H^2 + n(c-H^2)\Big).
\end{aligned} \tag{3.3}
$$

Since L is elliptic (by Lemma 2.1) and self-adjoint on compact manifold, we obtain from (3.3) that

$$
\begin{aligned}
0 \geq & \frac{n+2}{n}\int_M |\nabla f_\varepsilon|^2 dv + \frac{n+2}{n}\int_M |\nabla g_\varepsilon|^2 dv \\
& + \int_M |\phi|^2\Big(\frac{|\phi|^2}{p} - \frac{s}{2}|\phi|^2 - \frac{n(n-2)^2}{2(n-1)s}H^2 + n(c-H^2)\Big) dv.
\end{aligned} \tag{3.4}
$$

Hence, from (2.42), we have

$$
\begin{aligned}
0 \geq & \frac{n+2}{n}\frac{k_1}{1+t}\|f_\varepsilon\|_{2^*}^2 - \frac{n+2}{n}\frac{k_2}{t}\|f_\varepsilon\|_2^2 + \frac{n+2}{n}\frac{k_1}{1+t}\|g_\varepsilon\|_{2^*}^2 - \frac{n+2}{n}\frac{k_2}{t}\|g_\varepsilon\|_2^2 \\
& + \int_M |\phi|^2\Big(\frac{|\phi|^2}{p} - \frac{s}{2}|\phi|^2 - \frac{n(n-2)^2}{2(n-1)s}H^2 + n(c-H^2)\Big) dv
\end{aligned} \tag{3.5}
$$

Now, letting $\varepsilon \to 0$ in (3.5) and writing $f^2 = \sum_{i,j}(h_{ij}^{n+1})^2 - nH^2$ and $g^2 = \sum_{i,j,\beta \neq n+1}(h_{ij}^\beta)^2$, we get

$$
\begin{aligned}
0 \geq & \frac{n+2}{n}\frac{k_1}{1+t}\|f\|_{2^*}^2 - \frac{n+2}{n}\frac{k_2}{t}\|f\|_2^2 + \frac{n+2}{n}\frac{k_1}{1+t}\|g\|_{2^*}^2 - \frac{n+2}{n}\frac{k_2}{t}\|g\|_2^2 \\
& + \int_M |\phi|^2\Big(\frac{|\phi|^2}{p} - \frac{s}{2}|\phi|^2 - \frac{n(n-2)^2}{2(n-1)s}H^2 + n(c-H^2)\Big) dv
\end{aligned} \tag{3.6}
$$

Note that $f^2 + g^2 = |\phi|^2$ and $\|f\|_2^2 + \|g\|_2^2 = \|\phi\|_2^2$. Then, from (3.6), Minkowski inequality

$$
\|\phi\|_{2^*}^2 = \big\||\phi|^2\big\|_{\frac{2^*}{2}} = \|f^2+g^2\|_{\frac{2^*}{2}} \leq \|f^2\|_{\frac{2^*}{2}} + \|g^2\|_{\frac{2^*}{2}} = \|f\|_{2^*}^2 + \|g\|_{2^*}^2
$$

and Hölder's inequality

$$
\int_M S|\phi|^2 \leq \|S\|_{\frac{n}{2}}\|\phi\|_{2^*}^2,
$$

we obtain

$$
\begin{aligned}
0 \geq{}& \frac{n+2}{n}\frac{k_1}{1+t}\|\phi\|_{2^*}^2 - \frac{n+2}{n}\frac{k_2}{t}\|\phi\|_2^2 \\
&+ \int_M |\phi|^2\Big(\frac{|\phi|^2}{p} - \frac{s}{2}|\phi|^2 - \frac{n(n-2)^2}{2(n-1)s}H^2 + n(c-H^2)\Big)dv \\
\geq{}& \frac{n+2}{n}\frac{k_1}{1+t}\|\phi\|_{2^*}^2 - \frac{n+2}{n}\frac{k_2}{t}\|\phi\|_2^2 \\
&+ \int_M |\phi|^2\Big(-\frac{s}{2}S + \frac{ns}{2}H^2 - \frac{n(n-2)^2}{2(n-1)s}H^2 + n(c-H^2)\Big)dv \\
\geq{}& \left(\frac{ns}{2}H^2 - \frac{n(n-2)^2}{2(n-1)s}H^2 + n(c-H^2) - \frac{n+2}{n}\frac{k_2}{t}\right)\|\phi\|_2^2 \\
&+ \left(\frac{n+2}{n}\frac{k_1}{1+t} - \frac{s}{2}\|S\|_{\frac{n}{2}}\right)\|\phi\|_{2^*}^2. \qquad (3.7)
\end{aligned}
$$

Since the mean curvature H is bounded away from zero, if we set $|H| \geq k > 0$ and choose $t = t(s) = \frac{2sk_2(n-1)(n+2)}{n^2(n-2)^2k^2} \geq \frac{2sk_2(n-1)(n+2)}{n^2(n-2)^2H^2}$, we have

$$\frac{n(n-2)^2}{2(n-1)s}H^2 \geq \frac{n+2}{n}\frac{k_2}{t}$$

and

$$
\begin{aligned}
&\frac{ns}{2}H^2 - \frac{n(n-2)^2}{2(n-1)s}H^2 + n(c-H^2) - \frac{n+2}{n}\frac{k_2}{t} \\
&\geq \frac{ns}{2}H^2 - \frac{n(n-2)^2}{(n-1)s}H^2 + n(c-H^2) \\
&= \frac{n}{s}\left(\frac{H^2}{2}s^2 + (c-H^2)s - \frac{(n-2)^2}{n-1}H^2\right).
\end{aligned}
$$

So, taking

$$s > \alpha(n,H) = \frac{1}{H^2}\left(\sqrt{(c-H^2)^2 + \frac{2(n-2)^2}{n-1}H^4} - (c-H^2)\right)$$

and $t(s) = \frac{2sk_2(n-1)(n+2)}{n^2(n-2)^2k^2}$, we have

$$\frac{ns}{2}H^2 - \frac{n(n-2)^2}{2(n-1)s}H^2 + n(c-H^2) - \frac{n+2}{n}\frac{k_2}{t} \geq 0. \qquad (3.8)$$

Hence, if $c = 0$, we take $\beta(n) = \alpha(n,H) = \sqrt{1 + \frac{2(n-2)^2}{n-1}} + 1$; if $c > 0$, we take $\beta(n) = 2 + \frac{\sqrt{2}(n-2)}{\sqrt{n-1}} > \alpha(n,H)$. Therefore, if

$$\|S\|_{\frac{n}{2}} < C(n) = \sup_{s>\beta(n)} \frac{2(n+2)k_1}{ns(1+t(s))},$$

we obtain $|\phi|^2 \equiv 0$ and M^n is a totally umbilical.

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