Integral formulae for codimension-one foliated Finsler manifolds

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Abstract. We study extrinsic geometry of a codimension-one foliation $\mathcal{F}$ of a Finsler space (M, F), in particular, of a Randers space $(M, \alpha + \beta)$. Using a unit vector field ν orthogonal (in the Finsler sense) to the leaves of $\mathcal{F}$, we define a new Riemannian metric g on M, which for Randers case depends nicely on (α, β). For that g we derive several geometric invariants of $\mathcal{F}$ (e.g. the Riemann curvature and the shape operator) in terms of F; then under natural assumptions on β which simplify derivations, we express them in terms of invariants arising from α and β. Using our approach of [13], we produce the integral formulae for $\mathcal{F}$ of closed (M, F) and $(M, \alpha + \beta)$, which relate integrals of mean curvatures with those involving algebraic invariants obtained from the shape operator of $\mathcal{F}$ and the Riemann curvature in the direction ν. They generalize formulae by Brito-Langevin-Rosenberg (that total mean curvatures of any order for a foliated closed Riemannian space of constant curvature don't depend on a choice of $\mathcal{F}$).

M.S.C. 2010: 53C12, 53C21.
Key words: Finsler space; Randers norm; foliation; Riemann curvature; integral formula; shape operator; Cartan torsion; variation formula.

1 Introduction

Two recent decades brought increasing interest in Finsler geometry (see [2, 4, 15] and the bibliographies therein), in particular, in extrinsic geometry of hypersurfaces of Finsler manifolds (see the items above and, for example, [14]). Among all the Finsler structures, Randers metrics (introduced in [9] and being the closest relatives of Riemannian ones) play an important role.

Extrinsic geometry of foliated Riemannian manifolds is also of definite interest since some time (see [11, 12] and, again, the bibliographies therein). Among other topics of interest, one can find a number of papers devoted to so called *integral formulae* (see surveys in [12, 1]), which provide obstructions for existence of foliations

Balkan Journal of Geometry and Its Applications, Vol.21, No.1, 2016, pp. 76-102.

(or compact leaves of them) with given geometric properties. A series of integral formulae has been provided in [13]. They include the formulae in [10] that the total mean curvature of the leaves is zero, and generalize the formulae in [3], which show that total mean curvatures (of arbitrary order k) for codimension-one foliations on a closed $(m+1)$-dimensional manifold of constant sectional curvature K depend only on K, k, m and the volume of the manifold, not on a foliation. One of such formulae was used in [7] to prove that codimension-one foliations of a closed Riemannian manifold of negative Ricci curvature are far (in a sense defined there) from being umbilical.

In this paper we study extrinsic geometry of a codimension-one transversely oriented foliation $\mathcal{F}$ of a closed Finsler space (M, F), in particular, of a Randers space $(M, \alpha+\beta)$, α being the norm of a Riemannian structure a and β a 1-form of α-norm smaller than 1 everywhere on M. Using a unit normal ν (in the Finsler sense) to the leaves of $\mathcal{F}$ we define a new Riemannian structure g on M, which in Randers case depends nicely on α and β. For that g, we derive several geometric invariants of $\mathcal{F}$ (e.g. the Riemann curvature and the shape operator) in terms of F; under natural assumptions on β which simplify derivations, we express them in terms of corresponding invariants arising from α and some quantities related to β. Then, using the approach of [13], we produce the integral formulae for $\mathcal{F}$ on (M, F) and $(M, \alpha+\beta)$; some of them generalize the formulae in [3].

Our formulae relate integrals of σ_i's with those involving algebraic invariants (see Appendix) obtained from A_p ($p \in M$) – the shape operator of a foliation $\mathcal{F}$, R_p – the Riemann curvature in the direction ν normal to $\mathcal{F}$, and their products of the form $(R_p)^j A_p$, $j = 1, 2, \ldots$ In fact, we get a bit more: we produce an infinite sequence of such formulae for a smooth unit vector field ν on M involving these algebraic invariants. To simplify calculations, we work on locally symmetric ($\nabla R = 0$ with respect to g) Finsler manifolds, where our approach can be applied with the full force (Section 3). We show that our formulae reduce to these in [3] in the case of constant curvature and to those in [13] in the Riemannian case. Using Finsler geometry of Randers spaces we produce also (Section 4) integral formulae on codimension-one foliated Riemannian manifolds which involve not only A_p's and R_p's but also an auxiliary 1-form β.

We discuss a number of particular cases and provide consequences of our new formulae.

2 Preliminaries

Recall Euler's Theorem: If a function f on $\mathbb{R}^{m+1}$ is smooth away from the origin of $\mathbb{R}^{m+1}$ then the following two statements are equivalent:

– f is positively homogeneous of degree r, that is $f(\lambda\, y) = \lambda^r f(y)$ for all $\lambda > 0$;

– the radial derivative of f is r times f, namely, $f_{y^i}(y)\, y^i = r f(y)$.

The obvious consequence of Euler's Theorem helps us to represent several formulae in what follows:

Corollary 2.1. *If a smooth function f on $\mathbb{R}^{m+1} \setminus \{0\}$ obeys the 2-homogeneity condition $f(\lambda\, y) = \lambda^2 f(y)$ for $\lambda > 0$ then $f(y) = \frac{1}{2} f_{y^i y^j}(y)\, y^i y^j$ for smooth functions $f_{y^i y^j}$ on $\mathbb{R}^{m+1} \setminus \{0\}$.*

Proof. By Euler's Theorem, $f_{y^i}(y)\, y^i = 2f(y)$. Since $f_{y^i}(\lambda\, y) = \lambda f_{y^i}(y)$, by Euler's Theorem, we have $f_{y^i}(y) = f_{y^i y^j}(y) y^j$. □

2.1 The Minkowski and Randers norms

Definition 2.1 (see [15]). A *Minkowski norm* on a vector space $\mathbb{R}^{m+1}$ is a function $F : \mathbb{R}^{m+1} \to [0,\infty)$ with the following properties (of regularity, positive 1-homogeneity and strong convexity):

$\mathrm{M}_1 : F \in C^\infty(\mathbb{R}^{m+1} \setminus \{0\})$, $\mathrm{M}_2 : F(\lambda\, y) = \lambda F(y)$ for all $\lambda > 0$ and $y \in \mathbb{R}^{m+1}$,

M_3 : For any $y \in \mathbb{R}^{m+1} \setminus \{0\}$, the following symmetric bilinear form is positive definite on $\mathbb{R}^{m+1}$:

$$g_y(u, v) = \frac{1}{2}\,\frac{\partial^2}{\partial s\, \partial t}\,\big[F^2(y + su + tv)\big]_{|\, s=t=0}\,. \tag{2.1}$$

By (M_2), $g_{\lambda y} = g_y$ for all $\lambda > 0$. By (M_3), $\{y \in \mathbb{R}^{m+1} : F(y) \le 1\}$ is a strictly convex set. Note that

$$g_y(y, v) = \frac{1}{2}\,\frac{\partial}{\partial t}\,\big[F^2(y + tv)\big]_{|\, t=0}, \quad g_y(y, y) = F^2(y). \tag{2.2}$$

One can check that $F(u+v) \le F(u) + F(v)$ (the triangle inequality) and $F_{y^i}(y)\, u^i \le F(u)$ (the fundamental inequality) for all $y \in \mathbb{R}^{m+1} \setminus \{0\}$ and $u, v \in \mathbb{R}^{m+1}$. By Corollary 2.1, we have $F^2(y) = g_{ij}(y)\, y^i y^j$, where $g_{ij} = \frac{1}{2}\,[F^2]_{y^i y^j} = F F_{y^i y^j} + F_{y^i} F_{y^j}$ are smooth functions in $\mathbb{R}^{m+1} \setminus \{0\}$ which, in general, cannot be extended continuously to all of $\mathbb{R}^{m+1}$. The following symmetric trilinear form C for Minkowski norms is called the *Cartan torsion*:

(2.3)

$$C_y(u, v, w) = \frac{1}{2}\,\frac{\partial}{\partial t}\,\big[g_{y+tw}(u, v)\big]_{|\, t=0} \quad \text{where} \quad y \in \mathbb{R}^{m+1} \setminus \{0\},\ u, v, w \in \mathbb{R}^{m+1}\,.$$

The homogeneity of F implies the following:

$$C_y(u, v, w) = \frac{1}{4}\,\frac{\partial^3}{\partial r\, \partial s\, \partial t}\,\big[F^2(y + ru + sv + tw)\big]_{|\, r=s=t=0}, \quad C_{\lambda y} = \lambda^{-1} C_y \quad (\lambda > 0).$$

We have $C_y(y, \cdot\,, \cdot\,) = 0$. The *mean Cartan torsion* is given by $I_y(u) := \mathrm{Tr}\, C_y(\cdot\,, \cdot\,, u)$. Observe that

$$C_{ijk} := C(\partial_{y^i}, \partial_{y^j}, \partial_{y^k}) = \frac{1}{2}\,\frac{\partial}{\partial y^k}\, g_{ij} = \frac{1}{4}\,[F^2]_{y^i y^j y^k}, \qquad I_k = g^{ij} C_{ijk}.$$

Let (b_i) be a basis for $\mathbb{R}^{m+1}$ and (θ^i) the dual basis in $(\mathbb{R}^{m+1})^*$. The *Busemann-Hausdorff volume form* is defined by $\mathrm{d}V_F = \sigma_F(x)\, \theta^1 \wedge \cdots \wedge \theta^{m+1}$, where $\sigma_F = \frac{\mathrm{vol}\, \mathbb{B}^{m+1}}{\mathrm{vol}\, B^{m+1}}$. Here $\mathbb{B}^{m+1} := \{y \in \mathbb{R}^{m+1} : \|y\| < 1\}$ is a Euclidean unit ball, and $\mathrm{vol}\, B^{m+1}$ is the Euclidean volume of a strongly convex subset $B^{m+1} := \{y \in \mathbb{R}^{m+1} : F(y^i b_i) < 1\}$ (so that for the unit cubic $\mathcal{U} = [0,1]^{m+1}$, $\mathrm{vol}\, \mathcal{U} = 1$).

The *distortion* of F is defined by $\tau(y) = \log(\sqrt{\det g_{ij}(y)}/\sigma_F)$. It has the 0-homogeneity property: $\tau(\lambda y) = \tau(y)$ $(\lambda > 0)$, and $\tau = 0$ for Riemannian spaces.

The *angular form* is defined by $h_y(u, v) = g_y(u, v) - F(y)^{-2} g_y(y, u)\, g_y(y, v)$. Observe that $h_y(u, u) \ge g_y(u, u) - F(y)^{-2} g_y(y, y)\, g_y(u, u) = 0$ and equality holds if and only if $u || \, y$.

A vector $n \in \mathbb{R}^{m+1}$ is called a *normal* to a hyperplane $W \subset \mathbb{R}^{m+1}$ if $g_n(n, w) = 0\ (w \in W)$. There are exactly two normal directions to W, see [15], which are opposite when F is *reversible* (i.e., $F(-y) = F(y)$ for all $y \in \mathbb{R}^{m+1}$).

Definition 2.2. Let $a(\cdot\,,\cdot) = \langle\cdot\,,\cdot\rangle$ be a scalar product and $\alpha(y) = \|y\|_\alpha = \sqrt{\langle y, y\rangle}$ for $y \in \mathbb{R}^{m+1}$ the corresponding Euclidean norm on $\mathbb{R}^{m+1}$. If β is a linear form on $\mathbb{R}^{m+1}$ with $\|\beta\,\|_\alpha < 1$ then the following function F is called the *Randers norm*:

$$F(y) = \alpha(y) + \beta(y) = \sqrt{\langle y, y\rangle} + \beta(y). \tag{2.4}$$

For Randers norm (2.4) on $\mathbb{R}^{m+1}$, the bilinear form g_y obeys, see [15],

$$\begin{aligned} g_y(u, v) &= \alpha^{-2}(y)(1 + \beta(y))\,\langle u, v\rangle + \beta(u)\,\beta(v) \\ &- \alpha^{-3}(y)\,\beta(y)\,\langle y, u\rangle\,\langle y, v\rangle + \alpha^{-1}(y)\,\big(\beta(u)\,\langle y, v\rangle + \beta(v)\,\langle y, u\rangle\big)\,, \end{aligned} \tag{2.5}$$

$$\det g_y = (F(y)/\alpha(y))^{m+2} \det a. \tag{2.6}$$

Let $N \in \mathbb{R}^{m+1}$ be a unit normal to a hyperplane W in $\mathbb{R}^{m+1}$ with respect to $\langle\cdot\,,\cdot\rangle$, i.e.,

$$\langle N, w\rangle = 0 \quad (w \in W), \qquad \alpha(N) = \|N\|_\alpha = \sqrt{\langle N, N\rangle} = 1.$$

Let n be a vector F-normal to W, lying in the same half-space with N and such that $\|n\|_\alpha = 1$. Set

$$g(u, v) := g_n(u, v), \quad u, v \in \mathbb{R}^{m+1}.$$

Then $g(n, n) = F^2(n)$, see (2.2), and $F(n) = 1 + \beta(n)$.

The 'musical isomorphisms' $\sharp$ and $\flat$ will be used for rank one tensors and symmetric rank 2 tensors on $(\mathbb{R}^{m+1}, a)$ and Riemannian manifolds. For example, if β is a 1-form on $\mathbb{R}^{m+1}$ and $v \in \mathbb{R}^{m+1}$ then $\langle \beta^\sharp, u\rangle = \beta(u)$ and $v^\flat(u) = \langle v, u\rangle$ for any $u \in \mathbb{R}^{m+1}$.

Lemma 2.2. *If the Randers norm obeys $\beta(N) = 0$ (i.e., $\beta^\sharp \in W$) then*

$$n = c\,N - \beta^\sharp, \tag{2.7}$$

$$g(u, v) = c^2\big(\langle u, v\rangle - \beta(u)\,\beta(v)\big), \quad u, v \in W\,, \tag{2.8}$$

$$g(n, n) = c^4, \quad g(n, v) = 0, \tag{2.9}$$

where $c := (1 - \|\beta\,\|_\alpha^2)^{1/2} > 0$. The vector $\nu = c^{-2} n$ is an F-unit normal to W.

Proof. For arbitrary β and $y = n$ and $\alpha(n) = 1$, the formula (2.5) reads

$$g(u, v) = (1 + \beta(n))\langle u, v\rangle + \beta(u)\,\beta(v) - \beta(n)\,\langle n, u\rangle\,\langle n, v\rangle + \beta(u)\,\langle n, v\rangle + \beta(v)\,\langle n, u\rangle. \tag{2.10}$$

Assuming $u = n$, from (2.10) we find

$$g(n, v) = (1 + \beta(n))\,\langle n + \beta^\sharp,\, v\rangle. \tag{2.11}$$

Note that $|\beta(n)| = |\langle \beta^\sharp, n\rangle| \le \alpha(\beta^\sharp)\,\alpha(n) < 1$; hence, $1 + \beta(n) > 0$. We find from (2.11) with $v \in W$ that $n + \beta^\sharp = \hat c\,N$ for some $\hat c > 0$. Using $1 = \langle n, n\rangle = \hat c^{\,2} - 2\,\hat c\,\beta(N) + \|\beta\,\|_\alpha^2$, we get two values

$$\hat c = \beta(N) \pm (\beta(N)^2 + c^2)^{1/2}.$$

By condition $\beta(N)=0$ we have $\beta^\sharp\in W$, this yields $\hat c=c$ and (2.7). Thus,

$$\beta(n)=\beta(cN-\beta^\sharp)=-\|\beta\|^2_\alpha,\qquad 1+\beta(n)=c^2.$$

Finally, (2.8) follows from (2.10). □

Lemma 2.3. *Let the Randers norm obeys $\beta(N)=0$ (i.e., $\beta^\sharp\in W$). If $u,U\in W$ and*

$$g(u,v)=\langle U,v\rangle\quad\text{for all}\quad v\in W \tag{2.12}$$

then $\beta(u)=c^{-4}\beta(U)$ and

$$c^2\,u=U+c^{-2}\beta(U)\,\beta^\sharp. \tag{2.13}$$

Proof. By (2.8), we have

$$g(u,v)=c^2\langle u-\beta(u)\beta^\sharp,\,v\rangle.$$

Then from (2.12), since u,U and $\beta^\sharp$ belong to W, we obtain

$$u-\beta(u)\beta^\sharp=c^{-2}U.$$

Applying β we get $\beta(u)-\beta(u)\,\|\beta\,\|^2_\alpha=c^{-2}\beta(U)$, $\beta(u)=c^{-4}\beta(U)$ and then (2.13). □

2.2 Finsler spaces

Let M^{m+1} be a connected smooth manifold and TM its tangent bundle. The natural projection $\pi:TM_0\to M$, where $TM_0:=TM\setminus\{0\}$ is called the *slit tangent bundle*. A *Finsler structure* on M is a Minkowski norm F in tangent spaces T_pM, which smoothly depends on a point $p\in M$. Note that π_* maps the double tangent bundle T^2M into TM itself.

A *spray* on a manifold M is a smooth vector field $\mathbb{G}$ on TM_0 such that

$$\pi_*(\mathbb{G}_v)=v,\quad \mathbb{G}_{\lambda v}=\lambda\,(h_\lambda)_*(\mathbb{G}_v)\qquad(v\in TM_0,\ \lambda>0), \tag{2.14}$$

where $h_\lambda: v\mapsto\lambda\,v$ is the homothety of TM. The meaning of $(2.14)_1$ is that $\mathbb{G}$ is a *second-order vector field* over M, and $(2.14)_2$ is the homogeneous quadratic condition. In local coordinates (x^i), $\mathbb{G}$ is expressed as $\mathbb{G}(y)=y^i\partial_{x^i}-2G^i\partial_{y^i}$, where $G^i(\lambda\,y)=\lambda^2G^i(y)$ $(\lambda>0)$.

Using $\mathbb{G}$ we define the following notions: covariant derivative, parallel translation (and parallel vectors) along a curve, geodesics and curvature. A curve $\gamma(t)$ in TM_0 satisfying $\dot\gamma=\mathbb{G}_\gamma$ is an integral curve of $\mathbb{G}$; it is equal to the canonical lift of $c:=\pi\circ\gamma$. The *covariant derivative* of a vector field $u(t)$ along a curve $c(t)$ in M is given by $D_{\dot c}\,u=\{\dot u^i+\Gamma^i_{kj}(\dot c)\,\dot c^k\,u^j\}\,\partial_{x^i\,|\,c}$. Here $G^i=\frac12\,\Gamma^i_{kj}\,y^ky^j$ for smooth functions $\Gamma^i_{kj}=(G^i)_{y^ky^j}$ on TM_0, see Corollary 2.1. The following properties are obvious:

$$D_{\dot c}\,(u+v)=D_{\dot c}\,u+D_{\dot c}\,v,\quad D_{\dot c}\,(fu)=\dot c(f)\,u+f D_{\dot c}\,u,\quad D_{\lambda\dot c}\,u=\lambda D_{\dot c}\,u$$

for any $f\in C^\infty(M)$ and $\lambda>0$, see [15]. A vector field $u(t)$ along c is *parallel* if $D_{\dot c}\,u(t)\equiv0$, i.e.,

$$\dot u^i+\Gamma^i_{kj}(\dot c)\,\dot c^k\,u^j=0\quad(i\ge1).$$

A curve $c(t)$ in M is called a *geodesic* of $\mathbb{G}$ if it is a projection of an integral curve of $\mathbb{G}$; hence, $\ddot c = \mathbb{G}_{\dot c}$. A curve $c(t)$ is a geodesic if and only if the tangent vector $u = \dot c$ is parallel along itself: $D_{\dot c}\,\dot c = 0$. For a geodesic $c(t)$ we have the following quasilinear system of second order ODEs

$$\ddot c^{\,i} + 2G^{\,i}(\dot c) = 0, \quad i = 1, \dots, m+1\,.$$

A Finsler metric F on M induces a *Finsler spray* $\mathbb{G}$ on TM_0, whose geodesics are locally shortest paths connecting endpoints and have constant speed. Its geodesic coefficients are given by

$$G^{\,i} = \frac{1}{4}\, g^{il}\big([F^2]_{x^k y^l}\, y^k - [F^2]_{x^l}\big) = \frac{1}{4}\, g^{il}\big(2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\big)\, y^j y^k\,,$$

see [15]. Here $g_{ij}(y) = \frac12\,[F^2]_{y^i y^j}(y)$, compare (2.1). Then $\Gamma^i_{kj}(y) = \frac12\, g^{il}\big(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\big)$ are homogeneous of 0-degree functions on TM_0.

Remark 2.3. A Finsler metric on a manifold M is called a *Berwald metric* if in any local coordinate system (x, y) in TM_0, the Christoffel symbols Γ^i_{jk} are functions on M only, in which case the geodesic coefficients $G^i = \frac12\,\Gamma^i_{kj}(x)\, y^k y^j$ are quadratic in $y = y^i\partial_{x^i}$. On a Berwald space, the parallel translation along any geodesic preserves the Minkowski functionals; thus, such spaces can be viewed as Finsler spaces modeled on a single Minkowski space. Berwald metrics are characterized among Randers ones, $F = \alpha + \beta$, by the following criterion: β is parallel with respect to α, see [15, Theorem 2.4.1]. If β is a closed 1-form, then Finslerian geodesics are the same (as sets) as the geodesics of the metric a.

A Finsler manifold is *positively* (resp. *negatively*) *complete* if every geodesic $c(t)$ on $(0, t_0)$ can be extended for $(0, \infty)$ (resp. $(-\infty, 0)$), and F is *complete* if it is both positively and negatively complete. This property is satisfied by all closed Finsler manifolds. Let (M, F) be positively complete; hence, for any $p, q \in M$ there exists a globally minimizing geodesic from p to q, see also Hopf-Rinov theorem [15, p. 178]. Let c_y be a geodesic with $c_y(0) = p$ and $\dot c_y(0) = y \in T_pM$. The *exponential map* is defined by $\exp_p(y) = c_y(1)$. By homogeneity of $\mathbb{G}$ one has $c_y(t) = c_{ty}(1)$ for $t > 0$; hence, $\exp_p(ty) = c_y(t)$. Recall [14] that $\exp_p$ is smooth on TM_0 and C^1 at the origin with $d(\exp_p)_{|\,0} = \mathrm{id}_{\,T_pM}$.

Consider a geodesic $c(t)$, $0 \le t \le 1$. A C^∞ map $\mathcal{H} : (-\varepsilon, \varepsilon) \times [0, 1] \to M$ is called a *geodesic variation* of c if $\mathcal{H}(0, t) = c(t)$ and for each $s \in (-\varepsilon, \varepsilon)$, the curve $c_s(t) := \mathcal{H}(s, t)$ is a geodesic. For a geodesic variation $\mathcal{H}$ of c, the variation field $Y(t) := \frac{\partial \mathcal{H}}{\partial s}(0, t)$ along c satisfies the *Jacobi equation*:

$$D_{\dot c} D_{\dot c}\, Y + R_{\dot c}(Y) = 0 \tag{2.15}$$

for some $(y \in TM)$-dependent (1,1)-tensor R_y. Jacobi equation (2.15) serves as the definition of curvature. A vector field $Y(t)$ satisfying (2.15) along a geodesic $c(t)$ is called *Jacobi field*. We have $g_{\dot c}(Y(t), \dot c(t)) = \lambda^2(a + bt)$ and $g_{\dot c}(D_{\dot c}\, Y(t), \dot c(t)) = \lambda^2 b$ for some constants a, b and $\lambda{=}F(\dot c)$. The orthogonal component $Y^\perp(t) = Y(t) - (a + bt)\dot c(t)$ of the Jacobi field $Y(t)$ along $c(t)$ is also a Jacobi field such that $Y^\perp(t)$ and $D_{\dot c}\, Y^\perp(t)$ are $g_{\dot c}$-orthogonal to $\dot c(t)$. Define $R^{(1)}_{\dot c(t)} : T_{c(t)}M \to T_{c(t)}M$ by $R^{(1)}_{\dot c(t)}\,(u(t)) =$

$D_{\dot c(t)}[R_{\dot c(t)}\,(u(t))]$, where $u(t)$ is a parallel vector field along c. Similarly, we define $R^{(2)}_{\dot c(t)}$, $R^{(3)}_{\dot c(t)}$ etc. Thus, by (2.15), a spray defines transformations $R_y : T_pM \to T_pM$ called the *Riemann curvature in a direction* $y \in T_pM \setminus \{0\}$, and we have $R_y(y) = 0$ and $R_{\lambda y} = \lambda^2 R_y$ $(\lambda > 0)$. In coordinates, $R_y = R^i{}_k dx^k \partial_{x_i}$ and $R^i{}_k(y)\, y^k = 0$, where R^i_k's depend on the Finsler spray only [14]:

$$R^i{}_k = 2\,(G^i)_{x^k} - y^j\,(G^i)_{x^j\,y^k} + 2\,G^j\,(G^i)_{y^j\,y^k} - (G^i)_{y^j}\,(G^j)_{y^k}\,.$$

Moreover, $R^i{}_k = R_j{}^i{}_{kl}\, y^j\, y^l$ for local functions $\{R_j{}^i{}_{kl}\} = \frac{1}{2}\,(R^i{}_k)_{y^j y^l}$ on TM_0 (see Corollary 2.1) and

$$R_j{}^i{}_{kl} = (\Gamma^i_{jl})_{x^k} - (\Gamma^i_{jk})_{x^l} + \Gamma^m_{jl}\,\Gamma^i_{mk} - \Gamma^m_{jk}\,\Gamma^i_{ml}\,.$$

For the Finsler spray, R_y is g_y-self-adjoint: $g_y(R_y(u), v) = g_y(u, R_y(v)),\ u, v \in T_pM$.

For a plane $P \subset T_pM$ tangent to M and a vector $y \in P \setminus \{0\}$, the *flag curvature* $K(P, y)$ is given by

$$K(P, y) = \frac{g_y(R_y(u), u)}{g_y(y, y)g_y(u, u) - g_y(y, u)g_y(y, u)}\,,$$

where $u \in P$ is such that $P = \operatorname{span}\{y, u\}$; certainly, the value of $K(P, y)$ is independent of the choice of $u \in P$. If $K(P, y)$ is a scalar function on TM_0 (that holds in dimension two) then F is said to be of *scalar* (flag) *curvature*, in this case, $R_y(u) = K(\pi(y))\{g_y(y, y)u - g_y(y, y)y\}$ $(y, u \in TM_0)$. If $K = K(\pi(y))$ (i.e., the flag curvature is *isotropic*) and $m \ge 2$ then $K = \text{const}$, see [5, Lemma 7.1.1]. For each $K \in \mathbb{R}$ there exist many non-isometric Finsler metrics of constant scalar curvature K.

Let $\{e_i\}_{1\le i\le m+1}$ be a g_y-orthonormal basis for T_pM such that $e_{m+1} = y/F(y)$, and let $P_i = \operatorname{span}\{e_i, y\}$ for some $y \in T_pM$. Then $K(P_i, y) = F^{-2}(y)\, g_y(R_y(e_i), e_i)$. The *Ricci curvature* is a function on TM_0 defined as the trace of the Riemann curvature,

$$\operatorname{Ric}(y) = \sum\nolimits_{i=1}^m g_y(R_y(e_i), e_i) = F^2(y) \sum\nolimits_{i=1}^m K(P_i, y)$$

with the homogeneity property $\operatorname{Ric}(\lambda y) = \lambda^2 \operatorname{Ric}(y)$ $(\lambda > 0)$. In a coordinate system, by Corollary 2.1 we have $\operatorname{Ric}(y) = R_j{}^i{}_{ik}\, y^j\, y^k = \operatorname{Ric}_{jk}\, y^j\, y^k$. A Finsler space (M^{m+1}, F) is said to be of *constant Ricci curvature* λ (or, *Einstein*) if $\operatorname{Ric}(y) = m\lambda\, F^2(y)$ $(y \in TM_0)$, or $\operatorname{Ric}_{jk} = m\lambda\, g_{jk}$ in coordinates.

3 Codimension-one foliated Finsler spaces

Given a transversally oriented codimension-one foliation $\mathcal{F}$ of a Finsler manifold (M^{m+1}, F), there exists a globally defined F-normal (to the leaves) smooth vector field n which defines a Riemannian metric $g := g_n$ with the Levi-Civita connection ∇. We have $g(n, u) = 0$ $(u \in T\mathcal{F})$ and $g(n, n) = F^2(n)$, see (2.9). Then $\nu = n/F(n)$ is an F-unit normal.

3.1 The Riemann curvature and the shape operator

In this section we apply the variational approach to find a relationship between the Riemann curvature of F and g. It generalizes the following.

Proposition 3.1 (see [15])**.** *Let Y be a geodesic field on an open subset $\mathcal{U}$ in a Finsler space (M,F) and $\hat g := g_Y$ the induced metric on $\mathcal{U}$. Then the Riemann curvature of F and $\hat F := \sqrt{\hat g}$ obey $R_Y = \hat R_Y$. Moreover, Y is a geodesic field of $\hat F$ and for the Levi-Civita connection we have $D_Y X = \hat D_Y X$.*

For a codimension-one Riemannian foliation, a unit normal ν is a geodesic vector field; hence, by Proposition 3.1, transformations R_ν defined for F by (2.15) coincide with the Jacobi operator $R(\cdot,\nu)\nu$ of the metric g. Recall that the second differential is defined by $\nabla^2_{u,v} = \nabla_u\nabla_v - \nabla_{\nabla_u v}$ for any u,v.

Let Y_t ($|t|\le\varepsilon$) be a smooth family of F-unit vector fields on an open subset $\mathcal{U}$ in (M,F). Put $\dot Y_t = \partial_t Y_t$ and $\dot g_t = \partial_t g_t$, where $g_t := g_{Y_t}$ is a family of metrics on $\mathcal{U}$. By definition (2.3) of the Cartan torsion, we have

$$\dot g_t = 2C_{Y_t}(\,\cdot\,,\,\cdot\,,\dot Y_t). \tag{3.1}$$

Note that $\dot g_t(Y_t,\,\cdot) = 2C_{Y_t}(\,Y_t,\,\cdot\,,\dot Y_t) = 0$.

Proposition 3.2. *Let Y_t ($|t|\le\varepsilon$) doesn't depend on t at a point $p\in\mathcal{U}$ and $u,v\in T_pM$. Then*

$$\begin{aligned} -\partial_t R_t(u,Y_t,Y_t,v) &= C_Y(u,\nabla^t_v Y_t,\nabla^t_Y\dot Y_t) + C_Y(\nabla^t_u Y_t, v,\nabla^t_Y\dot Y_t)\\ &+\ C_Y(\nabla^t_Y Y_t, v,\nabla^t_u\dot Y_t) + C_Y(u,\nabla^t_Y Y_t,\nabla^t_v\dot Y_t)\\ &+\ C_Y(u,v,(\nabla^t)^2_{Y,Y}\dot Y_t) +\ 2(\nabla^t_Y C_{Y_t})(u,v,\nabla^t_Y\dot Y_t). \end{aligned} \tag{3.2}$$

The shape operators A_t (when $Y_p=\nu_p$) of $\mathcal{F}$ with respect to g_t and the volume forms $\mathrm{d}V_t$ at p obey

$$g_t(\partial_t A_t(u),v) \ =\ -C_\nu(u,v,\nabla^t_\nu\,\dot Y_t),\quad \partial_t(\mathrm{d}V_t)=0. \tag{3.3}$$

Proof. Put $\Pi(u,v) = \partial_t\nabla^t_u\,v$ for t-independent vector fields u,v. Then, see [16],

$$2\,g_t(\Pi(u,v),w) = (\nabla^t_v\,\dot g_t)(u,w) + (\nabla^t_u\,\dot g_t)(v,w) - (\nabla^t_w\,\dot g_t)(u,v), \tag{3.4}$$

and for arbitrary t-dependent vector fields X_t and Z_t we obtain

$$\partial_t\nabla^t_{X_t}Z_t = \Pi(X_t,Z_t) + \nabla^t_{X_t}(\partial_t Z_t) + \nabla^t_{\partial_t X_t}Z_t.$$

By definition,

$$R_t(u,Z_t)Y_t = \nabla^t_u(\nabla^t_{Z_t}Y_t) - \nabla^t_{Z_t}(\nabla^t_u Y_t) - \nabla^t_{[u,Z_t]}Y_t.$$

So,

$$\partial_t R_t(u,Z_t)Y_t = \partial_t(\nabla^t_u(\nabla^t_{Z_t}\,Y_t)) - \partial_t(\nabla^t_{Z_t}(\nabla^t_u\,Y_t)) - \partial_t(\nabla^t_{[u,Z_t]}\,Y_t).$$

Deriving the terms of the above,

$$\begin{aligned} \partial_t(\nabla^t_{Z_t}(\nabla^t_u\,Y_t)) &= \Pi(Z_t,\nabla^t_u\,Y_t) + \nabla^t_{Z_t}(\Pi(u,Y_t)) + \nabla^t_{Z_t}(\nabla^t_u\,\dot Y_t) + \nabla^t_{\dot Z_t}(\nabla^t_u\,Y_t),\\ \partial_t(\nabla^t_u(\nabla^t_{Z_t}\,Y_t)) &= \Pi(u,\nabla^t_{Z_t}\,Y_t) + \nabla^t_u(\Pi(Z_t,Y_t)) + \nabla^t_u(\nabla^t_{\dot Z_t}\,Y_t) + \nabla^t_u(\nabla^t_{Z_t}\,\dot Y_t),\\ \partial_t(\nabla^t_{[u,Z_t]}\,Y_t) &= \Pi([u,Z_t],Y_t) + \nabla^t_{[u,Z_t]}\,\dot Y_t + \nabla^t_{[u,\dot Z_t]}\,Y_t \end{aligned}$$

with $\dot Z_t = \partial_t Z_t$, we obtain a 'time-dependent' version of [16, Proposition 2.3.4],

$$\partial_t R_t(u, Z_t)Y_t = (\nabla^t_u \Pi)(Z_t, Y_t) - (\nabla^t_{Z_t} \Pi)(u, Y_t) + R_t(u, Z_t)\dot Y_t + R_t(u, \dot Z_t)Y_t.$$

We shall compute $\partial_t R_t(u, Y_t, Y_t, v) := \partial_t g_t(R_t(u, Y_t)Y_t, v)$ at p; thus, terms with $\dot Y$ will be canceled at the final stage. Assume at a 'time' t of our choice, $\nabla = \nabla^t$ and $\nabla u = \nabla v = 0$ at p. Then perform the following preparatory calculations at p:

$$\begin{aligned}
\frac12 Y\big((\nabla^t_u \dot g_t)(Y_t, v)\big) &= Y\big(u\,(C_{Y_t}(Y_t, v, \dot Y_t)) - C_{Y_t}(\nabla^t_u Y_t, v, \dot Y_t)\big)\\
&= -C_Y(\nabla_u Y_t, v, \nabla_Y \dot Y_t),\\
\frac12 Y\big((\nabla^t_{Y_t} \dot g_t)(u, v)\big) &= Y\big(Y_t\,(C_{Y_t}(u, v, \dot Y_t))\big) - Y(C_{Y_t}(\nabla^t_{Y_t} u, v, \dot Y_t))\\
&\quad - Y(C_{Y_t}(u, \nabla^t_{Y_t} v, \dot Y_t))\\
&= C_Y(u, v, \nabla_Y \nabla_{Y_t} \dot Y_t) + 2(\nabla_Y C_Y)(u, v, \nabla_Y \dot Y_t),\\
\frac12 Y\big((\nabla^t_v \dot g_t)(u, Y_t)\big) &= Y\big(v\,(C_{Y_t}(u, Y_t, \dot Y_t)) - C_{Y_t}(u, \nabla_v Y_t, \dot Y_t)\big)\\
&= -C_Y(u, \nabla_v Y_t, \nabla_Y \dot Y_t),\\
(\nabla_{\nabla_Y Y_t} \dot g_t)(u, v) &= 2C_Y(u, v, \nabla_{\nabla_Y Y_t} \dot Y_t),\\
(\nabla_u \dot g_t)(\nabla_Y Y_t, v) &= 2C_Y(\nabla_Y Y_t, v, \nabla_u \dot Y_t),\\
(\nabla_v \dot g_t)(u, \nabla_Y Y_t) &= 2C_Y(u, \nabla_Y Y_t, \nabla_v \dot Y_t)\,.
\end{aligned}$$

Using all of that and (3.1) we obtain at p:

$$\begin{aligned}
&\langle(\nabla_Y \Pi)(u, Y_t), v\rangle = \langle \nabla_Y\,(\Pi(u, Y_t)) - \Pi(u, \nabla_Y Y_t),\, v\rangle\\
&\quad= Y\langle \Pi(u, Y_t),\, v\rangle - \langle \Pi(u, \nabla_Y Y_t),\, v\rangle\\
&\quad= \frac12 Y\big[\,(\nabla^t_u \dot g_t)(Y_t, v) + (\nabla^t_{Y_t} \dot g_t)(u, v) - (\nabla^t_v \dot g_t)(u, Y_t)\,\big]\\
&\qquad - \frac12\big[\,(\nabla_{\nabla_Y Y_t} \dot g_t)(u, v) + (\nabla_u \dot g_t)(\nabla_Y Y_t, v) - (\nabla_v \dot g_t)(u, \nabla_Y Y_t)\,\big]\\
&\quad= C_Y(u, \nabla_v Y_t, \nabla_Y \dot Y_t) - C_Y(\nabla_u Y_t, v, \nabla_Y \dot Y_t)\\
&\qquad + 2(\nabla_Y C_{Y_t})(u, v, \nabla_Y \dot Y_t) + C_Y(u, v, \nabla_Y \nabla^t_{Y_t} \dot Y_t) - C_Y(u, v, \nabla_{\nabla_Y Y_t} \dot Y_t)\\
&\qquad - C_Y(\nabla_Y Y_t, v, \nabla_u \dot Y_t) + C_Y(u, \nabla_Y Y_t, \nabla_v \dot Y_t).
\end{aligned}$$

Here the terms with $C_Y(Y, \cdot, \cdot)$ were canceled on $\mathcal U$, and the identity $[Y_t, v]^\top = -(\nabla^t_v Y_t)^\top$ at p (where $^\top$ is the orthogonal to Y at p component of a vector) was applied. Similarly, we use at p

$$\begin{aligned}
&u\big[(\nabla^t_{Y_t} \dot g_t)(Y_t, v)\big] = -2C_Y(\nabla_Y Y_t, v, \nabla_u \dot Y_t), \quad u\big[(\nabla^t_v \dot g_t)(Y_t, Y_t)\big] = 0,\\
&(\nabla_{\nabla_u Y_t} \dot g)(Y, v) = 0, \quad (\nabla_v \dot g)(Y, \nabla_u Y_t) = 0,\\
&(\nabla_Y \dot g)(\nabla_u Y_t, v) = 2\,C_Y(\nabla_u Y_t, v, \nabla_Y \dot Y_t)
\end{aligned}$$

to find

$$\begin{aligned}
&\langle(\nabla_u \Pi)(Y_t, Y_t), v\rangle = \langle\nabla_u(\Pi(Y_t, Y_t)) - 2\Pi(Y_t, \nabla_u Y_t), v\rangle \\
&= u\langle\Pi(Y_t, Y_t), v\rangle - 2\,\langle\Pi(Y_t, \nabla_u Y_t), v\rangle \\
&= u\big[(\nabla^t_{Y_t}\, \dot g_t)(Y_t, v) - \frac{1}{2}\,(\nabla^t_v\, \dot g_t)(Y_t, Y_t)\big] \\
&\quad -(\nabla_{\nabla_u Y_t}\, \dot g)(Y_t, v) - (\nabla_Y\, \dot g)(\nabla_u Y_t, v) + (\nabla_v\, \dot g)(Y, \nabla_u Y_t) \\
&= -2C_Y(\nabla_Y Y_t, v, \nabla_u \dot Y_t) - 2\,C_Y(\nabla_u Y_t, v, \nabla_Y \dot Y_t).
\end{aligned}$$

Since $\dot Y = 0$ at p, we have

$$\begin{aligned}
&\partial_t R_t(u, Y_t, Y_t, v) = (\partial_t g)(R_t(u, Y_t)Y_t, v) + g(\partial_t R_t(u, Y_t)Y_t, v) \\
&= 2\,C_Y(R_t(u, Y_t)Y_t, v, \dot Y) + g(\partial_t R_t(u, Y_t)Y_t, v) = g(\partial_t R_t(u, Y_t)Y_t, v).
\end{aligned}$$

Finally, we have (3.2) at p for all $t \ge 0$. For the second fundamental form b_t of $\mathcal F$ (with respect to g_t), as in the proof of [12, Lemma 2.9], using (3.1), (3.4), $\dot g(p) = 0$ and $\dot Y(p) = 0$, we get at a point p:

$$\begin{aligned}
\partial_t b_t(u, v) &= \dot g(\nabla_u v, Y) + g(\partial_t \nabla_u v, Y) + g(\nabla_u v, \partial_t Y) \\
&= \frac{1}{2}\big((\nabla_u \dot g)(v, Y) + (\nabla_v \dot g)(u, Y) - (\nabla_Y \dot g)(u, v)\big) + g(\nabla_u v, \dot Y) \\
&= -\nabla_Y(C_Y(u, v, \dot Y)) = -C_Y(u, v, \nabla_Y \dot Y).
\end{aligned}$$

From this, using $b_t(u, v) = g_t(A_t(u), v)$, we get $(3.3)_1$:

$$g_t(A_t(u), v) = \partial_t b_t(u, v) - \dot g(A(u), v) = -C_\nu(u, v, \nabla_\nu \dot Y).$$

By the formula for the volume form of a t-dependent metric, $\partial_t(\mathrm{d}V_t) = \frac{1}{2}\,(\mathrm{Tr}\,\dot g)\,\mathrm{d}V_t$, see [16], and definition of the mean Cartan torsion, we get

$$\partial_t(\mathrm{d}V_t) = I_{Y_t}(\dot Y_t)\,\mathrm{d}V_t. \tag{3.5}$$

Next, $(3.3)_2$ follows from (3.5) and $\dot Y(p) = 0$. □

Let L be a leaf through a point $p \in M$, and ρ the local distance function to L in a neighborhood of p. Denote by $\hat\nabla$ the Levi-Civita connection of the (local again) Riemannian metric $\hat g := g_{\nabla\rho}$. Note that $\nabla\rho = \nu$ on L. The *shape operator* $A : T\mathcal F \to T\mathcal F$ (self-adjoint for g) is defined at $p \in M$ by (compare [15] with the opposite sign)

$$A(u) = -\hat\nabla_u\, \nu \quad (u \in T_p\mathcal F).$$

The shape operator $A^g : T\mathcal F \to T\mathcal F$ with respect to the metric g is defined at $p \in M$ by

$$A^g(u) = -\nabla_u\, \nu \quad (u \in T_p\mathcal F).$$

Note that $2\,g(\nabla_u\, \nu, \nu) = u(g(\nu, \nu)) = 0$ $(u \in T\mathcal F)$; hence, $\nabla_u\, \nu \in T\mathcal F$. The *mean curvature* function (of the leaves with respect to g) is defined by $H^g = \mathrm{Tr}\, A^g$. Recall that $\mathcal F$ is *g-totally umbilical* if $A^g = H^g I_m$, and is *g-totally geodesic* if $A^g \equiv 0$.

Corollary 3.3. *Let L be a hypersurface in an open set $\mathcal{U}\subset M$. If an F-unit vector field Y_t $(0\le t\le\varepsilon)$ is given in $\mathcal{U}$ and orthogonal to L then for the metric $g_t:=g_{Y_t}$ for all $u,v\in T_pL$ $(p\in L)$ we have*

$$\begin{aligned}\partial_t R_t(u,Y_t,Y_t,v) &= C_Y(A_t(u),v,\nabla^t_Y\dot Y_t)+C_Y(u,A_t(v),\nabla^t_Y\dot Y_t)\\ &\quad -C_Y(u,v,(\nabla^t)^2_{Y,Y}\dot Y_t)-2(\nabla^t_Y C_{Y_t})(u,v,\nabla^t_Y\dot Y_t),\end{aligned}\tag{3.6}$$

$$g(\partial_t A_t(u),v) = -C_Y(u,v,\nabla^t_Y\dot Y_t),\quad \partial_t(\mathrm{d}V_t)=0.\tag{3.7}$$

Proof. This follows from $\dot Y_t=0$ on L, the definition of A_t (for g_t) and (3.2)–(3.3). □

Definition 3.1. A vector field $\widehat Y$ defined in some neighborhood $\mathcal{U}\subset M$ of a point $p\in\mathcal{U}$ is called a *geodesic extension* of a vector $Y_p\in T_pM$ if $\widehat Y(p)=Y_p$ and the integral curves of $\widehat Y$ are geodesics of the Finsler metric. Similarly, we define a *geodesic extension* of a (e.g. normal) vector field along a hypersurface $L\subset\mathcal{U}$. In both cases, $\hat g:=g_{\widehat Y}$ is called the *osculating Riemannian metric* of F on $\mathcal{U}$.

We will use osculating metric (given locally) to express the Riemannian curvature of $g=g_\nu$ (for an unit F-normal ν to $\mathcal{F}$) in terms of Riemannian curvature and the Cartan torsion of F.

Given a vector field Y, let $C^\sharp_Y$ be a $(1,1)$-tensor g_Y-dual to the symmetric bilinear form $C_Y(\cdot\,,\cdot\,,\nabla_Y Y)$. Note that $C_n(\cdot\,,\cdot\,,\nabla_n n)=C_{c^2\nu}(\cdot\,,\cdot\,,c^4\nabla_\nu\nu)=c^2C_\nu(\cdot\,,\cdot\,,\nabla_\nu\nu)$.

Theorem 3.4. *Let ν be a unit normal to a codimension-one foliation of a Finsler space (M^{m+1},F). The Riemann curvatures (in the ν-direction) of F and $g=g_\nu$ are related by*

$$\begin{aligned}g((R_\nu-R^g_\nu)(u),v)&=-C_\nu\big(A^g(u)+\frac12 C^\sharp_\nu(u),v,\nabla_\nu\nu\big)\\ &\quad -C_\nu\big(u,A^g(v)+\frac12 C^\sharp_\nu(v),\nabla_\nu\nu\big)\\ &\quad +C_\nu\big(u,v,\nabla^2_{\nu,\nu}\nu-C^\sharp_\nu(\nabla_\nu\nu)\big)+2(\nabla_\nu C_\nu)(u,v,\nabla_\nu\nu)\quad(u,v\in T_pL).\end{aligned}\tag{3.8}$$

The shape operators and volume forms are related by

$$A-A^g=C^\sharp_\nu,\qquad \mathrm{d}V_g=e^{\tau(\nu)}\,\mathrm{d}V_F.\tag{3.9}$$

In particular, the traces are related by

$$\begin{aligned}\mathrm{Ric}_\nu-\mathrm{Ric}^g_\nu &= I_\nu(\nabla^2_{\nu,\nu}\nu-C^\sharp_\nu(\nabla_\nu\nu))+2(\nabla_\nu I_\nu)(\nabla_\nu\nu)\\ &\quad -\mathrm{Tr}\,\big(C^\sharp_\nu(C^\sharp_\nu+2A^g)\big),\\ \mathrm{Tr}\,A-\mathrm{Tr}\,A^g &= I_\nu(\nabla_\nu\nu).\end{aligned}\tag{3.10}$$

Proof. Let $\mathcal{U}$ be a "small" neighborhood of $p\in L$ such that any two geodesics starting from $L\cap\mathcal{U}$ in the ν-direction do not intersect in $\mathcal{U}$. Then for any $q\in\mathcal{U}$ there is a unique geodesic γ starting from L in the ν-direction such that $\gamma(s)=q$ for some $s\ge0$, in other words, $q=\exp_{\gamma(0)}(s\,\dot\gamma(0))$. Thus, $\widehat Y:q\to\dot\gamma(s)$ $(q\in\mathcal{U})$ is an F-unit geodesic vector field $(\nabla_{\widehat Y}\widehat Y=0)$ – a geodesic extension of $\nu_{\,|\,L}$.

Consider a family of vector fields $Y_t=t\,\widehat Y+(1-t)\,\nu$ $(0\le t\le1)$ on $\mathcal{U}$, define the Riemannian metrics $g_t:=g_{Y_t}$, g_1 being osculating, and denote by R_t their Riemann curvatures. Since $\dot Y_t=\widehat Y-\nu$ and $Y_{t\,|L}=\nu_{|L}=\widehat Y_{|L}$ for all t, we have $\dot Y_{t\,|L}=0$ and $g_{t\,|L}\equiv g_{\,|L}$. By (3.1) and (3.4), we get $\Pi_t(\nu,\nu)=\Pi_t(\nu,\widehat Y)=0$ on L; hence, $\nabla^t_\nu\,\nu$ and $\nabla^t_\nu\widehat Y$ restricted on L don't depend on t. Next, we find

$$g(\Pi(\nu,\nu),v)=C_\nu(u,v,\nabla_\nu(\widehat Y-\nu))=-C_\nu(u,v,\nabla_\nu\,\nu),\quad u,v\in TM_{\,|L},$$

i.e., $\Pi(\nu,u)=-C^\sharp_\nu(u)$. We calculate on L:

$$\begin{aligned}
g(\partial_t(\nabla^t_\nu\,u),v) &= \nabla^t_\nu(C_Y(u,v,\widehat Y-\nu))+\nabla^t_u(C_Y(\nu,v,\widehat Y-\nu))-\nabla^t_v(C_Y(u,\nu,\widehat Y-\nu))\\
&= (\nabla^t_\nu C_Y)(u,v,\widehat Y-\nu)+C_Y(u,v,\nabla^t_\nu(\widehat Y-\nu))\\
&+ (\nabla^t_u C_\nu)(n,v,\widehat Y-\nu)+C_\nu(\nabla^t_u\,\nu,v,\widehat Y-\nu)+C_\nu(\nu,v,\nabla^t_u(\widehat Y-\nu))\\
&- (\nabla^t_v C_\nu)(u,\nu,\widehat Y-\nu)-C_\nu(u,\nabla^t_v\,\nu,\widehat Y-\nu)-C_\nu(u,\nu,\nabla^t_v(\widehat Y-\nu))\\
&= C_\nu(u,v,\nabla^t_\nu(\widehat Y-\nu))=-C_\nu(u,v,\nabla_\nu\,\nu).
\end{aligned}$$

Since, $\partial_t(g(\nabla^t_\nu\,u,v))=g(\partial_t\nabla^t_\nu\,u,v)$ and $\partial_t(g(\nabla^t_u\,\nu,v))=g(\partial_t\nabla^t_u\,\nu,v)$ on L, we obtain

$$\begin{aligned}
g(\nabla^t_\nu\,u,v)&=g(\nabla_\nu\,u,v)-t\,C_\nu(u,v,\nabla_\nu\,\nu),\\
g(\nabla^t_u\,\nu,v)&=g(\nabla_u\,\nu,v)-t\,C_\nu(u,v,\nabla_\nu\,\nu).
\end{aligned}$$

Recall that $\nabla^2_{u,v}$ is tensorial in u,v. We show that $(\nabla^t)^2_{\nu,\nu}\,\widehat Y$ is t-independent on L:

$$\begin{aligned}
(\nabla^t)^2_{\widehat Y,\widehat Y}\,\widehat Y &= \nabla^t_n(\nabla^t_{\widehat Y}\,\widehat Y)=\nabla_\nu(\nabla^t_{\widehat Y}\,\widehat Y)-t\,C^\sharp_\nu(\nabla^t_\nu\,\widehat Y)\\
&= \nabla_\nu(\nabla^t_{\widehat Y}\,\widehat Y)=\nabla_\nu(\nabla_{\widehat Y}\,\widehat Y-t\,C^\sharp_\nu(\widehat Y))\\
&= \nabla^2_{\nu,\nu}\,\widehat Y-t\,(\nabla_\nu C^\sharp_\nu)(\widehat Y)-t\,C^\sharp_\nu(\nabla_\nu\widehat Y)=\nabla^2_{\nu,\nu}\,\widehat Y.
\end{aligned}$$

Thus, $(\nabla^2_{\nu,\nu}\,\widehat Y)_{\,|L}=(\widehat\nabla^2_{\nu,\nu}\,\widehat Y)_{\,|L}=0$. Using this and $(\nabla_\nu\widehat Y)_{\,|L}=0$, we find on L:

$$\begin{aligned}
\nabla^t_{Y_t}\,\dot Y_t &= -\nabla_\nu\,\nu,\\
(\nabla^t)^2_{Y_t,Y_t}\,\dot Y_t &= (\nabla^t)^2_{\nu,\nu}\,(\widehat Y-\nu)=\nabla^t_\nu\big(\nabla_\nu(\widehat Y-\nu)-t\,C^\sharp_\nu\,(\widehat Y-\nu)\big)\\
&= \nabla^2_{\nu,\nu}\,(\widehat Y-\nu)-t\,\nabla_\nu\,(C^\sharp_\nu(\widehat Y-\nu))-t\,C^\sharp_\nu(\nabla_\nu\,(\widehat Y-\nu))\\
&= -\nabla^2_{\nu,\nu}\,\nu+2\,t\,C^\sharp_\nu(\nabla_\nu\,\nu).
\end{aligned}$$

Then we obtain on L:

$$\begin{aligned}
C_{Y_t}(\cdot\,,\cdot\,,\nabla_{Y_t}\,\dot Y_t) &= C_\nu(\cdot\,,\cdot\,,\nabla_\nu(\widehat Y-\nu))=-C_\nu(\cdot\,,\cdot\,,\nabla_\nu\,\nu),\\
C_{Y_t}(\cdot\,,\cdot\,,\nabla^2_{Y_t,Y_t}\,\dot Y_t) &= C_\nu(\cdot\,,\cdot\,,\nabla^2_{\nu,\nu}(\widehat Y-\nu))=-C_\nu(\cdot\,,\cdot\,,\nabla^2_{\nu,\nu}\,\nu).
\end{aligned}$$

Next, we calculate on L, using $C_Z(Z,\cdot\,,\cdot\,)=0$ for $Z=\nabla_\nu\,\nu$,

$$\begin{aligned}
&(\nabla_{Y_t}C_{Y_t})(\cdot\,,\cdot\,,\nabla_{Y_t}\,\dot Y_t)=(\nabla_\nu C_{\,t\,\widehat Y+(1-t)\,\nu})(\cdot\,,\cdot\,,-\nabla_\nu\,\nu)\\
&=(\nabla_\nu C)_{\,\nu}(\cdot\,,\cdot\,,-\nabla_\nu\,\nu)+C_{\,(1-t)\nabla_\nu\,\nu}(\cdot\,,\cdot\,,-\nabla_\nu\,\nu)=-(\nabla_\nu C_{\,\nu})(\cdot\,,\cdot\,,\nabla_\nu\,\nu).
\end{aligned}$$

By the above and $(3.3)_1$, we obtain $(3.9)_1$. By Corollary 3.3, for all $t\in[0,1]$, and using $A_t=A^g+t\,C^\sharp_\nu$, see $(3.9)_1$, and $(\nabla^t)^2_{\nu,\nu}\,\nu=-\nabla^2_{\nu,\nu}\,\nu+2\,t\,C^\sharp_\nu(\nabla_\nu\,\nu)$, we obtain

$$\begin{aligned}
\partial_t R_t(u,\nu,\nu,v) &= -C_\nu(A_t(u),v,\nabla_\nu\,\nu)-C_\nu(u,A_t(v),\nabla_\nu\,\nu)\\
&\quad+C_\nu(u,v,(\nabla^t)^2_{\nu,\nu}\,\nu)+2(\nabla_\nu C_\nu)(u,v,\nabla_\nu\,\nu)\\
&= -C_\nu(A^g(u)+t\,C^\sharp_\nu(u),v,\nabla_\nu\,\nu)-C_\nu(u,A^g(u)+t\,C^\sharp_\nu(v),\nabla_\nu\,\nu)\\
&\quad+C_\nu(u,v,-\nabla^2_{\nu,\nu}\,\nu+2\,t\,C^\sharp_\nu(\nabla_\nu\,\nu))+\,2(\nabla_\nu C_\nu)(u,v,\nabla_\nu\,\nu)
\end{aligned}$$

for $u, v \in T_pL$, where the right hand side becomes linear in t. Integrating this by $t \in [0,1]$ yields (3.8). Finally, using the equality for volume forms, $\mathrm{d}\widehat{V} = \mathrm{d}V_g$, and definition of τ (see Section 2.1), we get $(3.9)_2$. $\square$

Since any geodesic vector field Y satisfies conditions

$$C_Y(u,v,\nabla_Y Y) = 0, \quad C_Y(u,v,\nabla^2_{Y,Y} Y) = 0 \quad (\forall\, u,v), \tag{3.11}$$

the following corollary generalizes Proposition 3.1.

Corollary 3.5. *If Y is a unit vector field on a Finsler space (M,F) and $g := g_Y$ a Riemannian metric on M with the Levi-Civita connection ∇ and conditions* (3.11)*, then $R_Y = R^g_Y$.*

Proof. By (3.11), we have $C^\sharp_Y = 0$ and

$$(\nabla_Y C_Y)(u,v,\nabla_Y Y) = \nabla_Y(C_Y(u,v,\nabla_Y Y)) - C_Y(u,v,\nabla^2_{Y,Y} Y) = 0.$$

If a vector field $\widehat{Y}$ is a local geodesic extension of $Y(p)$ then $R^g_Y = \hat{R}_Y$ (and $A^g = \hat{A}$) at p, see (3.8) and (3.9). Thus, the claim follows from Proposition 3.1. $\square$

3.2 Integral formulae

Let $\mathcal{F}$ is a codimension-one foliation of a closed Finsler space (M^{m+1}, F) with the Busemann-Hausdorff volume form $\mathrm{d}V_F$. Define a family of diffeomorphisms $\{\phi_t : M \to M,\ 0 \le t < \varepsilon\}$ ($\varepsilon > 0$ being small enough) by

$$\phi_t(p) = \exp_p(t\,\nu), \quad \text{where} \quad \nu \in T_pM \quad \text{is an } F\text{-unit normal to } \mathcal{F} \text{ at } p \in M.$$

Let $c(t)$ ($t \ge 0$) be an F-geodesic with $c(0) = p$ and $\dot c(0) = \nu(p)$. Any geodesic variation built of ϕ_t-trajectories determines an F-Jacobi field $Y(t)$ on c, and $A_p(Y(0)) = -[D_{\dot c(t)}\, Y(t)]_{\,|t=0}$, see [15, p. 225]. Recall that if vectors $u(t)$ and $v(t)$ are D-parallel along $c(t)$ then $g_{\dot c(t)}(u(t), v(t))$ is constant. Choose a positively oriented $g_{\nu(p)}$-orthonormal frame $(e^1, \dots, e^m)$ of $T_p\mathcal{F}$ and extend it by parallel translation to the frame $(E^1_t, \dots, E^m_t)$ of vector fields $g_{\dot c(t)}$-orthogonal to $\dot c(t)$ along $c(t)$. Denote also by $E^{m+1}_t = \dot c(t)$ the tangent vector field along $c(t)$. Denote by $Y^i(t)$ ($i \le m$) the Jacobi field along $c(t)$ satisfying $Y^i(0) = e^i$ and $D_{\dot c}\, Y^i(0) = A_p(e^i)$. Let $R(t)$ be the matrix with entries $g_{\dot c}(R_{\dot c}(E^i_t), E^j_t)$. Denote by $\mathbf{Y}(t)$ the $m \times m$ matrix consisting of the scalar products $g_{\dot c}(Y^i(t), E^j_t)$ ("*F-Jacobi tensor*"). Then $\mathbf{Y}(0) = I_m$ and $\mathbf{Y}'(0) = A_p$. It is known (see, for instance, [15, Sections 2.1 and 2.2]) that

$$|\, d\,\phi_t(p)| = \det \mathbf{Y}(t)\,,$$

where $|\, d\phi_t(p)|$ is the Jacobian of ϕ_t at p. Assume that $R^{(1)}_{\dot c(t)} \equiv 0$ for any F-geodesic $c(t)$ ($t \ge 0$) (e.g. (M,F) is *locally symmetric* with respect to F). For $t = 0$, we have $R^{(2)}_{\dot c(0)} \equiv R^{(3)}_{\dot c(t)} \equiv \ldots \equiv 0$. For short, write $R_p := R(0)$. Note that $\operatorname{Tr} R_p = \operatorname{Ric}(\nu(p))$. The F-Jacobi equation $\mathbf{Y}'' = -R(t)\mathbf{Y}$ implies that

$$\mathbf{Y}^{(2k)}(0) = (-R_p)^k, \quad \mathbf{Y}^{(2k+1)}(0) = (-R_p)^k A_p, \quad k = 0, 1, 2, \ldots$$

Hence, our Jacobi tensor has the form

$$\mathbf{Y}(t)=\sum\nolimits_{k=0}^{\infty}\mathbf{Y}^{(k)}(0)\,\frac{t^k}{k!}=I_m+tA_p-\frac{t^2}{2!}R_p-\frac{t^3}{3!}R_pA_p+\frac{t^4}{4!}R_p^2+\ldots.$$

Certainly, the radius of convergence of the series is uniformly bounded from below on M (by $1/\|R\|_F>0$). The volume of M is defined by $\mathrm{Vol}_F(M)=\int_M \mathrm{d}V_F$. Therefore – by Dominated Convergence Theorem – its integration together with Change of Variables Theorem yield the equality for any $t\ge 0$ small enough

$$\mathrm{Vol}_F(M)=\int_M \det\big(I_m+tA_p-\frac{t^2}{2!}R_p-\frac{t^3}{3!}R_pA_p+\frac{t^4}{4!}R_p^2+\ldots\big)\,\mathrm{d}V_F\,, \tag{3.12}$$

where $\mathrm{d}V_F$ is the volume form of F. Formula (3.12) together with Lemma 5.2 of Appendix imply our main result (which generalizes that of [13] valid for the Riemannian case). Note that the invariants $\sigma_\lambda(A_1,\ldots,A_k)$ of a set of real $m\times m$ matrices A_i are defined and discussed in Appendix.

Theorem 3.6. *If $\mathcal{F}$ is a codimension-one foliation on a closed Finsler manifold (M^{m+1},F), which is F-locally symmetric, then for any $0\le k\le m$ one has*

$$\int_M\sum\nolimits_{\|\lambda\|=k}\sigma_\lambda\left(B_1(p),\ldots B_k(p)\right)\,\mathrm{d}V_F=0, \tag{3.13}$$

where $B_{2k}(p)=\frac{(-1)^k}{(2k)!}\,(R_p)^k,\ B_{2k+1}(p)=\frac{(-1)^k}{(2k+1)!}\,(R_p)^kA_p$ *for* $p\in M$.

The formulae (3.13) for few initial values of k, $k=1,\ldots 3$, read as follows:

$$\int_M \sigma_1(A_p)\,\mathrm{d}V_F = 0, \tag{3.14}$$

$$\int_M \big(\sigma_2(A_p)-\frac{1}{2}\,\mathrm{Tr}\,R_p\big)\,\mathrm{d}V_F = 0, \tag{3.15}$$

$$\int_M \big(\sigma_3(A_p)-\frac{1}{2}\,\mathrm{Tr}(A_p)\,\mathrm{Tr}\,R_p+\frac{1}{3}\,\mathrm{Tr}(R_pA_p)\big)\,\mathrm{d}V_F = 0. \tag{3.16}$$

The formulae (3.14) and (3.15) are well known for arbitrary foliated Riemannian manifolds, see the Introduction. For $m=1$, (3.15) reduces to the integral of flag (Gauss) curvature, $\int_M K\,\mathrm{d}V_F=0$.

Remark 3.2. 1. The compactness of M in Theorem 3.6 can be replaced by weaker conditions: M is positively complete of finite F-volume, and has 'bounded geometry' in the following sense:

$$\sup\nolimits_{p\in M}\|R_p\|_F<\infty,\quad \sup\nolimits_{p\in M}\|A_p\|_F<\infty. \tag{3.17}$$

2. Similar formulae exist for codimension-one foliations of on arbitrary (non-locally symmetric with respect to F) Finsler manifolds. They are more complicated since they contain terms which depend on covariant derivatives of R_p. More precisely, they contain just terms of the form $R_p^{(k)}$, where $R_p^{(1)}=D_{\nu(p)}R_p$, $R_p^{(2)}=D_{\nu(p)}D_{\nu(p)}\,R_p$ and so on. For the F-Jacobi tensor $\mathbf{Y}(t)$ we get

$$\mathbf{Y}(t)=I_m+tA_p-\frac{t^2}{2!}R_p-\frac{t^3}{3!}(R_pA_p+R_p^{(1)})+\frac{t^4}{4!}(R_p^2-R_p^{(2)}-2R_p^{(1)}A_p)+\ldots$$

The t^3 term of (3.12) becomes, compare (3.16),

$$\int_M \big(\sigma_3(A_p) - \frac{1}{2}\mathrm{Tr}(R_p)\,\mathrm{Tr}(A_p) + \frac{1}{3}\,\mathrm{Tr}(R_p\,A_p) - \frac{1}{6}\,\mathrm{Tr}\,R_p^{(1)}\big)\,\mathrm{d}V_F = 0.$$

In general, the t^k term in (3.12) contains $R_p^{(j)}$'s with $j \le k-2$.

Corollary 3.7. *Let $\mathcal{F}$ be a codimension-one foliation on a F-locally symmetric complete Finsler manifold (M, F) of finite F-volume and bounded (in the sense of (3.17)) geometry. If* $\mathrm{rank}(A_p) \le 1$ *for all* $p \in M$ *(for example, $\mathcal{F}$ is F-totally geodesic) then the Riemannian curvature R_p vanishes identically provided that M has everywhere non-negative (or, non-positive) Ricci curvature* $\mathrm{Ric}_p = \mathrm{Tr}\,R_p$.

Proof. Since in this case $\sigma_2(A_p) = 0$, integral formula (3.15) implies the claim. □

Given a unit normal ν to $\mathcal{F}$, denote by Q_R the symmetric $(0,2)$-tensor in the rhs of (3.8). Then, see (3.10),

$$\mathrm{Tr}\,Q_R = I_\nu(\nabla^2_{\nu,\nu}\nu + C^\sharp_\nu(\nabla_\nu\,\nu)) + 2(\nabla_\nu I_\nu)(\nabla_\nu\,\nu) - \mathrm{Tr}\,\big(C^\sharp_\nu(C^\sharp_\nu + 2\,A^g)\big).$$

Define the 1-form θ_g by the equality

$$\theta_g(X) = g([X,\nu],\nu) \qquad (X \in TM).$$

Note that $\nabla_\nu\,\nu = \theta_g^\sharp$ is the mean curvature of ν-curves with respect to g. Comparing (3.13) for F and g, we obtain a series of integral formulas, the first two of which are given in the following.

Theorem 3.8. *Let* $\tau(\nu) = \mathrm{const}$ *on a codimension-one foliated Finsler space* (M, F). *Then*

$$\int_M I_\nu(\nabla_\nu\,\nu)\,\mathrm{d}V_F \;=\; 0, \tag{3.18}$$

$$\int_M \big(\sigma_2(C^\sharp_\nu) + (\mathrm{Tr}\,A^g)(\mathrm{Tr}\,C^\sharp_\nu) - \mathrm{Tr}(A^g C^\sharp_\nu) - \frac{1}{2}\,\mathrm{Tr}\,Q_R\big)\,\mathrm{d}V_F \;=\; 0. \tag{3.19}$$

Proof. By $(3.9)_1$, $A = A^g + C^\sharp_\nu$, where $A = A_p$. Thus, (3.18) follows from (3.14), using $(3.9)_2$ and Theorem 3.4. Note that by (5.4) with $k = 1$ and (5.6) (of Appendix), and by (3.10), we have

$$\begin{aligned}\sigma_2(A_p) &= \sigma_2(A^g) + \mathrm{Tr}(A^g)\,\mathrm{Tr}\,C^\sharp_\nu - \mathrm{Tr}(A^g C^\sharp_\nu),\\ \mathrm{Ric}_\nu &= \mathrm{Tr}\,R_p = \mathrm{Ric}^g_\nu + \mathrm{Tr}\,Q_R.\end{aligned}$$

Thus, (3.19) follows from (3.15), using $(3.9)_2$ and (5.6) with $k = 2$ (of Appendix). □

3.3 Examples

Finsler manifolds of constant flag curvature. We provide examples, these of (M, F) with constant flag curvature $K(\nu, P)$ on M, i.e., such that $R_p = K\,I_m$ for some $K \in \mathbb{R}$.

a) For (M, F) with zero flag curvature, $R_p = 0$, and we obtain the Jacobi tensor of a simple form, linear in t: $\mathbf{Y}(t) = I_m + tA_p$ $(t \ge 0)$. Then (3.12) reduces to

$\mathrm{Vol}_F(M) = \int_M \det(I_m + tA_p)\,\mathrm{d}V_F$. From this we obtain the Finsler generalization of the case $K = 0$ of [3, Theorem 1.1], i.e.,

$$\int_M \sigma_k(A_p)\,\mathrm{d}V_F = 0, \quad k > 0. \tag{3.20}$$

b) Assume now that the flag curvature $K(\nu, P)$ of (M, F) is constant and positive, say $K = 1$. Then $R_p = I_m$ and one can rewrite the Taylor series for $\mathbf{Y}(t)$ $(t \geq 0)$ in the form $\mathbf{Y}(t) = \cos t\,\big(I_m + (\tan t)A_p\big)$. Change of Variables Theorem for integration implies that the equality

$$\mathrm{Vol}_F(M) = (\cos t)^m \int_M \det\big(I_m + (\tan t)A_p\big)\,\mathrm{d}V_F$$

holds for arbitrary $t \geq 0$ small enough. One can use the substitution $\tan t \to \tilde t$ and the identity $\cos^2 t = (1 + \tilde t^2)^{-1}$ for further derivations.

c) The case of negative constant flag curvature $K(\nu, P)$ of M, say $K = -1$, is similar to the case (b). One can use the substitution $\tanh(t) \to \tilde t$ and the identity $\cosh^2 t = (1 - \tilde t^2)^{-1}$ for derivations.

The above yields the following extension of Theorem 1.1 in [3].

Corollary 3.9. *Let $\mathcal{F}$ be a transversally oriented codimension-one foliation on a Finsler manifold (M^{m+1}, F) of finite F-volume and $\sup_{p\in M} \|A_p\|_F < \infty$ (e.g. closed) with a unit normal ν and condition $R_p = K I_m$. Then, for any $0 \leq k \leq m$,*

$$\int_M \sigma_k(A_p)\,\mathrm{d}V_F = \begin{cases} K^{k/2}\binom{m/2}{k/2}\,\mathrm{Vol}_F(M), & m,\ k \text{ even},\\ 0, & m \text{ or } k \text{ odd}.\end{cases} \tag{3.21}$$

Remark 3.3. By Theorem 8.2.4 in [8], if a Finsler manifold M is closed and has constant negative curvature then it is Randers.

If (M, F) is F-locally symmetric and the leaves of $\mathcal{F}$ are *F-totally geodesic* (i.e., $A_p = 0$) then

$$\mathbf{Y}^{(2k+1)}(0) = 0, \quad \mathbf{Y}^{(2k)}(0) = (-R_p)^k.$$

Finally we get the F-Jacobi tensor $\mathbf{Y}(t) = I_m - \frac{t^2}{2!}R_p + \frac{t^4}{4!}R_p^2 - \frac{t^6}{6!}R_p^3 + \ldots$, and (3.13) reduces to

$$\int_M \sum\nolimits_{\|\lambda\|=k} \sigma_\lambda\big(-\frac{1}{2!}\,R_p,\ \frac{1}{4!}\,R_p^2, \ldots, \ \frac{(-1)^k}{(2k)!}R_p^k\big)\,\mathrm{d}V_F = 0.$$

For codimension-one F-totally geodesic foliations on arbitrary positively complete (or closed) Finsler manifolds of finite F-volume, we get

$$\int_M \mathrm{Tr}\,R_p\ \mathrm{d}V_F = 0, \quad \int_M \mathrm{Tr}\,R_p^{(1)}\,\mathrm{d}V_F = 0,$$
$$\int_M \big(\sigma_2(R_p) + \frac{1}{6}\ \mathrm{Tr}\,R_p^2 - \frac{1}{6}\ \mathrm{Tr}\,R_p^{(2)}\big)\,\mathrm{d}V_F = 0, \tag{3.22}$$

and so on. Equalities (3.22) imply directly the following statement (see also Corollary 3.7).

Corollary 3.10. *Let $\mathcal{F}$ be a codimension-one F-totally geodesic foliation on a F-locally symmetric positively complete Finsler manifold (M,F) of finite F-volume and with condition* $(3.17)_1$. *Then R_p vanishes identically provided that either M has everywhere non-negative (or, non-positive) Ricci curvature* Ric, *or $\sigma_2(R_p)$ is non-negative.*

It has been observed in [7] that codimension-one foliations of compact negatively-Ricci curved Riemannian spaces are far (in a sense) from being totally umbilical. In the case of an *F-totally umbilical foliation*, $A_p = H\,I_m$, therefore on a locally symmetric Finsler space (M,F) the following can be derived from (3.15) – (3.16) etc. with the use of Lemma 5.1 of Appendix:

$$\int_M \big((m-1)(m-2)H^2 - \operatorname{Tr} R_p\big)\,\mathrm{d}V_F = 0, \tag{3.23}$$

$$\int_M H\big(\frac{m(m-1)(m-2)}{3m-2}H^2 - \operatorname{Tr} R_p\big)\,\mathrm{d}V_F = 0. \tag{3.24}$$

These integrals for k even ((3.23), (3.24), etc.) contain polynomials depending on H^2 only. If all the coefficients of such polynomials are positive, then the polynomials are positive for all values of H and one may easily get obstructions for existence of totally umbilical foliations on some Finsler manifolds.

4 Codimension-one foliated Randers spaces

Let $\mathcal{F}$ be a transversally oriented codimension-one foliation of M^{m+1} equipped with a Randers metric

$$F(y) = \sqrt{a(y,y)} + \beta(y), \quad \|\beta\,\|_\alpha < 1, \quad \beta^\sharp \in \Gamma(T\mathcal{F}).$$

As before, let us write $a(\cdot,\cdot) = \langle\cdot,\cdot\rangle$. Let N be a unit a-normal vector field to $\mathcal{F}$, i.e., $\langle N,N\rangle = 1$ and $\langle N,v\rangle = 0$ for $v \in T\mathcal{F}$, and n an F-normal vector field to $\mathcal{F}$ with the property $\langle n,n\rangle = 1$. Denote by $\bar\nabla$ the Levi-Civita connection of the Riemannian metric a and by ∇ the Levi-Civita connection of the Riemannian metric $g = g_n$ on M. According to [4, (1.15) and (1.19)] we have

$$\tau(y) \;=\; (m+2)\log\sqrt{(1+\beta(y)/\alpha(y))\,c^{-2}}\,, \tag{4.1}$$

$$I_y(v) \;=\; \frac{m+2}{2F(y)}\Big(\beta(v) - \frac{\langle v,y\rangle\,\beta(y)}{\alpha^2(y)}\Big)\,. \tag{4.2}$$

In particular, $\tau(n) = 0$ and $I_n(v) = \frac{m+2}{2\,c^4}\,\langle\,\beta^\sharp - (c^2-1)\,n,\ v\rangle$. Remark that for Randers spaces

$$C_n(u,v,w) = \frac{1}{m+2}\,\big(I_n(u)\,h_n(v,w) + I_n(v)\,h_n(u,w) + I_n(w)\,h_n(u,v)\big)\,,$$

where the angular form h_n is given by

$$h_n(u,v) = c^2\big(\langle u,v\rangle - \langle u,n\rangle\,\langle v,n\rangle\big)\,, \tag{4.3}$$

see [4, (1.11) and (1.20)]. Since $\sigma_F = c^{m+2}\sqrt{\det a_{ij}}$, see [4, p. 6], and $\sqrt{\det g_{ij}(n)} = c^{m+2}\sqrt{\det a_{ij}}$, see (2.6), the volume form of F and canonical volume forms of Riemannian metrics g and a obey

$$\mathrm{d}V_F = c^{m+2}\mathrm{d}V_a, \qquad \mathrm{d}V_g = c^{m+2}\mathrm{d}V_a, \qquad \mathrm{d}V_F = \mathrm{d}V_g. \tag{4.4}$$

Let $Z = \nabla_\nu\,\nu$ (which is dual of θ_g in Sect. 3.2) and $\bar Z = \bar\nabla_N N$ be the curvature vectors of ν-curves and N-curves for Riemannian metrics g and a, respectively.

4.1 The shape operators of g and a

The shape operators of $\mathcal{F}$ with respect to the metrics a and g are defined as follows:

$$\bar A(u) = -\bar\nabla_u N, \quad A^g(u) = -\nabla_u\,\nu\,,$$

where $u\in T\mathcal{F}$ and $\nu = c^{-2}n = c^{-1}(N - c^{-1}\beta^\sharp)$ with $c=\sqrt{1-\|\beta\|^2_\alpha}>0$.

The derivative $\bar\nabla u : TM\to TM$ is defined by $(\bar\nabla u)(v) = \bar\nabla_v\, u = \bar\nabla_v\, u$, where $v\in TM$. The conjugate derivative $(\bar\nabla u)^t : TM\to TM$ is defined by $\langle(\bar\nabla\, u)^t(v), w\rangle = \langle v, (\bar\nabla\, u)(w)\rangle$ for all $v,w\in TM$. The *deformation tensor* $\overline{\mathrm{Def}}$,

$$2\,\overline{\mathrm{Def}}_u = \bar\nabla u + (\bar\nabla u)^t,$$

measures the degree to which the flow of a vector field $u\in\Gamma(TM)$ distorts the metric a. The same notation $\overline{\mathrm{Def}}_u$ will be used for its dual (with respect to a) $(1,1)$-tensor. Set $\overline{\mathrm{Def}}_u^\top(v) = (\,\overline{\mathrm{Def}}_u(v))^\top$. For $\beta\ne0$, let

$$\bar A(\beta^\sharp)^{\perp\beta} = \bar A(\beta^\sharp) - \langle\bar A(\beta^\sharp),\beta^\sharp\rangle\beta^\sharp\cdot\|\beta^\sharp\|_\alpha^{-2}$$

be the projection of $\bar A(\beta^\sharp)$ on $(\beta^\sharp)^\perp$. Note that $\lim_{\beta\to0}\bar A(\beta^\sharp)^{\perp\beta}=0$.

Proposition 4.1. *Let $\beta(N)=0$ on M. Then on $T\mathcal{F}$ we have*

$$c\,A^g = \bar A - c^{-2}(c\,N-\beta^\sharp)(c)I_m + c^{-1}(\overline{\mathrm{Def}}_{\beta^\sharp})^\top_{|T\mathcal{F}} + U_1^\flat\otimes\beta^\sharp + U_2\otimes\beta\,, \tag{4.5}$$

where

$$\begin{aligned} U_1 &= -\frac12\,c^{-2}\big((c\,N-\beta^\sharp)(c)\,\beta^\sharp - 2\,c^{-1}(\,\overline{\mathrm{Def}}_{\beta^\sharp}\,\beta^\sharp)^\top - \bar\nabla^\top_{N-c^{-1}\beta^\sharp}\beta^\sharp \\ &\quad + c\,\bar Z + c\,\beta(\bar Z)\,\beta^\sharp - \bar A(\beta^\sharp)^{\perp\beta}\big), \\ U_2 &= \frac12\,\big(\bar\nabla^\top_{N-c^{-1}\beta^\sharp}\,\beta^\sharp - c\,\bar Z - \bar A(\beta^\sharp)^{\perp\beta}\,\big)\,. \end{aligned} \tag{4.6}$$

Proof. By the well-known formula for Levi-Civita connection of g, using equalities $g(u,n)=0=g(v,n)$ and $g([u,v],n)=0$, we have

$$2\,g(\nabla_u\,n, v) = n(g(u,v)) + g([u,n],v) + g([v,n],u) \quad (u,v\in T\mathcal{F}). \tag{4.7}$$

One may assume $\bar\nabla^\top_X u = \bar\nabla^\top_X v = 0$ for all $X\in T_pM$ at a given point $p\in M$. Using (2.11) with $u=[u,n]$ and $v=v$, we obtain

$$\begin{aligned} n(g(u,v)) &= n(c^2(\langle u,v\rangle - \beta(u)\,\beta(v)))\\ &= n(c^2)(\langle u,v\rangle - \beta(u)\beta(v)) - c^2\beta(u)(\bar\nabla_n\,\beta)(v) - c^2(\bar\nabla_n\,\beta)(u)\beta(v),\\ g([u,n],v) &= c^2\big(\langle [u,n],v\rangle + \beta(v)\langle [u,n]),\,n\rangle\big)\\ &= -c^2\langle c\,\bar A(u) + \bar\nabla_u\,\beta^\sharp, v\rangle + c^3\langle \bar A(\beta^\sharp)+c\bar Z,\,u\rangle\,\beta(v),\\ g([v,n],u) &= c^2\big(\langle [v,n],u\rangle + \beta(u)\langle [v,n]),\,n\rangle\big)\\ &= -c^2\langle c\,\bar A(v) + \bar\nabla_v\,\beta^\sharp, u\rangle + c^3\langle \bar A(\beta^\sharp)+c\bar Z,\,v\rangle\,\beta(u). \end{aligned}$$

Substituting the above into (4.7), we find

$$\begin{aligned} &2\,g(\nabla_u\,n,v) = n(c^2)\big(\langle u,v\rangle - \beta(u)\beta(v)\big) - 2\,c^3\langle \bar A(u),v\rangle - 2\,c^2\langle \overline{\mathrm{Def}}_{\beta^\sharp}(u),v\rangle\\ &-c^2(\bar\nabla_n\,\beta)(u)\,\beta(v) - c^2\beta(u)(\bar\nabla_n\,\beta)(v) + c^3\langle \bar A(\beta^\sharp)+c\bar Z,\,u\rangle\,\beta(v)\\ &\qquad + c^3\beta(u)\langle \bar A(\beta^\sharp)+c\bar Z,\,v\rangle. \end{aligned} \tag{4.8}$$

From (4.8), assuming $g(\nabla_u\,n,v) = \langle \mathfrak{D}(u),\,v\rangle$ and using Lemma 2.3, we get

$$-2\,c^4A^g(u) \;=\; 2\,\mathfrak{D}(u) + c^{-2}\langle 2\,\mathfrak{D}(u),\,\beta^\sharp\rangle\,\beta^\sharp\,, \tag{4.9}$$

where $\mathfrak{D}: T\mathcal F\to T\mathcal F$ is a linear operator, and

$$\begin{aligned} 2\,\mathfrak{D}(u) \;&=\; n(c^2)\,(u-\beta(u)\beta^\sharp) - 2\,c^3\bar A(u) - 2\,c^2\,(\,\overline{\mathrm{Def}}_{\beta^\sharp}(u))^\top\\ &-\; c^2(\bar\nabla^\top_n\,\beta)(u)\,\beta^\sharp - c^2\beta(u)\bar\nabla^\top_n\,\beta^\sharp + c^3\langle \bar A(\beta^\sharp)+c\bar Z,\,u\rangle\,\beta^\sharp\\ &+\; c^3\beta(u)(\bar A(\beta^\sharp)+c\bar Z). \end{aligned} \tag{4.10}$$

From (4.10) we get

$$\begin{aligned} 2\,\langle \mathfrak{D}(u),\beta^\sharp\rangle \;&=\; n(c^2)\,c^2\beta(u) - 2\,c^3\langle \bar A(\beta^\sharp),u\rangle - 2\,c^2\,\langle \overline{\mathrm{Def}}_{\beta^\sharp}(\beta^\sharp),u\rangle\\ &-\; c^2(1-c^2)\,(\bar\nabla^\top_n\,\beta)(u) + c^3n(c)\beta(u) + c^3(1-c^2)\,\langle \bar A(\beta^\sharp)+c\bar Z,\,u\rangle\\ &+\; c^3\langle \bar A(\beta^\sharp)+c\bar Z,\beta^\sharp\rangle\,\beta(u). \end{aligned} \tag{4.11}$$

From (4.9) – (4.11) we obtain

$$\begin{aligned} &c\,A^g = \bar A - c^{-1}\,(N-c^{-1}\,\beta^\sharp)(c)\,I_m c^{-1}\,(\overline{\mathrm{Def}}_{\beta^\sharp})^\top_{|T\mathcal F}\\ &-\frac12\,c^{-2}\,\big((c\,N-\beta^\sharp)(c)\,\beta^\sharp - 2\,c^{-1}(\,\overline{\mathrm{Def}}_{\beta^\sharp}\,\beta^\sharp)^\top - \bar\nabla^\top_{N-c^{-1}\beta^\sharp}\,\beta^\sharp + c\,\bar Z + c\,\langle \bar Z,\,\beta^\sharp\rangle\,\beta^\sharp\\ &-\bar A(\beta^\sharp) + \langle \bar A(\beta^\sharp),\beta^\sharp\rangle\,\beta^\sharp\big)^\flat\otimes\beta^\sharp + \frac12\,\big(\bar\nabla^\top_{N-c^{-1}\beta^\sharp}\,\beta^\sharp - c\bar Z - \bar A(\beta^\sharp)\big)\otimes\beta. \end{aligned}$$

From the above the expected (4.5) – (4.6) follow. □

Corollary 4.2. *Let* $\beta(N)=0$. *If* $\|\beta\|_\alpha = \mathrm{const}$ *then on* $T\mathcal F$ *we have*

$$\begin{aligned} &c\,A^g = \bar A + c^{-1}\,(\overline{\mathrm{Def}}_{\beta^\sharp})^\top_{|T\mathcal F} + \frac12\,\big(\bar\nabla^\top_{N-c^{-1}\beta^\sharp}\,\beta^\sharp - c\bar Z - \bar A(\beta^\sharp)^{\perp\beta}\big)\otimes\beta\\ &+\frac12\,c^{-2}\,\big(2\,c^{-1}\overline{\mathrm{Def}}{}^\top_{\beta^\sharp}(\beta^\sharp) + \bar\nabla^\top_{N-c^{-1}\beta^\sharp}\,\beta^\sharp + \bar A(\beta^\sharp)^{\perp\beta}\\ &- c\,\bar Z - c\,\langle \bar Z,\,\beta^\sharp\rangle\,\beta^\sharp\big)^\flat\otimes\beta^\sharp. \end{aligned} \tag{4.12}$$

If, in particular, $\bar\nabla\beta = 0$ (*i.e.,* F *is a Berwald structure*) *then*

$$(4.13)\quad c\,A^g = \bar A - \frac12\big(\bar A(\beta^\sharp)^{\perp\beta} + c\bar Z\big)\otimes\beta + \frac12\,c^{-2}\big(\bar A(\beta^\sharp)^{\perp\beta} - c\,\bar Z - c\,\langle\bar Z,\,\beta^\sharp\rangle\,\beta^\sharp\big)^\flat\otimes\beta^\sharp.$$

4.2 The Riemann curvature of g and a

In this section we study relationship between Riemann curvature of two metrics, g and a, on a Randers space.

Proposition 4.3. *For a codimension-one foliation of* M *with Riemannian metrics* g *and* a *we have*

$$(4.14)\qquad Z = c^{-2}\bar Z - c^{-3}\,\bar\nabla^\top c + c^{-4}\,\beta(\bar Z - c^{-1}\,\bar\nabla^\top c)\,\beta^\sharp,$$

$$(4.15)\qquad C_n^\sharp = c^{-2}\bar C + c^{-4}(\beta\circ\bar C)\otimes\beta^\sharp,$$

where

$$\begin{aligned} 2\,\bar C &= \mathrm{Sym}(\beta\otimes\bar Z) + c^{-3}\big(c\,\beta(\bar Z) - 2\,\beta^\sharp(c) - n(c)\big)\big(I_m - \beta\otimes\beta^\sharp\big)\\ &- c^{-1}\mathrm{Sym}(\beta\otimes\bar\nabla^\top c) + c^{-1}(\beta^\sharp(c) + n(c))\big(I_m - 3\,\beta\otimes\beta^\sharp\big). \end{aligned}$$

We also have

$$(4.16)\quad \langle\bar\nabla_u\bar Z, v\rangle = \langle\bar\nabla_v\bar Z, u\rangle,\quad g(\nabla_u Z, v) = g(\nabla_v Z, u)\quad (u, v\in T\mathcal F),$$

$$(4.17)\quad \bar R_N = (\overline{\mathrm{Def}}_{\bar Z})^\top_{|T\mathcal F} + \bar\nabla_N\bar A - \bar A^2 - \bar Z^\flat\otimes\bar Z,\quad R^g_\nu = (\mathrm{Def}_Z)^\top_{|T\mathcal F} + \nabla_\nu A - A^2 - Z^\flat\otimes Z.$$

Proof. Extend $X\in T_p\mathcal F$ at a point $p\in M$ onto a neighborhood of p with the property $(\bar\nabla_Y X)^\top = 0$ for any $Y\in T_pM$. By the well known formula for the Levi-Civita connection, we obtain at p:

$$g(Z, X) = g([X,\nu],\nu).$$

Then, using the equalities $\nu = c^{-1}N - c^{-2}\beta^\sharp$ and $[X, fY] = X(f)Y + f[X,Y]$, we calculate

$$\begin{aligned} g([X,\nu],\nu) &= c^{-4}X(c)\,g(N,\beta^\sharp) - c^{-3}X(c)\,g(N,N)\\ &+ c^{-2}g([X,N],\,N) - c^{-3}g([X,N],\beta^\sharp). \end{aligned}$$

Note that

$$[X,N] = \bar\nabla_X N - \bar\nabla_N X = -\bar A(X) - \langle\bar\nabla_N X,\,N\rangle\,N = -\bar A(X) + \langle\bar Z,\,X\rangle\,N$$

and $N = c\nu + c^{-1}\beta^\sharp$. Then, by Lemma 2.2 and the equalities

$$\begin{aligned} g(\beta^\sharp,\beta^\sharp) &= c^2(\langle\beta^\sharp,\,\beta^\sharp\rangle - \beta(\beta^\sharp)^2) = c^4(1-c^2),\\ g(N,\beta^\sharp) &= g(c\nu + c^{-1}\beta^\sharp,\,\beta^\sharp) = c^{-1}g(\beta^\sharp,\beta^\sharp) = c^3(1-c^2),\\ g(N,N) &= g(c\nu + c^{-1}\beta^\sharp,\,c\nu + c^{-1}\beta^\sharp) = c^2 + c^{-2}g(\beta^\sharp,\beta^\sharp) = c^2(2-c^2), \end{aligned}$$

we obtain

$$\begin{aligned} g([X,N],\,N) &= -\langle\bar A(\beta^\sharp),\,X\rangle + \langle\bar Z, X\rangle\,g(N,N) = c^2\langle(2-c^2)\bar Z - c\bar A(\beta^\sharp),\ X\rangle,\\ g([X,N],\beta^\sharp) &= -\langle\bar A(\beta^\sharp),\,X\rangle + \langle\bar Z, X\rangle\,g(N,\beta^\sharp) = c^3\langle(1-c^2)\bar Z - c\bar A(\beta^\sharp),\ X\rangle. \end{aligned}$$

Hence,

$$g(Z,X) \;=\; -c^{-1}X(c) + \langle \bar Z, X\rangle = \langle \bar Z - c^{-1}\bar\nabla c,\ X\rangle.$$

By Lemma 2.3, we get (4.14). From (4.2) – (4.3), (4.14) and a bit of help from Maple program we find

$$\begin{aligned}
2\,C_n(u,v,Z) \;&=\; \langle \bar Z, u\rangle\,\beta(v) + \langle \bar Z, v\rangle\,\beta(u)\\
&+\; c^{-3}(c\,\beta(\bar Z) - 2\,\beta^\sharp(c) - n(c))\big(\langle u,v\rangle - \beta(u)\beta(v)\big)\\
&-\; c^{-1}(u(c)\,\beta(v) + v(c)\,\beta(u)) + c^{-1}(\beta^\sharp(c) + n(c))\big(\langle u,v\rangle - 3\,\beta(u)\,\beta(v)\big).
\end{aligned}$$

Using $g(C^\sharp_n(u),v) = \langle \bar C(u), v\rangle$, where $C^\sharp_n$ is g-dual to $C_n(\cdot,\cdot,\nabla_n\, n)$, and

$$\begin{aligned}
2\,\bar C(u) \;&=\; \langle \bar Z, u\rangle\,\beta^\sharp + \beta(u)\bar Z + c^{-3}(c\,\beta(\bar Z) - 2\,\beta^\sharp(c) - n(c))\big(u - \beta(u)\beta^\sharp\big)\\
&-\; c^{-1}(u(c)\,\beta^\sharp + \beta(u)\,\bar\nabla^\top c) + c^{-1}(\beta^\sharp(c) + n(c))\big(u - 3\,\beta(u)\,\beta^\sharp\big),
\end{aligned}$$

we apply Lemma 2.3 to get (4.15).

We shall prove (4.16) and (4.17) for a. It is sufficient to show that

$$\text{(4.18)}\quad \langle \bar R(u,N)N, v\rangle = \langle (\bar\nabla_N\,\bar A - \bar A^2)(u), v\rangle - \langle \bar Z, u\rangle\langle \bar Z, v\rangle + \langle \bar\nabla_u\,\bar Z, v\rangle, \quad u,v\in T\mathcal{F}.$$

Since the left hand side of (4.18) is symmetric, we obtain $\langle \bar\nabla_u \bar Z, v\rangle = \langle \bar\nabla_v \bar Z, u\rangle$, see $(4.17)_1$ and $(4.16)_1$. Indeed,

$$\begin{aligned}
&\langle \bar R(u,N)N, v\rangle = \langle \bar\nabla_u\bar\nabla_N N, v\rangle - \langle \bar\nabla_N\bar\nabla_u N, v\rangle - \langle \bar\nabla_{\bar\nabla_u N - \bar\nabla_N u} N, v\rangle\\
&= \langle \bar\nabla_u \bar Z, v\rangle + \langle \bar\nabla_N(\bar A(u)), v\rangle - \langle \bar A^{\,2}(u), v\rangle + \langle \bar\nabla_{\langle \bar\nabla_N\, u, N\rangle N}\, N, v\rangle - \langle \bar A(\bar\nabla^\top_N\, u), v\rangle\\
&= \langle (\bar\nabla_N\,\bar A - \bar A^{\,2})(u), v\rangle - \langle \bar Z, u\rangle\langle \bar Z, v\rangle + \langle \bar\nabla_u\,\bar Z, v\rangle,
\end{aligned}$$

that completes the proof of (4.18). The proof of $(4.16)_2$ and $(4.17)_2$ (for the metric g) is similar. □

By (4.15), the equality $C^\sharp_n = 0$ is independent of the condition $\bar\nabla\beta = 0$. Moreover, we have the following.

Corollary 4.4. Let $m > 3$ and $c = \text{const}$. Then $C^\sharp_n = 0$ if and only if $\bar Z = 0$.

Proof. By our assumptions, $\bar C = \frac12\,\mathrm{Sym}(\beta\otimes\bar Z) + \frac12\,c^{-2}\,\beta(\bar Z)\big(I_m - \beta\otimes\beta^\sharp\big)$. Hence, $C^\sharp_n = 0$ reads

$$\beta(\bar Z) I_m = \beta(\bar Z)\,\beta\otimes\beta^\sharp - c^2\,\mathrm{Sym}(\beta\otimes\bar Z) - 2\,(\beta\circ\bar C)\otimes\beta^\sharp.$$

Since the matrix $\beta(\bar Z)I_m$ is conformal, while the matrix in the right hand side of above equality has the form $\omega\otimes\beta^\sharp - c^2\bar Z^{\,\perp\beta}\otimes\beta$ and rank ≤ 3, for $m > 3$ we obtain

$$\beta(\bar Z) = 0, \quad \mathrm{Sym}(\beta\otimes\bar Z) + 2\,c^{-2}(\beta\circ\bar C)\otimes\beta^\sharp = 0.$$

By the first condition, $\bar Z\perp\beta^\sharp$; thus, the second condition yields $\bar Z = 0$ (that is, $\mathcal{F}$ is a Riemannian foliation for the metric a) and $\bar C = 0$. The converse claim follows directly from (4.15) and the definition of $\bar C$. □

Remark 4.1. In [15] and [5] one may find coordinate presentations of R_y through $\bar R_y$ for all $y\in TM$. For example, if $\bar\nabla\beta = 0$ (i.e., F is a Berwald structure) then $R_y(u) = \bar R_y(u)$ for all u. Alternative formulas with relationship between R_ν and $\bar R_\nu$ follow from (4.17), where A^g and Z are expressed using $\bar A$ and $\bar Z$ given in Propositions 4.1 and 4.3.

4.3 Around the Reeb and Brito-Langevin-Rosenberg formula

Based on (3.13) and (3.21), one may produce a sequence of similar formulae for Randers spaces. We will discuss first two of them (i.e., $k = 1, 2$).

Remark 4.2. In [10], G. Reeb proved that the total mean curvature of the leaves of a codimension-one foliation on a closed Riemannian manifold equals zero. Note that $\operatorname{Tr}\overline{\operatorname{Def}}^{\top}_{\beta^\sharp} = \overline{\operatorname{div}}\,\beta^\sharp + \beta(\bar Z)$, where $\bar Z = \bar\nabla_N N$ is the curvature vector of N-curves for the metric a. Using notations of Appendix, we find from (4.6),

$$\beta(U_1) = -\frac{2-c^2}{2\,c}\,N(c) - \frac{1}{2}\,\beta^\sharp(c) - \frac{2-c^2}{2\,c}\,\beta(\bar Z), \quad \beta(U_2) = -\frac{1}{2}\,(c\,N - \beta^\sharp)(c) - \frac{1}{2}\,c\,\beta(\bar Z).$$

Hence,

$$\beta(U_1) + \beta(U_2) = -c^{-1}(N(c) + \beta(\bar Z)).$$

Tracing (4.5), we get

$$c\,\sigma_1(A^g) = \sigma_1(\bar A) - (m+1)\,c^{-1}N(c) + m\,c^{-2}\beta^\sharp(c) + c^{-1}\,\overline{\operatorname{div}}\,\beta^\sharp.$$

The volume forms of g and a obey $\mathrm{d}V_g = c^{m+2}\,\mathrm{d}V_a$, see (4.4). Using the Reeb formula for metric g,

$$\int_M \sigma_1(A^g)\,\mathrm{d}V_g = 0,$$

the equality $\overline{\operatorname{div}}(c^m\beta^\sharp) = c^m\,\overline{\operatorname{div}}\,\beta^\sharp + \beta^\sharp(c^m)$ and the Divergence Theorem, we get

$$\int_M \big(c^{m+1}\sigma_1(\bar A) - N(c^{m+1})\big)\,\mathrm{d}V_a = 0\,, \tag{4.19}$$

which for $\beta = 0$ is the Reeb formula for metric a. Remark that (4.19) is a particular case of a general formula for any $f \in C^2(M)$, see [12, Lemma 2.5]:

$$\int_M (f\,\sigma_1(\bar A) - N(f))\,\mathrm{d}V_a = 0.$$

The next results concern Brito-Langevin-Rosenberg type formulas for foliated Randers spaces.

The *Newton transformations* $T_k(A)$ $(0 \le k \le m)$ of an $m \times m$ matrix A (see [12]) are defined either inductively by $T_0(A) = I_m$, $T_k(A) = \sigma_k(A)I_m - A\,T_{k-1}(A)$ $(k \ge 1)$ or explicitly as

$$T_k(A) = \sigma_k(A)I_m - \sigma_{k-1}(A)\,A + \ldots + (-1)^k A^k, \quad 0 \le k \le m,$$

and we have $T_k(\lambda\,A) = \lambda^k\,T_k(A)$ for $\lambda \ne 0$. Observe that if a rank-one matrix $A := U \otimes \beta$ (and similarly for $A := \omega \otimes \beta^\sharp$) has zero trace, i.e., $\beta(U) = 0$, then

$$A^2 = U(\beta^\sharp)^t \cdot U(\beta^\sharp)^t = U\beta(U)\,(\beta^\sharp)^t = \beta(U)\,A = 0.$$

Note that for $c = \text{const}$ we have, see (4.15), $C^\sharp_n = c^{-2}\bar C + c^{-4}(\beta \circ \bar C) \otimes \beta^\sharp$, where $C^\sharp_n = c^2 C^\sharp_\nu$ and

$$2\,\bar C = \operatorname{Sym}(\beta \otimes \bar Z) + c^{-2}\beta(\bar Z)(I_m - \beta \otimes \beta^\sharp).$$

Theorem 4.5. *Let $(M^{m+1},\alpha+\beta)$ be a codimension-one foliated closed Randers space with constant sectional curvature $\bar K$ of a. If a nonzero vector field $\beta^\sharp\in\Gamma(T\mathcal{F})$ obeys $\bar\nabla\beta=0$, then $\bar K=0$ and for $1\le k\le m$ we have*

$$\begin{aligned}\int_M \Big(\sum\nolimits_{j>0}\sigma_{k-j,j}(\bar A, c\,C^\sharp_\nu)+\langle T_{k-1}(\bar A+c\,C^\sharp_\nu)(\beta^\sharp),\,U_1\rangle\\ +\big\langle T_{k-1}(\bar A+c\,C^\sharp_\nu+U_1^\flat\otimes\beta^\sharp)(U_2),\,\beta^\sharp\big\rangle\Big)\,\mathrm{d}V_a=0,\end{aligned}\tag{4.20}$$

where $U_1=\frac12\,c^{-2}(\bar A(\beta^\sharp)-c\bar Z)$, $U_2=-\frac12(\bar A(\beta^\sharp)+c\bar Z)$. Moreover, if $m>3$ and $\bar Z=0$ then

$$\int_M \big\langle\big(c^{-2}T_{k-1}(\bar A)-T_{k-1}(\bar A+\frac12\,c^{-2}\bar A(\beta^\sharp)^\flat\otimes\beta^\sharp)\big)(\bar A(\beta^\sharp)),\ \beta^\sharp\big\rangle\,\mathrm{d}V_a=0.\tag{4.21}$$

Proof. By our assumptions, $c=\mathrm{const}$ and $\bar R(x,y)z=\bar K(\,\langle y,z\rangle\,x-\langle x,z\rangle\,y\,)$. Hence, on $T\mathcal{F}$

$$\bar R_N=\bar K I_m,\quad \bar R_{\beta^\sharp}=(1-c^2)\bar K I_m,\quad \bar R(\cdot,N)\beta^\sharp=0.$$

If $\bar\nabla\beta=0$ then $\bar R(U,\beta^\sharp,\beta^\sharp,U)=0$ and $\bar K(U\wedge\beta^\sharp)=0$ for all $U\perp\beta^\sharp$; hence, in our case, $\bar K=0$. By Remark 4.1, $R_y=\bar R_y$ for all $y\in TM_0$; hence, $R_y=0$. Since $\bar\nabla\beta^\sharp=0$, we obtain $\beta(\bar Z)=0$ and $\langle\bar A(\beta^\sharp),\beta^\sharp\rangle=0$:

$$\begin{aligned}\langle\beta^\sharp,\,\bar Z\rangle&=\langle\beta^\sharp,\,\bar\nabla_N N\rangle=-\langle\bar\nabla_N\,\beta^\sharp,\,N\rangle=0,\\ \langle\bar A(\beta^\sharp),\beta^\sharp\rangle&=-\langle\beta^\sharp,\,\bar\nabla_{\beta^\sharp}\,N\rangle=\langle\bar\nabla_{\beta^\sharp}\,\beta^\sharp,\,N\rangle=0.\end{aligned}$$

By (3.9) and Corollary 4.2,

$$c\,A=c\,A^g+c\,C^\sharp_\nu=\bar A+c\,C^\sharp_\nu+A_1+A_2,$$

where $A_1=U_1^\flat\otimes\beta^\sharp$ and $A_2=U_2\otimes\beta$ are rank ≤ 1 matrices (since $\langle U_i,\beta^\sharp\rangle=0$). By Corollary 5.5 of Appendix, we have

$$\begin{aligned}c^k\sigma_k(A)&=\sigma_k(\bar A)+\sum\nolimits_{j>0}\sigma_{k-j,j}(\bar A,c\,C^\sharp_\nu)+U_1\big(T_{k-1}(\bar A+c\,C^\sharp_\nu)(\beta^\sharp)\big)\\ &+\ \beta\big(T_{k-1}(\bar A+c\,C^\sharp_\nu+A_1)(U_2)\big).\end{aligned}\tag{4.22}$$

Recall that $\mathrm{d}V_F=c^{m+2}\,\mathrm{d}V_a$, see (4.4). Comparing (3.21) (when $K=0$) with

$$\int_M\sigma_k(\bar A_p)\,\mathrm{d}V_a=0,$$

we find (4.20). By Corollary 4.4, if $m>3$, $\bar Z=0$ then $C^\sharp_\nu=0$; hence, (4.20) yields (4.21). □

Example 4.3. For $k=1$, (4.20) yields the Reeb type formula

$$\int_M\sigma_1(C^\sharp_\nu)\,\mathrm{d}V_a=0.$$

Corollary 4.6. *Let $(M^{m+1},\alpha+\beta)$, $m>3$, be a codimension-one foliated closed Randers space with constant sectional curvature $\bar K$ of a. If $\bar Z=0$ and a nonzero vector field $\beta^\sharp\in\Gamma(T\mathcal{F})$ obeys $\bar\nabla\beta=0$ then $\bar K=0$ and $\bar A(\beta^\sharp)=0$ at any point of M. If, in addition, $\mathcal{F}$ is totally umbilical ($\bar A=\bar H\cdot I_m$) then $\mathcal{F}$ is totally geodesic.*

Proof. For $k=2$, the integrand in (4.21) reduces to $\frac{c^2-1}{4\,c^2}\,\|\bar A(\beta^\sharp)\|^2$. Thus, when $c\ne 1$, the claim follows.

Nevertheless, we will give alternative proof with use of integral formula (3.15). Our Randers space $(M,\alpha+\beta)$ is now Berwald. For the rank 1 matrices $A_1=U_1^\flat\otimes\beta^\sharp$ and $A_2=U_2\otimes\beta$, where $U_1=\frac12\,c^{-2}\bar A(\beta^\sharp)$ and $U_2=-\frac12\bar A(\beta^\sharp)$ and $\langle\bar A(\beta^\sharp),\beta^\sharp\rangle=0$, see (4.13) with $\bar Z=0$, we have

$$
\begin{aligned}
\operatorname{Tr}(A_1A_2) &= \langle U_1,U_2\rangle\,\beta(\beta^\sharp)=\frac{c^2-1}{4\,c^2}\,\|\bar A(\beta^\sharp)\|^2_\alpha,\\
\operatorname{Tr}(\bar AA_1) &= \langle U_1,\bar A(\beta^\sharp)\rangle=\frac{1}{2\,c^2}\,\|\bar A(\beta^\sharp)\|^2_\alpha,\\
\operatorname{Tr}(\bar AA_2) &= \langle U_2,\bar A(\beta^\sharp)\rangle=-\frac12\,\|\bar A(\beta^\sharp)\|^2_\alpha.
\end{aligned}
$$

Thus, $\operatorname{Tr}(A_1A_2+\bar AA_1+\bar AA_2)=\frac{1-c^2}{4\,c^2}\|\bar A(\beta^\sharp)\|^2$. By the identity for square matrices

$$
\begin{aligned}
\sigma_2(\textstyle\sum_i A_i) &= \frac12\,\operatorname{Tr}^2(\textstyle\sum_i A_i)-\frac12\,\operatorname{Tr}((\textstyle\sum_i A_i)^2)\\
&= \textstyle\sum_i\sigma_2(A_i)+\sum_{i<j}\big((\operatorname{Tr}A_i)(\operatorname{Tr}A_j)-\operatorname{Tr}(A_iA_j)\big)\,,
\end{aligned}
$$

and $\sigma_2(A_1)=\sigma_2(A_2)=0$, by the above and since $c\,A=c\,A^g=\bar A+A_1+A_2$, we get

$$
c^2\sigma_2(A) \;=\; c^2\sigma_2(A^g)=\sigma_2(\bar A)+\frac14\,(c^{-2}-1)\,\|\bar A(\beta^\sharp)\|^2_\alpha.
$$

From the integral formulae, (3.20), for F and for Riemannian metric a,

$$
\int_M\sigma_2(\bar A)\,{\rm d}V_a=0,\qquad \int_M\sigma_2(A)\,{\rm d}V_F=0,
$$

where the volume forms are related by ${\rm d}V_F=c^{m+2}{\rm d}V_a$, see (2.6), we find that $(c^{-2}-1)\int_M\|\bar A(\beta^\sharp)\|^2_\alpha\,{\rm d}V_a=0$. Since $c\ne1$ (for $\beta\ne0$), we obtain $\bar A(\beta^\sharp)=0$. □

Similar integral formulae exist for codimension one totally umbilical (i.e., $\bar A=\bar H I_m$, where $\bar H=\frac1m\operatorname{Tr}\bar A$) and totally geodesic foliations. Notice that non-flat closed Riemannian manifolds of constant curvature do not admit such foliations.

Corollary 4.7. *Let $\mathcal F$ be a codimension-one totally umbilical* (*for the metric a*) *foliation of a closed Randers space $(M^{m+1},\alpha+\beta)$ with constant sectional curvature $\bar K$ of a. If a nonzero vector field $\beta^\sharp\in\Gamma(T\mathcal F)$ obeys $\bar\nabla\beta^\sharp=0$ then $\bar K=0$, $\mathcal F$ is totally geodesic and for $1\le k\le m$* (*for $k=1$, see also Example 4.3*) *we have*

$$
\begin{aligned}
&\int_M\Big(c^k\sigma_k(C^\sharp_\nu)-\frac12\,c^{-1}\,\langle T_{k-1}(c\,C^\sharp_\nu)(\beta^\sharp),\,\bar Z\rangle\\
&-\frac{c}{2}\,\big\langle T_{k-1}(c\,C^\sharp_\nu-\frac12\,c^{-1}\bar Z^\flat\otimes\beta^\sharp)(\bar Z),\,\beta^\sharp\big\rangle\Big)\,{\rm d}V_a=0.
\end{aligned}\tag{4.23}
$$

Proof. Since $\langle\bar A(\beta^\sharp),\beta^\sharp\rangle=0$ (see the proof of Theorem 4.5), we obtain $\bar H=0$. Thus, (4.23) follows from (4.20) with $\bar A=0$ and $\beta(\bar Z)=0$. □

Remark 4.4. In results of this section, a closed manifold can be replaced by a complete manifold of finite volume with bounded geometry, see conditions (3.17).

5 Appendix: Invariants of a set of matrices

Here, we collect the properties of the invariants $\sigma_\lambda(A_1,\ldots,A_k)$ of real matrices A_i that generalize the elementary symmetric functions of a single symmetric matrix A. Let S_k be the group of all permutations of k elements. Given arbitrary quadratic $m\times m$ real matrices $A_1,\ldots A_k$ and the unit matrix I_m, one can consider the determinant $\det(I_m+t_1A_1+\ldots+t_kA_k)$ and express it as a polynomial of real variables $\mathbf{t}=(t_1,\ldots t_k)$. Given $\lambda=(\lambda_1,\ldots\lambda_k)$, a sequence of nonnegative integers with $|\lambda|:=\lambda_1+\ldots+\lambda_k\le m$, we shall denote by $\sigma_\lambda(A_1,\ldots,A_k)$ its coefficient at $\mathbf{t}^\lambda=t_1^{\lambda_1}\cdot\ldots t_k^{\lambda_k}$:

$$\det(I_m+t_1A_1+\ldots+t_kA_k)=\sum\nolimits_{|\lambda|\le m}\sigma_\lambda(A_1,\ldots A_k)\,\mathbf{t}^\lambda. \tag{5.1}$$

Evidently, the quantities σ_λ are invariants of conjugation by $GL(m)$-matrices:

$$\sigma_\lambda(A_1,\ldots A_k)=\sigma_\lambda(QA_1Q^{-1},\ldots QA_kQ^{-1}) \tag{5.2}$$

for all A_i's, λ's and nonsingular $m\times m$ matrices Q. Certainly, $\sigma_i(A)$ (for a single symmetric matrix A) coincides with the i-th elementary symmetric polynomial of the eigenvalues $\{k_j\}$ of A.

In the next lemma, we collect properties of these invariants.

Lemma 5.1 (see [13]). *For any $\lambda=(\lambda_1,\ldots\lambda_k)$ and any $m\times m$ matrices A_i, A and B one has*

(I) $\sigma_\lambda(0,A_2,\ldots A_k)=0$ *if* $\lambda_1>0$ *and* $\sigma_{0,\hat\lambda}(A_1,\ldots A_k)=\sigma_{\hat\lambda}(A_2,\ldots A_k)$ *where* $\hat\lambda=(\lambda_2,\ldots\lambda_k)$,

(II) $\sigma_\lambda(A_{s(1)},\ldots A_{s(k)})=\sigma_{\lambda\circ s}(A_1,\ldots A_k)$, *where* $s\in S_k$ *and* $\lambda\circ s=(\lambda_{s(1)},\ldots\lambda_{s(k)})$,

(III) $\sigma_\lambda(I_m,A_2,\ldots A_k)=\binom{m-|\hat\lambda|}{\lambda_1}\sigma_{\hat\lambda}(A_2,\ldots A_k)$,

(IV) $\sigma_{\lambda_1,\lambda_2,\hat\lambda}(A,A,A_3,\ldots A_k)=\binom{\lambda_1+\lambda_2}{\lambda_1}\,\sigma_{\lambda_1+\lambda_2,\hat\lambda}(A,A_3,\ldots A_k)$,

(V) $\sigma_{1,\hat\lambda}(A+B,A_2,\ldots A_k)=\sigma_{1,\hat\lambda}(A,A_2,\ldots A_k)+\sigma_{1,\hat\lambda}(B,A_2,\ldots A_k)$ *and*
$\sigma_\lambda(aA_1,A_2,\ldots A_k)=a^{\lambda_1}\sigma_\lambda(A_1,A_2,\ldots A_k)$ *if* $a\in\mathbb{R}\setminus\{0\}$.

The invariants defined above can be used in calculation of the determinant of a matrix $B(t)$ expressed as a power series $B(t)=\sum_{i=0}^\infty t^iB_i$. Indeed, if one wants to express $\det(B(t))$ as a power series in t, then the coefficient at t^j depends only on the part $\sum_{i\le j}t^iB_i$ of $B(t)$.

Lemma 5.2 ([13]). *If $B(t)$, $t\in\mathbb{R}$, is the $m\times m$ matrix given by $B(t)=\sum_{i=0}^\infty t^iB_i$, $B_0=I_m$ then*

$$\det(B(t))=1+\sum\nolimits_{k=1}^\infty\big(\sum\nolimits_{\lambda,\|\lambda\|=k}\sigma_\lambda(B_1,\ldots B_k)\big)\,t^k, \tag{5.3}$$

where $\|\lambda\|=\lambda_1+2\lambda_2+\ldots+k\lambda_k$ *for* $\lambda=(\lambda_1,\ldots\lambda_k)$. $\square$

Since $\det:\mathcal{M}(m)\to\mathbb{R}$, $\mathcal{M}(m)\approx\mathbb{R}^{m^2}$ being the space of all $m\times m$-matrices, is a polynomial function, the series in (5.3) is convergent for all $t\in(-r_0,r_0)$, where $r_0=1/\limsup_{k\to\infty}\|B_k\|^{1/k}$ is the radius of convergence of the series $B(t)$.

By the First Fundamental Theorem of Matrix Invariants, see [6], all the invariants σ_λ can be expressed in terms of the traces of the matrices involved and their products.

Lemma 5.3 ([13]). *For arbitrary matrices B, C and $k,l>0$ we have*

$$\sigma_{k,l}(B,C)=\sigma_k(B)\,\sigma_l(C)-\sum\nolimits_{i=1}^{\min(k,l)}\sigma_{k-i,l-i,i}(B,C,BC).$$

In particular, for $l=1$, it follows that

$$\sigma_{k,1}(B,C) \;=\; \sum\nolimits_{i=0}^{k}(-1)^i\sigma_{k-i}(B)\,\mathrm{Tr}(B^iC)=\mathrm{Tr}(T_k(B)C). \tag{5.4}$$

Lemma 5.4. *Let A,C be $m\times m$ matrices and* $\mathrm{rank}\,A=1$. *Then*

$$\sigma_k(C+A)=\sigma_k(C)+\mathrm{Tr}(T_{k-1}(C)A). \tag{5.5}$$

Proof. There exists a nonsingular matrix Q such that $\tilde A=QAQ^{-1}$ has one nonzero element, $\tilde a_{1i}\neq 0$ for some i (the simplest rank one matrix). By (5.2), $\sigma_{k,l}(\tilde C,\tilde A)=\sigma_{k,l}(C,A)$ where $\tilde C=QCQ^{-1}$. By Laplace's formula (which expresses the determinant of a matrix in terms of its minors), $\det(I_m+t\tilde C+s\tilde A)$ is a linear function in $s\in\mathbb{R}$; hence, see (5.1), $\sigma_{k,l}(\tilde C,\tilde A)=0$ for $l>1$. By the above, $\sigma_{k,l}(C,A)=0$ for $l>1$ and all k. Using the identity, see [13],

$$\sigma_k(C_1+C_2)=\sum\nolimits_{i=0}^{k}\sigma_{\,k-i,i}(C_1,C_2), \tag{5.6}$$

we find that

$$\sigma_k(C+A) \;=\; \sigma_k(C)+\sigma_{k-1,1}(C,A).$$

By (5.4), $\sigma_{k-1,1}(C,A)=\mathrm{Tr}(T_{k-1}(C)A)$ and (5.5) follows. □

Corollary 5.5. *Let C,D,A_i be $m\times m$ matrices and* $\mathrm{rank}\,A_i=1$ $(1\le i\le s)$. *Then*

$$\begin{aligned}\sigma_k(C+D+A_1+\dots A_s)&=\sigma_k(C)+\sum\nolimits_{j>0}\sigma_{k-j,j}(C,D)\\ &+\ \mathrm{Tr}(T_{k-1}(C+D)A_1)+\ldots+\mathrm{Tr}(T_{k-1}(C+D+A_1+\ldots+A_{s-1})A_s).\end{aligned} \tag{5.7}$$

Proof. This follows from Lemma 5.4 and (5.4). For $s=1$, we obtain

$$\begin{aligned}\sigma_k(C+D+A_1)&\overset{(5.5)}{=}\sigma_k(C+D)+\mathrm{Tr}(T_{k-1}(C+D)A_1)\\ &\overset{(5.6)}{=}\sigma_k(C)+\sum\nolimits_{j>0}\sigma_{k-j,j}(C,D)+\mathrm{Tr}(T_{k-1}(C+D)A_1).\end{aligned}$$

Then, by induction for s, (5.7) follows. □

Let C_i and P_i be m-vectors (columns) and I_m the identity m-matrix and $1\le i\le j\le m$. Note that $C_iP_j^t$ are $m\times m$-matrices of rank 1 with

$$\sigma_1(C_iP_j^t)=C_i^tP_j=P_j^tC_i,\quad \sigma_2(C_iP_j^t)=0,$$
$$(I_m+C_iP_j^t)^{-1}=I_m-(1+C_i^tP_j)^{-1}\,C_iP_j^t.$$

Lemma 5.6. *We have* $\det(I_m+\sum_{i=1}^k C_iP_i^t)=1+\det(\{C_i^tP_j\}_{1\le i,j\le k})$. *For example,*

$$\det(I_m+C_1P_1^t)=1+C_1^tP_1\,,$$
$$\det(I_m+C_1P_1^t+C_2P_2^t)=1+C_1^tP_1+C_2^tP_2+C_1^tP_1\cdot C_2^tP_2-C_1^tP_2\cdot C_2^tP_1\,,$$

and so on.

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