On the Brill-Noether theory of curves in a weighted projective plane

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Abstract. We study the gonality and the existence of low degree pencils on curves with a model on a weighted projective plane, when their singularities are only ordinary nodes or ordinary cusps and they are general in the weighted projective plane.

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1 Introduction

In this paper we consider the first steps of the Brill-Noether theory of curves on a weighted projective plane ([7], [8], [1]) (a very classical topic, but as far as we know the results of this note are new). See [2], [3], [4], [5], [6], [9] for smooth and singular plane curves.

Fix positive integers a, b, c and let $\mathbb{P} := \mathbb{P}(a, b, c)$ denote the weighted projective space with weights a, b, c. Up to isomorphisms of the ambient weighted projective plane we may assume that any 2 of the integer a, b, c are coprimes ([1, Proposition 3C.5], [7, Proposition 1.3]). We may assume $a \le b \le c$. Since $(a, b) = (b, c) = (a, b) = 1$, we are in one of the following cases:

1. $a = b = c = 1$;
2. $a = b = 1$, $c > 1$;
3. $a < b < c$, $(a, b) = 1$, $(a, c) = 1$, $(b, c) = 1$.

In the first case we have $\mathbb{P} \cong \mathbb{P}^2$. In the second case $\mathbb{P}$ is embedded as a cone over a rational normal curve of $\mathbb{P}^c$ and the blowing up of the vertex of the cone gives the Hirzebruch surface F_c ([1, page 124], [8, 1.2.3]). In this case it seems easier to work directly on F_c (the case $b = 1$ of Theorem 1.2 is true by [10]). Hence from now on we assume $a < b < c$ and $(a, b) = (a, c) = (b, c) = 1$.

We fix variables x_1, x_2, x_3 and give weight a to x_1, b to x_2 and c to x_3. For all integers $t \ge 0$ let $K[x_1, x_2, x_3]_{a,b,c;t}$ be the linear subspace of $K[x_1, x_2, x_3]$ generated

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by the monomials $x_1^{a_1}x_2^{a_2}x_3^{a_3}$ with $a_i \ge 0$ for all i and $aa_1 + ba_2 + ca_3 = t$, i.e. the monomials with weight t. We recall that $\mathbb{P}$ has only quotient singularities (if $a = 1 < b$, $\mathrm{Sing}(\mathbb{P}) = \{(0:1:0),(0:0:1)\}$, if $a > 1$, then $\mathrm{Sing}(\mathbb{P}) = \{(1:0:0),(0:1:0),(0:0:1)\}$, that the set of all rational equivalence classes of Weil divisors is a free abelian group of rank 1 ([1, Corollary 5.8]), that $\mathcal{O}_{\mathbb{P}}(t)$, $t \in \mathbb{Z}$, is the set of all rank one reflexive sheaves on $\mathbb{P}$, that $h^1(\mathcal{O}_{\mathbb{P}}(t)) = 0$ for all $t \in \mathbb{Z}$, $h^0(\mathcal{O}_{\mathbb{P}}(t)) = K[x_1,x_2,x_3]_{a,b,c;t}$ for all $t \ge 0$, that $\mathcal{O}_{\mathbb{P}}(t)$ is locally free if and only if $t \equiv 0 \pmod{abc}$. The line bundle $\mathcal{O}_{\mathbb{P}}(abc)$ is very ample ([1, Remark 3]). Hence for all $t > 0$ a general element of $|\mathcal{O}_{\mathbb{P}}(tabc)|$ is a smooth and connected curve. Fix a positive integer d and take $C \in |\mathcal{O}_{\mathbb{P}}(dabc)|$ such that C is smooth. Since C is a Cartier divisor of $\mathbb{P}$ and C is smooth, we have $C \cap \mathrm{Sing}(\mathbb{P}) = \emptyset$. Hence each $\mathcal{O}_C(t)$, $t \in \mathbb{Z}$, is a line bundle. We have $\mathcal{O}_{\mathbb{P}}(1) \cdot \mathcal{O}_{\mathbb{P}}(1) = \frac{1}{abc}$ in the rational Chow ring of $\mathbb{P}$ (use [11, Corollary A.2] or that the covering map $\mathbb{P}^2 \to \mathbb{P}$ is the quotient by the group $\mu_a \times \mu_b \times \mu_c$ and hence it has degree abc). Since $\omega_{\mathbb{P}} \cong \mathcal{O}_{\mathbb{P}}(-a-b-c)$ ([1, Corollary 6B.8], [7, Theorem 5.2], [8, 3.3.4 and 3.5.2]), the adjunction formula gives $\omega_C \cong \mathcal{O}_C(dabc-a-b-c)$ ([1, Corollary 6.B9], [8, 3.5.2]) Hence C has genus $1 + d(dabc - a - b - c)/2$. Since $h^1(\mathcal{O}_{\mathbb{P}}(t)) = 0$ for all t, for each integer $w \ge 0$ the restriction map $\rho_w : H^0(\mathcal{O}_{\mathbb{P}}(w)) \to H^0(\mathcal{O}_C(w))$ is surjective. Hence $h^0(\mathcal{O}_C(t)) = \dim(K[x_1,x_2,x_3]_{a,b,c;t})$ for all $t < dabc$. In particular we have $h^0(\mathcal{O}_C(ab)) = 2$. Hence C has gonality at most $\deg(\mathcal{O}_C(ab)) = dab$ (use again that $\mathcal{O}_{\mathbb{P}}(1) \cdot \mathcal{O}_{\mathbb{P}}(1) = \frac{1}{abc}$). The line bundle $\mathcal{O}_C(ab)$ is spanned, because $(0:0:1)$ is the only base point of $|\mathcal{O}_{\mathbb{P}}(1)|$ and $(0:0:1) \notin C$.

Our first result is non-trivial only if $c \gg ab$.

Theorem 1.1. *Let $C \in |\mathcal{O}_{\mathbb{P}}(dabc)|$ be a smooth curve. Assume $dabc - a - b - c > 0$ and $(a,b,d) \neq (1,2,1)$. Let $w\colon C \to \mathbb{P}^1$ be the morphism induced by $|\mathcal{O}_C(ab)|$. Let z be any positive integer such that $(z-2)ab \le dabc - a - b - c$. Then there is no degree z morphism $u\colon C \to \mathbb{P}^1$ such that u is not partially composed with w, i.e. such that the morphism $(w,u)\colon C \to \mathbb{P}^1 \times \mathbb{P}^1$ is birational onto its image.*

The condition " $dabc - a - b - c > 0$ " is equivalent to assuming that C has genus ≥ 2. The result is sharp, in the sense that it fails (just by 1) in the omitted case $(a,b,d) = (1,2,1)$ (see Remark 2.1).

In the case $a = 1$, we prove the following result.

Theorem 1.2. *Assume $a = 1 < b$. Let $C \in |\mathcal{O}_{\mathbb{P}}(dac)|$ be a smooth curve. Then C has gonality db and $\mathcal{O}_C(b)$ is the unique line bundle L on C such that $h^0(L) \ge 2$ and $\deg(L) \le db$.*

In section 3 we consider the case of singular curves. We consider both the spanned line bundles of minimal degree on the singular curve and the case of the normalization of an integral curve.

2 Proof of Theorems 1.1 and 1.2

Remark 2.1. Let $C \in |\mathcal{O}_{\mathbb{P}}(dabc)|$ be a smooth curve of genus $g \ge 2$. Assume $(a,b,c) = (1,2,1)$ (the case excluded in the statement of Theorem 1.1). Since $b = 2$ and $(b,c) = 1$, c is odd. We have $g = 1 + (c-3)/2$. The spanned line bundle $\mathcal{O}_C(2)$ has degree 2 and hence C is hyperelliptic. There is a degree z spanned line bundle whose associated morphism is not composed with the hyperelliptic involution if and only if $z \geq g+1 = 2+(c-3)/2$.

Proof of Theorem 1.1: Assume the existence of such a morphism and take z minimal for which it exists. Set $R := u^*(\mathcal{O}_{\mathbb{P}^1}(1))$. R is a spanned line bundle of degree z and in particular $h^0(R) \geq 2$. Let $g = 1 + d(dabc-a-b-c)/2$ be the genus of C.

First assume $z > g$, i.e. $z-2 \geq g-1$. We get $d(dabc-a-b-c)ab/2 \leq dabc-a-b-c$. Since $dabc-a-b-c > 0$, we get $d = 1$ and $ab = 2$, i.e. $d = 1$, $a = 1$, $b = 2$. We excluded this case in the statement of Theorem 1.1.

Now assume $z \leq g$ and hence $h^1(R) > 0$. Fix a general fiber of u. Since $h^1(R) > 0$ and $\omega_C \cong \mathcal{O}_C(dabc-a-b-c)$, we have $h^1(\mathcal{I}_Z(dabc-a-b-c)) > 0$. Assume for the moment that Z is reduced (this is always the case in characteristic zero). Fix an ordering $P_1, \dots, P_z$ of the points of the support of Z. Since R is spanned and $h^1(R) > 0$, we have $h^1(\mathcal{O}_C(Z')) = h^1(\mathcal{O}_C(Z))$ for each $Z' \subset Z$ with $\deg(Z') = z-1$. Take $Z' = \{P_1, \dots, P_{z-1}\}$. Since u is not composed with w and Z is general, for each P_i there is $D_i \in |\mathcal{O}_{\mathbb{P}}(ab)|$ such that $Z \cap D_i = \{P_i\}$. Since $|\mathcal{O}_{\mathbb{P}}(ab)|$ is spanned outside $\mathrm{Sing}(\mathbb{P})$, P_1 imposes one condition to $|\mathcal{O}_{\mathbb{P}}(ab)|$. D_1 shows that the set $\{P_1, P_2\}$ imposes 2 independent conditions to $|\mathcal{O}_{\mathbb{P}}(ab)|$. D_2 shows that the set $\{P_1, P_2, P_3\}$ imposes 3 independent conditions to $|\mathcal{O}_{\mathbb{P}}(2ab)|$. And so on. We get that Z' imposes $z-1$ independent conditions to $|\mathcal{O}_{\mathbb{P}}(z-2)(ab)|$. Since $(z-2)ab \leq dabc-a-b-c$, we get $h^1(\mathcal{I}_{Z'}(dabc-a-b-c)) = 0$, a contradiction.

Now assume that Z is not reduced, i.e. that u is not separable. We get that the base field has characteristic $p > 0$. Since the base field is algebraically closed, we also get that it is composed with a Frobenius of $\mathbb{P}^1$, contradicting the minimality of z. $\square$

Proof of Theorem 1.2: We have $\mathrm{Sing}(\mathbb{P}) = \{(0:1:0), (0:0:1)\}$. Since $C \in |\mathcal{O}_{\mathbb{P}}(dbc)|$, it is a Cartier divisor of $\mathbb{P}$. Since C is smooth, then $(0:0:1) \notin C$. Hence $\mathcal{O}_C(b)$ is a spanned line bundle of degree db. Since $h^1(\mathcal{O}_{\mathbb{P}}(b-dbc)) = 0$, we have $h^0(\mathcal{O}_C(b)) = 2$. Take a line bundle L with minimal degree $z \leq db$ with $h^0(L) \geq 2$ and assume $L \neq \mathcal{O}_C(b)$. Fix a general $Z \in |L|$. As in last part of the proof of Theorem 1.1 we reduce to the case in which Z is reduced. Since L is spanned, we may assume $Z \cap \{z_0 = 0\} = \emptyset$. We fix an ordering $P_1, \dots, P_z$ of the points of Z and set $Z' := \{P_1, \dots, P_{z-1}\}$. As in the proof of Theorem 1.1 to get a contradiction it is sufficient to prove that $h^1(\mathcal{I}_{Z'}(dbc-1-b-c)) = 0$. Since $z \leq db$, we have $(z-2)c \leq (db-2)c \leq dbc-1-b-c$ and so it is sufficient to find $D_i \in |\mathcal{O}_{\mathbb{P}}(c)|$, $1 \leq i \leq z-2$, such that $P_i \in D_i$ and $P_{i+1} \notin D_i$. Fix $i \in \{1, \dots, z-2\}$. If there is $T \in |\mathcal{O}_{\mathbb{P}}(b)|$ with $P_i \in T$ and $P_{i+1} \notin T$, say T with equation $u(z_0, z_1) \in K[z_0, z_1, z_2]$, then we take as D_i the divisor with $z_0^{c-b}u(z_0, z_1)$ as its equations. Now assume that D_{i+1} is contained in every element of $|\mathcal{I}_{P_i}(b)|$ and fix $T \in |\mathcal{I}_{P_i}(b)|$. Since $P_i \notin \{(0:1:0), (0:0:1)\}$, T is the only element of $|\mathcal{O}_{\mathbb{P}}(b)|$ containing P_i. Let M be a general element of $|\mathcal{I}_{P_i}(c)|$. Set $e := \lfloor c/b \rfloor$. We have $\dim(K[x_0, x_1, x_2]_{1,b,c;c-b}) = e$ and $\dim(K[x_0, x_1, x_2]_{1,b,c;c}) = e+2$ and so $h^0(\mathcal{O}_{\mathbb{P}}(c-b)) \leq h^0(\mathcal{O}_{\mathbb{P}}(c)-2$. Hence T is not a component of M. We have $P_i \in T \cap M$. Since $\mathcal{O}_{\mathbb{P}}(b) \cdot \mathcal{O}_{\mathbb{P}}(c) = 1$, P_i is a smooth point of $\mathbb{P}$ and $P_i \in T \cap M$, P_i is the only element of $\mathbb{P} \setminus \mathrm{Sing}(\mathbb{P})$ contained in $M \cap T$. Hence $P_{i+1} \notin M$. Take $D_i := M$. $\square$

To check the key assumption of Theorem 1.1 the following well-known result may be useful.

Lemma 2.1. *Take a smooth and connected curve $C \subset \mathbb{P}$ such that $(0:0:1) \notin C$ and assume the existence of $D \in |\mathcal{O}_{\mathbb{P}}(ab)|$, $D \neq C$, such that the scheme $C \cap D$ has 1 connected component with multiplicity 2 and $\deg(w) - 2$ connected components with multiplicity 1. Let $w\colon C \to \mathbb{P}^1$ be the morphism induced by $|\mathcal{O}_{\mathbb{P}}(ab)|$. Then w is not composed with an involution, i.e. there are no triple (X, w_1, w_2) with X a connected smooth curve, $w_1\colon C \to X$, $w_2\colon X \to \mathbb{P}^1$, $w = w_2 \circ w_1$, $\deg(w_1) \geq 2$ and $\deg(w_2) \geq 2$.*

Proof. If $ab = 2$ (i.e. if $(a, b) = (1, 2)$), then w is not composed. In the general case we use that the monodromy group of w is the full symmetric group (see [12, Proposition 2.1] for a characteristic free proof, but remember that the monodromy group is 1-transitive just because C is an integral curve). □

3 Singular curves

We only look at integral curves T, which are contained in the smooth locus of $\mathbb{P}$ and hence that are Cartier divisors of $\mathbb{P}$. Let T be any such curve. There are many different Brill-Noether theories for integral singular curves. If we only look at spanned line bundles, then the proofs of Theorems 1.1 and 1.2 only require minimal modifications.

Theorem 3.1. *Let $C \in |\mathcal{O}_{\mathbb{P}}(dabc)|$ be an integral curve. Assume $dabc - a - b - c > 0$, i.e. assume that C has arithmetic genus ≥ 2, and $(a, b, d) \neq (1, 2, 1)$. Let $w\colon C \to \mathbb{P}^1$ be the morphism induced by $|\mathcal{O}_C(ab)|$. Fix a positive integer z such that $(z - 2)ab \leq dabc - a - b - c$ and there is a degree z spanned line bundle R on C. Let $u\colon C \to \mathbb{P}^y$, $y := h^0(R) - 1$, be the morphism induced by $H^0(R)$. In positive characteristic assume that either u is separable or that the algebraic group $Pic^0(C)$ has no unipotent part. Then the morphism $(w, u)\colon C \to \mathbb{P}^y \times \mathbb{P}^1$ is not birational onto its image.*

Theorem 3.2. *Assume $a = 1 < b$. Let $C \in |\mathcal{O}_{\mathbb{P}}(dac)|$ be a integral curve such that $C \cap Sing(\mathbb{P}) = \varnothing$. In positive characteristic assume that either u is separable or that the algebraic group $Pic^0(C)$ has no unipotent part. Then $\mathcal{O}_C(b)$ is the unique line bundle R on C such that $h^0(R) \geq 2$, R is spanned and $\deg(R) \leq db$.*

Proofs of Theorems 3.1 and 3.2: Take any spanned line bundle R on C with $h^0(R) \geq 2$ and call Z the zero-locus of a general section of R. Set $z := \deg(Z)$. Since R is spanned, we have $Z \cap \mathrm{Sing}(C) = \varnothing$. In characteristic zero Z is reduced and we may continue the proofs of Theorems 1.1 and 1.2. Now assume $p := \mathrm{char}(K) > 0$ and that Z is not reduced. Set $B := Z_{\mathrm{red}}$. Let $u\colon C \to \mathbb{P}^y$, $y := h^0(R) - 1$, be the morphism induced by $H^0(R)$. Since Z is general, it is not reduced if and only if u is not separable and, if p^e, $e > 0$, is the inseparable degree of u, then each connected component of Z has degree p^e and $Z = p^e B$ (this equality is non-ambiguous, because $B \subset C_{\mathrm{reg}}$). Varying Z in $|L|$ we get infinitely many effective divisors B which, multiplied by p^e, are linearly equivalent. By assumption the p^e-torsion of $\mathrm{Pic}^0(C)$ is finite. Hence C has a line bundle A of degree z/p^e with $h^0(A) \geq 2$, a contradiction. □

Let $Y \subset \mathbb{P}$ be an integral curve with $Y \cap \mathrm{Sing}(\mathbb{P}) = \varnothing$ and only ordinary nodes and ordinary cusps as its singularities. Set $S := \mathrm{Sing}(Y)$ and $s := \sharp(S)$. Since $Y \cap \mathrm{Sing}(\mathbb{P}) = \varnothing$, Y is a Cartier divisor of $\mathbb{P}$ and hence there is an integer $d > 0$ such that $Y \in |\mathcal{O}_{\mathbb{P}}(dabc)|$. The adjunction formula, gives $\omega_Y \cong \mathcal{O}_Y(dabc - a - b - c)$. Since

$h^1(\mathcal{O}_{\mathbb{P}}(-a-b-c)) = 0$, the restriction map $H^0(\mathcal{O}_{\mathbb{P}}(dabc)) \to H^0(\omega_Y)$ is surjective. Let $f\colon C \to Y$ be the normalization map. Since $Y \cap \mathrm{Sing}(\mathbb{P}) = \varnothing$, for each $x \in \mathbb{Z}$ the sheaf $\mathcal{O}_C(x) := f^*(\mathcal{O}_Y(x))$ is a line bundle. Since Y have only nodes and ordinary cusps as its singularities, we have $p_a(C) = p_a(Y) - s$ and $H^0(\omega_C)$ is induced by the linear system $|\mathcal{I}_S(dabc-a-b-c)|$ on $\mathbb{P}$. Since $\mathcal{O}_Y(ab)$ is a spanned line bundle, C has gonality at most dab. Let $w\colon C \to \mathbb{P}^1$ denote the morphism induced by $f^*(\mathcal{O}_Y(ab))$. We have $h^0(\mathcal{O}_C(ab)) = 2$ if and only if $h^1(\mathcal{I}_S(dabc-a-b-c-ab)) = 0$.

Theorem 3.3. *Assume $(z-2)ab \le dabc-a-b-c$, $s+z \le 2+d(dabc-a-b-c)/2$ and that $S \subset \mathbb{P}$ is a general subset with cardinality s. Then there is no degree z morphism $u\colon C \to \mathbb{P}^1$ such that the morphism $(w,u)\colon C \to \mathbb{P}^1 \times \mathbb{P}^1$ is birational onto its image.*

Theorem 3.4. *Assume $a = 1 < b$, $s + db \le 2 + d(dabc-a-b-c)/2$ and that S is general in $\mathbb{P}$. Then $\mathcal{O}_C(b)$ is the only line bundle L on C with $\deg(L) \le db$ and $h^0(L) \ge 2$.*

Proofs of Theorems 3.3 and 3.4: Fix a spanned line bundle L on C with $z := \deg(L) \ge 2$ and call $w\colon C \to \mathbb{P}^1$ the morphism induced by $f^*(H^0(\mathcal{O}_Y(ab))$. Take a general $Z \in |L|$. Since L is spanned, we have $Z \cap f^{-1}(\mathrm{Sing}(Y)) = \varnothing$. Hence f induces an isomorphism between Z and $f(Z)$. We assume that Z is a reduced (see the last part of the proof of Theorem 1.1). We fix an ordering the points $P_1, \dots, P_z$ of $f(Z)$. Set $Z' := \{P_1, \dots, P_{z-1}\}$. As in the proof of Theorem 1.1 to get a contradiction it is sufficient to prove that $h^1(\mathcal{I}_{S\cup Z'}(dabc-a-b-c)) > 0$. Since S is general, it is sufficient to prove that $h^1(\mathcal{I}_{Z'}(dabc-a-b-c)) = 0$ and that $h^0(\mathcal{I}_{Z'}(dabc-a-b-c)) \ge s$. The vanishing of $h^1(\mathcal{I}_{Z'}(dabc-a-b-c))$ is done as in the proof of Theorem 3.1. Since $h^1(\mathcal{I}_{Z'}(dabc-a-b-c)) = 0$, we have $h^0(\mathcal{I}_{Z'}(dabc-a-b-c)) = p_a(Y) - z + 1$. Hence it is sufficient to assume $s \le p_a(Y) - z + 1$ □

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References

[1] M. Beltrametti, L. Robbiano, *Introduction to the theory of weighted projective spaces*, Exposition. Math. 4, 2 (1986), 111-162.

[2] M. Coppens, *The gonality of general smooth curves with a prescribed plane nodal model*, Math. Ann. 289, 1 (1991), 89-93.

[3] M. Coppens, *Free linear systems on integral Gorenstein curves*, J. Algebra 145 (1992), 209-218.

[4] M. Coppens, *The existence of base point free linear systems on smooth plane curves*, J. Algebraic Geom. 4, 1 (1995), 1-15.

[5] S. Greco, G. Raciti, *The Lüroth semigroup of plane algebraic curves*, Pacific J. Math. 151, 1 (1991), 43-56.

[6] S. Greco, G. Raciti, *Gap orders of rational functions on plane curves with few singular points*, Manuscripta Math. 70, 4 (1991), 441-447.

[7] C. Delorme, *Espaces projectifs anisotropes*, Bull. Soc. Math. France 103, 2 (1975), 203-223.

[8] I. Dolgachev, *Weighted projective varieties*, Group actions and vector fields (Vancouver, B.C., 1981), 34-71, Lecture Notes in Math., 956, Springer, Berlin, 1982.

[9] R. Hartshorne, *Generalized divisors on Gorenstein curves and a theorem of Noether*, J. Math. Kyoto Univ. 26 (1986), 375-386.

[10] G. Martens, *The gonality of curves on a Hirzebruch surface*, Arch. Math. (Basel) 67, 4 (1996), 349-352.

[11] S. Mori, *On a generalization of complete intersections*, J. Math. Kyoto Univ. 15, 3 (1975), 619–646.

[12] J. Rathmann, *The uniform position principle for curves in characteristic* p, Math. Ann. 276, 4 (1987), 565–579.

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