Totally null surfaces in neutral Kähler 4-manifolds

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Abstract. We study the totally null surfaces of the neutral Kähler metric on certain 4-manifolds. The tangent spaces of totally null surfaces are either self-dual (α-planes) or anti-self-dual (β-planes) and so we consider α-surfaces and β-surfaces. The metric of the examples we study, which include the spaces of oriented geodesics of 3-manifolds of constant curvature, are anti-self-dual, and so it is well-known that the α-planes are integrable and α-surfaces exist. These are holomorphic Lagrangian surfaces, which for the geodesic spaces correspond to totally umbilic foliations of the underlying 3-manifold. The β-surfaces are less known and our interest is mainly in their description. In particular, we classify the β-surfaces of the neutral Kähler metric on TN, the tangent bundle to a Riemannian 2-manifold N. These include the spaces of oriented geodesics in Euclidean and Lorentz 3-space, for which we show that the β-surfaces are affine tangent bundles to curves of constant geodesic curvature on S^2 and H^2, respectively. In addition, we construct the β-surfaces of the space of oriented geodesics of hyperbolic 3-space.

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1 Introduction

Neutral Kähler 4-manifolds exhibit remarkably different behavior than their positive-definite counterparts. The failure of the complex structure J to tame the symplectic structure Ω means that 2-planes in the tangent space of a point can be both holomorphic and Lagrangian. Under favorable conditions (namely the vanishing of the self-dual conformal curvature) such planes are integrable and there exist holomorphic Lagrangian surfaces.

In the space $L(M)$ of oriented geodesics of a 3-manifold of constant curvature M (on which a natural neutral Kähler structure exists) such surfaces play a distinctive role: they correspond to totally umbilic foliations of M (see [2, 4, 5]).

Holomorphic Lagrangian planes are totally null, that is, the induced metric identically vanishes on the plane. Moreover, with respect to the Hodge star operator of

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the neutral metric, the self-dual 2-forms vanish on these planes. There exists however another class of totally null planes, upon which the anti-self-dual forms vanish. The former planes are referred to as α-planes, while the latter are β-planes.

In this note we consider the β-surfaces in certain neutral Kähler 4-manifolds, which include spaces $L(M)$ of oriented geodesics of 3-manifolds M of constant curvature. In the cases of $M = E^3, E^3_1, H^3$ we compute the β-surfaces explicitly and show that they include $L(E^2), L(H^2)$. In particular, we prove:

Main Theorem. *A β-surface in $L(E^3)$ is an affine tangent bundle over a curve of constant geodesic curvature in (S^2, g_{rnd}).*

A β-surface in $L(E^3_1)$ is an affine tangent bundle over a curve of constant geodesic curvature in $(H^2 g_{hyp})$.

A β-surface in $L(H^3)$ is a piece of a torus which, up to isometry, is either

1. *$L(H^2)$, where $H^2 \subset H^3$, or*
2. *$\mathcal{C}_1 \times \mathcal{C}_2 \subset \; S^2 \times S^2 - \bar{\Delta}$, where $\mathcal{C}_1$ is a circle given by the intersection of the 2-sphere and a plane containing the north pole, and $\mathcal{C}_2$ is the image of $\mathcal{C}_1$ under reflection in the horizontal plane through the origin.*

In the next section we discuss self-duality for planes in neutral Kähler 4-manifolds and their properties. We then turn to the neutral metric on TN and the special case $L(E^3)$ and $L(E^3_1)$. In the final section we characterize the β-surfaces in $L(H^3)$.

2 Neutral metrics on 4-manifolds

2.1 Self-dual and anti-self-dual 2-forms

Consider the neutral metric G on $\mathbb{R}^4$ given in standard coordinates (x^1, x^2, x^3, x^4) by

$$ds^2 = (dx^1)^2 + (dx^2)^2 - (dx^3)^2 - (dx^4)^2.$$

Throughout, we denote $\mathbb{R}^4$ endowed with this metric by $\mathbb{R}^{2,2}$.

The space of 2-forms on $\mathbb{R}^{2,2}$ is a 6-dimensional linear space that splits naturally with respect to the Hodge star operator $*$ of G into two 3-dimensional spaces: $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, the space of self-dual and anti-self-dual 2-forms. Thus, if $\omega \in \Lambda^2$, then $\omega = \omega_+ + \omega_-$, where $*\omega_+ = \omega_+$ and $*\omega_- = -\omega_-$.

We can easily find a basis for Λ^2_+ and Λ^2_-. First, define the *double null* basis of 1-forms:

$$\Theta^1 = dx^1 + dx^3, \qquad \Theta^2 = dx^2 - dx^4, \qquad \Theta^3 = -dx^2 - dx^4, \qquad \Theta^4 = dx^1 - dx^3,$$

so that the metric is

$$ds^2 = \Theta^1 \otimes \Theta^4 - \Theta^2 \otimes \Theta^3.$$

Proposition 2.1. *If $\omega \in \Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, with $\omega = \omega_+ + \omega_-$, then*

$$\omega_+ = a_1 \Theta^1 \wedge \Theta^2 + b_1 \Theta^3 \wedge \Theta^4 + c_1(\Theta^1 \wedge \Theta^4 - \Theta^2 \wedge \Theta^3),$$

$$\omega_- = a_2 \Theta^1 \wedge \Theta^3 + b_2 \Theta^2 \wedge \Theta^4 + c_2(\Theta^1 \wedge \Theta^4 + \Theta^2 \wedge \Theta^3),$$

for $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R}$.

Proof. This follows from computing the Hodge star operator acting on 2-forms:

$$*(\Theta^1\wedge\Theta^4)=-\Theta^2\wedge\Theta^3,\qquad *(\Theta^2\wedge\Theta^4)=-\Theta^2\wedge\Theta^4,\qquad *(\Theta^1\wedge\Theta^3)=-\Theta^1\wedge\Theta^3,$$

$$*(\Theta^3\wedge\Theta^4)=\Theta^3\wedge\Theta^4,\qquad *(\Theta^1\wedge\Theta^2)=\Theta^1\wedge\Theta^2,$$

which completes the proof. □

2.2 Totally null planes

Definition 2.1. A plane $P\subset\mathbb{R}^{2,2}$ is *totally null* if every vector in P is null with respect to G, and the inner product of any two vectors in P is zero.

A plane P is *self-dual* if $\omega_+(P)=0$ for all $\omega_+\in\Lambda^2_+$, and *anti-self-dual* if $\omega_-(P)=0$ for all $\omega_-\in\Lambda^2_-$. Self-dual planes are also called *α-planes*, while anti-self-dual planes are called *β-planes*.

Proposition 2.2. *A plane P is totally null iff P is either self-dual or anti-self-dual.*

Proof. Suppose all self-dual forms vanish on P and let $\{V,W\}$ be a basis for P. Let (e_1,e_2,e_3,e_4) be the vector basis of $\mathbb{R}^{2,2}$ that is dual to $(\Theta^1,\Theta^2,\Theta^3,\Theta^4)$ and $V=V^je_j$, $W=W^je_j$. Since all of the self-dual 2-forms vanish on P, we have from the expression of ω_+ in Proposition 2.1 that

$$V^1W^2=W^1V^2,\qquad V^3W^4=W^3V^4, \tag{2.1}$$

$$V^1W^4-V^2W^3=W^1V^4-W^2V^3. \tag{2.2}$$

We can assume without loss of generality that V and W are orthogonal: $G(V,W)=0$, which in frame components says that

$$V^1W^4+W^1V^4=V^2W^3+W^2V^3.$$

Combining this with equation (2.2) we have that

$$V^1W^4=V^2W^3,\qquad W^1V^4=W^2V^3. \tag{2.3}$$

Multiplying the first equation of (2.3) by W^1 we have $V^1W^4W^1=V^2W^3W^1$, which, by virtue of the first equation of (2.1), is $V^1W^4W^1=W^2W^3V^1$. Thus

$$G(W,W)V^1=2(W^1W^4-W^2W^3)V^1=0.$$

Similarly, multiplying the first equation of (2.3) by W^2, and the second equation by W^3 and W^4, applying equations (2.1), we find that

$$G(W,W)V^2=G(W,W)V^3=G(W,W)V^4=0.$$

Thus, either $G(W,W)=0$ or $V=0$. Since the latter is not true, we conclude that W is a null vector.

On the other hand, multiplying the second equation of (2.3) by V^1 and V^2, and the first by V^3 and V^4, utilizing equations (2.1), we have

$$G(V,V)W^1 = G(V,V)W^2 = G(V,V)W^3 = G(V,V)W^4 = 0.$$

Thus V is also a null vector, and the plane spanned by V and W is totally null, as claimed. An analogous argument establishes that a plane on which all anti-self-dual 2-forms vanish is totally null.

Conversely, suppose that a plane P is totally null. That is, in terms of a vector basis V and W as before

$$V^1V^4 = V^2V^3, \qquad W^1W^4 = W^2W^3, \tag{2.4}$$

$$V^1W^4 + V^4W^1 - V^2W^3 - V^3W^2 = 0. \tag{2.5}$$

Multiplying equation (2.5) by V^1,V^3, W^1 and W^3, yields, with the aid of equations (2.4):

$$V^2(V^3W^1 - V^1W^3) = V^1(V^3W^2 - V^1W^4), \tag{2.6}$$

$$V^4(V^3W^1 - V^1W^3) = V^3(V^3W^2 - V^1W^4), \tag{2.7}$$

$$W^2(V^1W^3 - V^3W^1) = W^1(V^2W^3 - V^4W^1), \tag{2.8}$$

$$W^4(V^1W^3 - V^3W^1) = W^3(V^2W^3 - V^4W^1). \tag{2.9}$$

Now, adding V^1 times equation (2.8), W^1 times equation (2.6), V^3 times equation (2.9) and W^3 times equation (2.7) and using equation (2.5), we obtain

$$(V^1W^2 - V^2W^1 + V^3W^4 - V^4W^3)(V^1W^3 - V^3W^1) = 0. \tag{2.10}$$

By a similar manipulation we find that

$$(V^1W^2 - V^2W^1 + V^3W^4 - V^4W^3)(V^2W^4 - V^4W^2) = 0. \tag{2.11}$$

Now suppose that P, in addition to being totally null, is Lagrangian. If $J(V)$ is not in P, then, since $G(W, J(V)) = \Omega(W,V) = 0$, the metric would be identically zero on the 3-space spanned by $\{V, W, J(V)\}$. For a non-degenerate metric G on $\mathbb{R}^{2,2}$ this is not possible. Thus $J(V) \in P$ and so P is a complex plane. It follows easily that P is self-dual.

On the other hand, suppose that the totally null plane P is not Lagrangian. Then $\Omega(V,W) \neq 0$ or

$$V^1W^2 - V^2W^1 + V^3W^4 - V^4W^3 \neq 0.$$

By equations (2.10) and (2.11), we have $V^1W^3 - V^3W^1 = V^2W^4 - V^4W^2 = 0$. Moreover, substituting these in (2.6) to (2.9) we conclude that $V^1W^4 - V^4W^1 + V^2W^3 - V^3W^2 = 0$. Then, by Proposition 2.1 we must have $\omega_-(V,W) = 0$, which completes the result. □

2.3 Kähler structures on $\mathbb{R}^{2,2}$

Up to an overall sign, there are two complex structures on $\mathbb{R}^{2,2}$ that are compatible with the metric G:

$$\begin{cases} J(X^1, X^2, X^3, X^4) = (-X^2, X^1, -X^4, X^3), \\ J'(X^1, X^2, X^3, X^4) = (-X^2, X^1, X^4, -X^3). \end{cases}$$

By compatibility we mean that $G(J\cdot, J\cdot) = G(\cdot, \cdot)$, and similarly for J'.

We can utilize these and define two symplectic forms by $\Omega = G(\cdot, J\cdot)$ and $\Omega' = G(\cdot, J'\cdot)$. That is

$$\Omega = dx^1 \wedge dx^2 - dx^3 \wedge dx^4, \qquad \Omega' = dx^1 \wedge dx^2 + dx^3 \wedge dx^4.$$

Thus, the symplectic 2-form Ω is self-dual while Ω' is anti-self-dual. Moreover, we have the following result:

Proposition 2.3. *An α-plane is holomorphic and Lagrangian with respect to (J, Ω), while a β-plane is holomorphic and Lagrangian with respect to (J', Ω').*

Proof. The proof follows from arguments similar to those of Proposition 2.2. □

Given a null vector V in $\mathbb{R}^{2,2}$, the planes spanned by $\{V, J(V)\}$ and $\{V, J'(V)\}$ are easily seen to be totally null. More explicitly, the set of totally null planes is, in fact, the disjoint union $S^1 \coprod S^1$, which can be parameterized as follows. For $a, b \in \mathbb{R}$, $\phi \in [0, 2\pi)$ and $\epsilon = \pm 1$, consider the vector in $\mathbb{R}^{2,2}$ given by

$$V^\epsilon_\phi(a, b) = (a\cos\phi + b\sin\phi, a\sin\phi - b\cos\phi, a, -\epsilon b)\,.$$

Let P^ϵ_ϕ be the plane containing $V^\epsilon_\phi(a, b)$ as a and b vary over $\mathbb{R}$. Then a quick check shows that P^+_ϕ is self-dual, while P^-_ϕ is anti-self-dual.

An alternative way of visualising the null planes is as follows.

Definition 2.2. The neutral *null cone* is the set of null vectors in $\mathbb{R}^{2,2}$:

$$\mathcal{C} = \{X \in \mathbb{R}^{2,2} \mid G(X, X) = 0\}.$$

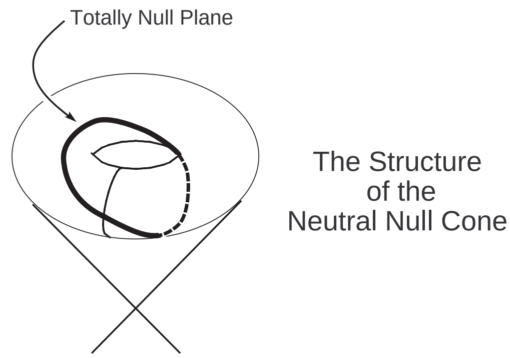
The null cone is a cone over a torus, in distinction to the lorentz $\mathbb{R}^{3,1}$ case where the null cone is a cone over a 2-sphere. To see the torus, simply note that the map $f : \mathbb{R} \times S^1 \times S^1 \to \mathcal{C}$

$$f(a, \theta_1, \theta_2) = (a\cos\theta_1, a\sin\theta_1, a\cos\theta_2, a\sin\theta_2)$$

parameterizes the null vectors as a cone.

Since every vector that lies in a totally null plane is null, we can picture a null plane as a cone over a circle in $\mathcal{C}$. A straight-forward calculation shows that:

Proposition 2.4. *A totally null plane is a cone over either a (1,1)-curve or a (1,-1)-curve on the torus, the former for an α-plane, the latter for a β-plane.*



By rotating around the meridian we see that the set of totally null planes is $S^1 \coprod S^1$.

2.4 Neutral Kähler surfaces

Let (M, G, J, Ω) be a smooth neutral Kähler 4-manifold. Thus M is a smooth 4-manifold, G is a neutral metric, while J is a complex structure that is compatible with G and $\Omega(\cdot,\cdot) = G(J\cdot,\cdot)$ is a closed non-degenerate (symplectic) 2-form.

The existence of a unitary frame at a point of M implies that it is possible to apply the algebra of the last section pointwise on M, and we therefore have $S^1 \cup S^1$ worth of totally null planes at each point. On a compact 4-manifold, the existence of an oriented 2-dimensional distribution implies topological restrictions on M [6], and so not every compact 4-manifold admits a neutral Kähler structure. However, the examples we consider are non-compact and the neutral Kähler structure will be given explicitly.

On any (pseudo)-Riemannian 4-manifold (M, G) the Riemann curvature tensor can be considered as an endomorphism of $\Lambda^2(M)$. The splitting $\Lambda^2(M) = \Lambda^2_+(M) \oplus \Lambda^2_-(M)$ with respect to the Hodge star operator $*$ yields a block decomposition of the Riemann curvature tensor

$$\mathrm{Riem} = \left(\begin{array}{cc} \mathrm{Weyl}^+ + \frac{1}{12}\mathrm{R} & \mathrm{Ric} \\ \mathrm{Ric} & \mathrm{Weyl}^- + \frac{1}{12}\mathrm{R} \end{array}\right),$$

where Ric is the Ricci tensor, R is the scalar curvature and $\mathrm{Weyl}^\pm$ are the self- and anti-self-dual Weyl curvature tensors [1].

Definition 2.3. A (pseudo)-Riemannian 4-manifold (M, G) is *anti-self-dual* if the self-dual part of the Weyl conformal curvature tensor vanishes: $\mathrm{Weyl}^+ = 0$.

A well-known result of Penrose states:

Theorem 2.5. *[8] The α-surfaces of a neutral Kähler 4-manifold (M, G) are integrable iff (M, G) is anti-self-dual.*

3 Neutral Kähler metrics on TN

Let (N, g) be a Riemannian 2-manifold and consider the total space TN of the tangent bundle to N. Choose conformal coordinates ξ on N so that $ds^2 = e^{2u}d\xi d\bar{\xi}$ for some function $u = u(\xi, \bar{\xi})$, and the corresponding complex coordinates (ξ, η) on TN obtained by identifying

$$(\xi, \eta) \leftrightarrow \eta\frac{\partial}{\partial\xi} + \bar{\eta}\frac{\partial}{\partial\bar{\xi}} \in T_\xi N.$$

The coordinates (ξ, η) define a natural complex structure on TN by

$$J\left(\frac{\partial}{\partial\xi}\right) = i\frac{\partial}{\partial\xi} \qquad\qquad J\left(\frac{\partial}{\partial\eta}\right) = i\frac{\partial}{\partial\eta}.$$

In [4] a neutral Kähler structure was introduced on TN. In the above coordinate system, the symplectic 2-form is

$$\Omega = 2e^{2u}\mathbb{R}\mathrm{e}\left(d\eta \wedge d\bar{\xi} + 2\eta\partial_\xi u\; d\xi \wedge d\bar{\xi}\right), \tag{3.1}$$

while the neutral metric $\mathbb{G}$ is

$$\mathbb{G} = 2e^{2u}\mathbb{I}\mathrm{m}\left(d\bar{\eta}d\xi - 2\eta\partial_\xi u\; d\xi d\bar{\xi}\right). \tag{3.2}$$

Here we have introduced the notation ∂_ξ for differentiation with respect to ξ.
note:
When $u = 0$, we retrieve the neutral Kähler metric on $\mathbb{R}^4 = \mathrm{T}\mathbb{R}^2$, where

$$\xi = \tfrac{1}{2}\left[x^1 + x^3 + i(x^2 + x^4)\right], \qquad \eta = \tfrac{1}{2}\left[x^2 - x^4 + i(-x^1 + x^3)\right],$$

or

$$x^1 = \tfrac{1}{2}\left[\xi + \bar{\xi} + i(\eta - \bar{\eta})\right], \qquad x^2 = \tfrac{1}{2}\left[-i(\xi - \bar{\xi}) + \eta + \bar{\eta}\right],$$
$$x^3 = \tfrac{1}{2}\left[\xi + \bar{\xi} - i(\eta - \bar{\eta})\right], \qquad x^4 = \tfrac{1}{2}\left[-i(\xi - \bar{\xi}) - \eta - \bar{\eta}\right].$$

Proposition 3.1. *The double null basis for (TN, G) is*

$$\Theta^1 = 2\mathbb{R}e(d\xi), \qquad \Theta^2 = 2e^{2u}\mathbb{R}e\left(d\eta + 2\eta\partial_\xi u\; d\xi\right),$$
$$\Theta^3 = 2\mathbb{I}m(d\xi), \qquad \Theta^4 = 2e^{2u}\mathbb{I}m\left(d\eta + 2\eta\partial_\xi u\; d\xi\right).$$

Proof. A straight-forward check shows that

$$ds^2 = \Theta^1 \otimes \Theta^4 - \Theta^2 \otimes \Theta^3,$$

as claimed. □

The coordinate expressions for self-dual and anti-self-dual 2-forms on TN are

Proposition 3.2. *If $\omega \in \Lambda^2(TN) = \Lambda^2_+(TN) \oplus \Lambda^2_-(TN)$, with $\omega = \omega_+ + \omega_-$, then*

$$\omega_+ = a_1(d\xi \wedge d\eta + d\bar{\xi} \wedge d\bar{\eta}) + b_1[d\xi \wedge d\bar{\eta} + d\bar{\xi} \wedge d\eta + 2(\bar{\eta}\partial_{\bar{\xi}}u - \eta\partial_\xi u)d\xi \wedge d\bar{\xi}]$$
$$+ ic_1(d\xi \wedge d\eta - d\bar{\xi} \wedge d\bar{\eta}),$$

$$\omega_- = ia_2 d\xi \wedge d\bar{\xi} + ib_2[d\xi \wedge d\bar{\eta} - d\bar{\xi} \wedge d\eta + 2(\bar{\eta}\partial_{\bar{\xi}}u + \eta\partial_\xi u)d\xi \wedge d\bar{\xi}]$$
$$+ ic_2(d\eta \wedge d\bar{\eta} + 2\eta\partial_\xi u d\xi \wedge d\bar{\eta} + 2\bar{\eta}\partial_{\bar{\xi}}u d\bar{\xi} \wedge d\eta + 4\eta\bar{\eta}\partial_\xi u\partial_{\bar{\xi}}u d\xi \wedge d\bar{\xi}),$$

for $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R}$.

3.1 α-surfaces in TN

We first note that

Proposition 3.3. *The neutral Kähler metric G on TN is anti-self-dual.*

Proof. A calculation using the coordinate expression (3.2) of the metric shows that the only non-vanishing component of the conformal curvature tensor is

$$W_{\xi\bar{\xi}}{}^{\eta\bar{\eta}} = i(\eta\partial_\xi\kappa - \bar{\eta}\partial_{\bar{\xi}}\kappa),$$

where κ is the Gauss curvature of (N, g). Thus, from Proposition 3.2, for any $\omega_+ \in \Lambda^2_+(TN)$, $W(\omega_+) = 0$. That is, the metric is anti-self-dual. $\square$

By applying Theorem 2.5 we have:

Corollary 3.4. *There exists α-surfaces, i.e. holomorphic Lagrangian surfaces, in (TN, J, Ω).*

3.2 β-surfaces in TN

Proposition 3.5. *An immersed surface $\Sigma \subset TN$ is a β-surface iff locally it is given by $(s,t) \to (\xi(s,t), \eta(s,t))$ where*

$$\xi = se^{iC_0} + \xi_0, \qquad \eta = (te^{iC_0} + \eta_0)e^{-2u},$$

for $C_0 \in \mathbb{R}$ and $\xi_0, \eta_0 \in C$.

Proof. By Proposition 3.2 surface $f : \Sigma \to TN$ is a β-*surface* iff

$$f^*(d\xi \wedge d\bar{\xi}) = 0, \qquad f^*(d(\eta e^{2u}) \wedge d(\bar{\eta}e^{2u})) = 0, \tag{3.3}$$

and

$$f^*(d\xi \wedge d(\bar{\eta}e^{2u}) - d\bar{\xi} \wedge d(\eta e^{2u})) = 0. \tag{3.4}$$

The first equation of (3.3) implies that the map $(s,t) \to \xi(s,t)$ is not of maximal rank, and as it cannot be of rank zero (as this would mean that Σ is a fibre of $\pi : TN \to N$, and is therefore an α-surface) it must be of rank 1. By the implicit function theorem either

$$\xi(s,t) = \xi(s, t(s)) \qquad \text{or} \qquad \xi(s,t) = \xi(s(t), t).$$

Without loss of generality, we will assume the former: $\xi = \xi(s)$.

Similarly, the second equation of (3.3) implies that either

$$\eta e^{2u} = \psi(s,t) = \psi(s, t(s)) \qquad \text{or} \qquad \eta e^{2u} = \psi(s,t) = \psi(s(t), t).$$

Here, we must have the latter $\eta e^{2u} = \psi(t)$, or else the surface Σ would be singular. Turning now to equation of (3.4), we have

$$\frac{d\xi}{ds}\frac{d\bar{\psi}}{dt} = \frac{d\bar{\xi}}{ds}\frac{d\psi}{dt}.$$

By separation of variables we see that

$$\frac{d\xi}{ds} = e^{2iC_0}\frac{d\bar{\xi}}{ds}, \qquad \frac{d\psi}{ds} = e^{2iC_0}\frac{d\bar{\psi}}{ds},$$

for some real constant C_0. These can be integrated to

$$\xi = h_1(s)e^{iC_0} + \xi_0, \qquad \eta = (h_2(t)e^{iC_0} + \eta_0)e^{-2u},$$

for complex constants ξ_0 and η_0 and real functions h_1 and h_2 of s and t, respectively. Finally, we can reparameterize s and t so that $h_1 = s$ and $h_2 = t$, as claimed. □

3.3 The oriented geodesic spaces TS^2 and TH^2

In the cases where $N = S^2$ or $N = H^2$ endowed with a metric of constant Gauss curvature ($e^{2u} = 4(1\pm\xi\bar{\xi})^{-2}$), the above construction yields the neutral Kähler metric on the space $L(E^3)$ of oriented affine lines or on the space $L(E^3_1)$ of future-pointing time-like lines, in E^3 or E^3_1 (respectively) [5].

In what follows we consider only the Euclidean case, although analogous results hold for the Lorentz case. We define the map Φ which sends $L(E^3)\times\mathbb{R}$ to E^3 as follows: Φ takes an oriented line γ and a real number r to that point in E^3 which lies on γ and is an affine parameter distance r from the point on γ closest to the origin.

Proposition 3.6. *[4] The map can be written as $\Phi((\xi,\eta),r) = (z,t) \in C\oplus\mathbb{R} = E^3$ where the local coordinate expressions are:*

$$\begin{cases} z = \frac{2(\eta-\bar{\eta}\xi^2)+2\xi(1+\xi\bar{\xi})r}{(1+\xi\bar{\xi})^2}, \quad t = \frac{-2(\eta\bar{\xi}+\bar{\eta}\xi)+(1-\xi^2\bar{\xi}^2)r}{(1+\xi\bar{\xi})^2}, \\ eta = \frac{1}{2}(z-2t\xi-\bar{z}\xi^2), \quad r = \frac{\bar{\xi}z+\xi\bar{z}+(1-\xi\bar{\xi})t}{1+\xi\bar{\xi}}. \end{cases}$$

For α-surfaces, we have

Proposition 3.7. *A holomorphic Lagrangian surface in TS^2 corresponds to the oriented normals to totally umbilic surfaces in E^3 i.e. round spheres or planes.*

On the other hand:

Proposition 3.8. *A β-surface in TS^2 is an affine tangent bundle over a curve of constant geodesic curvature in (S^2, g_{rnd}).*

Proof. By Proposition 3.5, the β-surfaces are given by

$$\xi = se^{iC_0} + \xi_0, \qquad \eta = (1+\xi\bar{\xi})^2(te^{iC_0} + \eta_0).$$

Clearly this is a real line bundle over a curve on S^2. By a rotation this can be simplified to

$$\xi = s + \xi_0 e^{-iC_0}, \qquad \eta = (1+\xi\bar{\xi})^2(t + \eta_0 e^{-iC_0}),$$

and after an affine reparameterization of s and t we can set

$$\xi = s + iC_1, \qquad \eta = (1+\xi\bar{\xi})^2(t + iC_2).$$

Projecting onto S^2 we get the curve $\xi = s + iC_1$ with unit tangent $\vec{T}$ and normal vector $\vec{N}$ (with respect to the round metric)

$$\vec{T} = \frac{(1+\xi\bar{\xi})}{2\sqrt{2}}\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\bar{\xi}}\right), \qquad \vec{N} = \frac{i(1+\xi\bar{\xi})}{2\sqrt{2}}\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\bar{\xi}}\right).$$

Considered as a set of vectors on S^2, the β-surface is

$$\begin{aligned}\eta\frac{\partial}{\partial\xi} + \bar{\eta}\frac{\partial}{\partial\bar{\xi}} &= (1+\xi\bar{\xi})^2(t+iC_2)\frac{\partial}{\partial\xi} + (1+\xi\bar{\xi})^2(t-iC_2)\frac{\partial}{\partial\bar{\xi}} \\ &= 2\sqrt{2}(1+\xi\bar{\xi})(t\vec{T} + C_2\vec{N}).\end{aligned}$$

These form a real line bundle over the base curve - which do not pass through the origin in the fibre of TS^2 for $C_2 \neq 0$. For $C_2 = 0$, this is exactly the tangent bundle to the curve. The geodesic curvature of this curve is

$$\begin{aligned}g(\vec{N}, \nabla_{\vec{T}}\vec{T}) &= N_k T^j(\partial_j T^k + \Gamma^k_{jl}T^l) \\ &= N_k T^j \partial_j T^k + N^k T^j T^l(2\partial_j g_{lk} - \partial_k g_{jl}) = \sqrt{2}C_1,\end{aligned}$$

which completes the proof. □

A similar calculation establishes:

Proposition 3.9. *A β-surface in TH^2 is an affine tangent bundle over a curve of constant geodesic curvature in (H^2, g_{hyp}).*

We also have the following:

Corollary 3.10. *Given an affine plane P in E^3, the set $L(E^2)$ of oriented lines contained in P is a β-surface in TS^2.*

Proof. By Proposition 3.5, the β-surfaces are given by

$$\xi = se^{iC_0} + \xi_0, \qquad \eta = (1+\xi\bar{\xi})^2(te^{iC_0} + \eta_0).$$

Isometries of E^3 induce isometries on TS^2 and hence preserve β-surfaces. Thus we can translate and rotate P so that it is vertical and contains the t-axis. Thus we can consider the β-surface Σ with $\xi_0 = \eta_0 = 0$, and then using the map Φ we find the two parameter family of oriented lines in E^3 to be

$$z = \frac{2[(1-s^4)t + sr]}{1+s^2}e^{iC_0}, \qquad t = \frac{-4s(1+s^2)t + (1-s^2)r}{1+s^2}.$$

This is a vertical plane containing the t-axis, and Σ consists of all the oriented lines in this plane. □

4 Oriented geodesics in hyperbolic 3-space

We briefly recall the basic construction of the canonical neutral Kähler metric on the space $L(H^3)$ of oriented geodesics of H^3 - further details can be found in [2].

Consider the 4-manifold $P^1 \times P^1$ endowed with the canonical complex structure $J = j \oplus j$ and complex coordinates μ_1 and μ_2. If we let $\overline{\Delta} = \{(\mu_1, \mu_2) : \mu_1\bar{\mu}_2 = -1\}$ then $L(H^3) = P^1 \times P^1 - \overline{\Delta}$. We introduce the neutral Kähler metric and symplectic form on $L(H^3)$ by

$$G = -i\left[\frac{1}{(1+\mu_1\bar{\mu}_2)^2}d\mu_1 \otimes d\bar{\mu}_2 - \frac{1}{(1+\bar{\mu}_1\mu_2)^2}d\bar{\mu}_1 \otimes d\mu_2\right], \tag{4.1}$$

and

$$\Omega = -\left[\frac{1}{(1+\mu_1\bar{\mu}_2)^2}d\mu_1 \wedge d\bar{\mu}_2 + \frac{1}{(1+\bar{\mu}_1\mu_2)^2}d\bar{\mu}_1 \wedge d\mu_2\right]. \tag{4.2}$$

Proposition 4.1. *A double null basis for* $(L(H^3),\ G)$ *is*

$$\Theta^1 = Re\left(\frac{d\mu_1}{1+\mu_1\bar{\mu}_2} - \frac{d\mu_2}{1+\bar{\mu}_1\mu_2}\right), \qquad \Theta^2 = Re\left(\frac{d\mu_1}{1+\mu_1\bar{\mu}_2} + \frac{d\mu_2}{1+\bar{\mu}_1\mu_2}\right),$$

$$\Theta^3 = -Im\left(\frac{d\mu_1}{1+\mu_1\bar{\mu}_2} - \frac{d\mu_2}{1+\bar{\mu}_1\mu_2}\right), \qquad \Theta^4 = -Im\left(\frac{d\mu_1}{1+\mu_1\bar{\mu}_2} + \frac{d\mu_2}{1+\bar{\mu}_1\mu_2}\right).$$

Proof. A straight-forward computation shows that

$$ds^2 = \Theta^1 \otimes \Theta^4 - \Theta^2 \otimes \Theta^3,$$

as claimed □

The coordinate expressions for self-dual and anti-self-dual 2 forms on $L(H^3)$ are easily found to be:

Proposition 4.2. *If* $\omega \in \Lambda^2(L(H^3)) = \Lambda^2_+(L(H^3)) \oplus \Lambda^2_-(L(H^3))$, *with* $\omega = \omega_+ + \omega_-$, *then*

$$\omega_+ = (a_1+ic_1)\frac{d\mu_1 \wedge d\mu_2}{|1+\bar{\mu}_1\mu_2|^2} + (a_1-ic_1)\frac{d\bar{\mu}_1 \wedge d\bar{\mu}_2}{|1+\bar{\mu}_1\mu_2|^2} + b_1\left[\frac{d\mu_1 \wedge d\bar{\mu}_2}{(1+\mu_1\bar{\mu}_2)^2} + \frac{d\bar{\mu}_1 \wedge d\mu_2}{(1+\bar{\mu}_1\mu_2)^2}\right],$$

$$\omega_- = -i(a_2+c_2)\frac{d\mu_1 \wedge d\bar{\mu}_1}{|1+\bar{\mu}_1\mu_2|^2} - i(a_2-c_2)\frac{d\mu_2 \wedge d\bar{\mu}_2}{|1+\bar{\mu}_1\mu_2|^2} + ib_2\left[\frac{d\mu_1 \wedge d\bar{\mu}_2}{(1+\mu_1\bar{\mu}_2)^2} - \frac{d\bar{\mu}_1 \wedge d\mu_2}{(1+\bar{\mu}_1\mu_2)^2}\right].$$

4.1 α-surfaces in $L(H^3)$

Once again, the neutral metric on $L(H^3)$ is anti-self-dual, indeed, it is conformally flat, and so there exists α-surfaces in $L(H^3)$. These are found to be the normal congruence to the totally umbilic surfaces in H^3:

Proposition 4.3. *[3] A smooth surface* Σ *in* $L(H^3)$ *is totally null iff* Σ *is the oriented normal congruence of*

1. *a geodesic sphere, or*
2. *a horosphere, or*
3. *a totally geodesic surface*

in H^3.

4.2 β-surfaces in $L(H^3)$

Proposition 4.4. *Let Σ be a β-surface in $L(H^3)$. Then Σ is a piece of a torus which, up to isometry, is either*

1. *$L(H^2)$, where $H^2 \subset H^3$, or*
2. *$\mathcal{C}_1 \times \mathcal{C}_2 \subset S^2 \times S^2 - \bar{\Delta}$, where the $\mathcal{C}_1$ is a circle given by the intersection of the 2-sphere and a plane containing the north pole, and $\mathcal{C}_2$ is the image of $\mathcal{C}_1$ under reflection in the horizontal plane through the origin.*

Proof. Let $f : \Sigma \to L(H^3)$ be an immersed β-surface. Then for every anti-self-dual 2-form ω_- we have $f^*\omega_- = 0$. Then we obtain the following equations

$$f^*(d\mu_1 \wedge d\bar{\mu}_1) = 0, \qquad f^*(d\mu_2 \wedge d\bar{\mu}_2) = 0, \tag{4.3}$$

$$f^*\left(\frac{d\mu_1 \wedge d\bar{\mu}_2}{(1+\mu_1\bar{\mu}_2)^2} - \frac{d\bar{\mu}_1 \wedge d\mu_2}{(1+\bar{\mu}_1\mu_2)^2}\right) = 0. \tag{4.4}$$

The first equation of (4.3) implies that the map $(u, v) \mapsto \mu_1(u, v)$ is not of maximal rank and since it cannot be of rank zero (otherwise Σ would be an α-surface) it must be of rank 1. By the implicit function theorem either

$$\mu_1(u, v) = \mu_1(u, v(u)) \qquad \text{or} \qquad \mu_1(u, v) = \mu_1(u(v), v).$$

Without loss of generality, we will assume the former: $\mu_1 = \mu_1(u)$.

Similarly, the second equation of (4.3) implies that

$$\mu_2(u, v) = \mu_2(u, v(u)) \qquad \text{or} \qquad \mu_2(u, v) = \mu_2(u(v), v).$$

Here, we must have $\mu_2 = \mu_2(v)$, or else the surface Σ would be singular.

The equation (4.4) yields

$$\ln\mu_2 - \ln\bar{\mu}_2 + \ln(1+\bar{\mu}_1\mu_2) - \ln(1+\mu_1\bar{\mu}_2) = h_1(u) + h_2(v), \tag{4.5}$$

$$\ln\bar{\mu}_1 - \ln\mu_1 + \ln(1+\bar{\mu}_1\mu_2) - \ln(1+\mu_1\bar{\mu}_2) = w_1(u) + w_2(v), \tag{4.6}$$

for some complex functions h_1, h_2, w_1, w_2.

If $h_i = a_i e^{i\phi_i}$ for $i = 1, 2$, where $a_1 = a_1(u)$, $\phi_1 = \phi_1(u)$ and $a_2 = a_2(v)$, $\phi_2 = \phi_2(v)$ are real functions, we obtain

$$h_1(u) = ia_1 \qquad\qquad h_2(v) = ia_2.$$

By a similar argument, there are real functions $b_1 = b_1(u)$ and $b_2 = b_2(v)$ such that (4.5) and (4.6) become

$$\ln\mu_2 - \ln\bar{\mu}_2 + \ln(1+\bar{\mu}_1\mu_2) - \ln(1+\mu_1\bar{\mu}_2) = i(a_1(u) + a_2(v)), \tag{4.7}$$

$$\ln\bar{\mu}_1 - \ln\mu_1 + \ln(1+\bar{\mu}_1\mu_2) - \ln(1+\mu_1\bar{\mu}_2) = i(b_1(u) + b_2(v)). \tag{4.8}$$

Finally from combining equations (4.7) and (4.8) we have

$$\ln\left(\frac{1+\bar{\mu}_1\mu_2}{1+\mu_1\bar{\mu}_2}\right) = -2i(f(u)+g(v)).$$

We are thus led to consider the curves $\mathcal{C}_1, \mathcal{C}_2$ on S^2 given locally by non-constant functions $\mu_1 : \mathbb{R} \to S^2 : u \mapsto \mu_1(u)$ and $\mu_2 : \mathbb{R} \to S^2 : v \mapsto \mu_2(v)$ which satisfy

$$1+\mu_1\bar{\mu}_2 = (1+\bar{\mu}_1\mu_2)e^{2i(f+g)},$$

for $f = f(u)$ and $g = g(v)$.

If we switch to polar coordinates $\mu_1 = \lambda_1(u)e^{i\theta_1(u)}$ and $\mu_2 = \lambda_2(v)e^{i\theta_2(v)}$, this reduces to

$$\sin[f(u)+g(v)] = \lambda_1(u)\lambda_2(v)\sin[\theta_1(u)-f(u)-\theta_2(v)-g(v)]. \tag{4.9}$$

By a rotation we can set μ_2 to zero for some $v = v_0$, that is, $\lambda_2(v_0) = 0$. We find from equation (4.9) that

$$\sin[f(u)+g(v_0)] = 0,$$

and so letting $g_0 = g(v_0)$, we conclude that $f = -g_0$. Putting this back into (4.9) we have

$$\sin[g(v)-g_0] = \lambda_1(u)\lambda_2(v)\sin[\theta_1(u)-\theta_2(v)-g(v)+g_0]. \tag{4.10}$$

Thus for a fixed $u = u_0$ we have

$$\lambda_1(u_0)\lambda_2(v)\sin[\theta_1(u_0)-\theta_2(v)-g(v)+g_0] = \lambda_1(u)\lambda_2(v)\sin[\theta_1(u)-\theta_2(v)-g(v)+g_0],$$

or, for $v \neq v_0$

$$\lambda_1(u_0)\sin[\theta_1(u_0)-\theta_2(v)-g(v)+g_0] = \lambda_1(u)\sin[\theta_1(u)-\theta_2(v)-g(v)+g_0]. \tag{4.11}$$

Differentiating this relationship with respect to v yields

$$\begin{aligned} &\lambda_1(u_0)\cos[\theta_1(u_0)-\theta_2(v)-g(v)+g_0]\,\partial_v(\theta_2+g) \\ &\qquad = \lambda_1(u)\cos[\theta_1(u)-\theta_2(v)-g(v)+g_0]\,\partial_v(\theta_2+g). \end{aligned} \tag{4.12}$$

If $\partial_v(\theta_2+g) \neq 0$, then we can cancel this factor and square both sides of equations (4.11) and (4.12) to find that $\lambda_1 = \lambda_1(u_0)$. However, from the functional relation in equation (4.10), this means that θ_1 is also constant. Thus μ_1 would be constant, which is not true.

We conclude that $\partial_v(\theta_2+g) = 0$, or equivalently, $g(v) = -\theta_2(v)+g_1$. Substituting this back into equation (4.10) we have

$$\sin[\theta_2(v)+C_0] = \lambda_1(u)\lambda_2(v)\sin[\theta_1(u)+C_0],$$

where $C_0 = g_0 - g_1$.

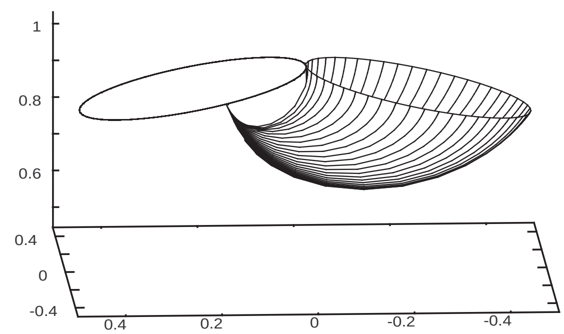
One solution of this equation is $\theta_1 = \theta_2 = -C_0$, which is the case $\Sigma = L(H^2)$, where $H^2 \subset H^3$. Otherwise, we can separate variables

$$\frac{\sin[\theta_2(v)+C_0)]}{\lambda_2(v)} = \lambda_1(u)\sin[\theta_1(u)+C_0] = C_1 \neq 0.$$

This yields
$$\mu_1 = \frac{C_1 e^{i\theta_1(u)}}{\sin[\theta_1(u) + C_0]}, \qquad \mu_2 = \frac{\sin[\theta_2(v) + C_0] e^{i\theta_2(v)}}{C_1}.$$
By a rotation of S^2 we can set C_0 to zero, and with a natural choice of parameterization of the curves, the final form is
$$\mu_1 = \frac{C_1 e^{iu}}{\sin u}, \qquad \mu_2 = \frac{\sin v \; e^{iv}}{C_1},$$
for $u, v \in [0, 2\pi)$.

These are the tori of part (2) in the statement. To see that they are circles note that if we view S^2 in $\mathbb{R}^3$ given by
$$x = \frac{\mu + \bar{\mu}}{1 + \mu\bar{\mu}}, \qquad y = \frac{-i(\mu - \bar{\mu})}{1 + \mu\bar{\mu}}, \qquad z = \frac{1 - \mu\bar{\mu}}{1 + \mu\bar{\mu}},$$
then the first curve parameterizes the intersection of S^2 with the plane $y + C_1(z-1) = 0$, while the second is the intersection with the plane $y - C_1(z+1) = 0$. □

In the ball model of H^3 these 2-parameter families of geodesics can be visualized as the set of geodesics that begin on a circle on the boundary and end on another circle of the same radius on the boundary, the two circles having a single point of intersection, as illustrated below.



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