

Gauge Bianchi Identities in Higher Order Lagrange Spaces

Adrian Sandovici

*Dedicated to Prof.Dr. Constantin UDRIȘTE
on the occasion of his sixtieth birthday*

Abstract

In the previous papers ([12], [13]) we put the bases of a gauge theory which can be applied any time we are dealing with physical phenomena that depend on the coordinates of the k – osculator bundle.

In the above mentioned papers we studied the strength fields of the second order and the Lagrangians generated only by strength fields. Also, for a full Lagrangian L_0 we determined the conserved currents and the corresponding conservation laws.

In this paper we shall define $(k+1)$ gauge covariant derivatives of strength fields and we shall determine four types of gauge Bianchi identities.

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Key words: gauge transformations, gauge Miron connection, higher order Lagrange spaces, generalized Lagrange metric of order k , gauge Bianchi identities.

1 Introduction

The higher order Lagrange spaces constitute an adequate geometrical framework for the development of an integrated gauge theory of the physical fields. Also, the form of interactions between some matter fields can be determined by postulating invariance under a certain group of transformations. In monograph [8] R. Miron gives an original construction of the geometry of higher order Lagrange spaces based on the k – osculator bundle notion. Generalizing some results given in [1], [5], [6], [9] and [14], R. Miron and Gh. Munteanu put the bases of a gauge theory on higher order Lagrange spaces ([8], [10]).

In the paper [12] we studied the strength fields of the second order on the geometrical model given by $GOSC^{(2)}M$. Also, we studied Lagrangians involved gauge fields, defined through strength fields. A full Lagrangian was defined as the sum of the Lagrangian of gauge fields and local gauge invariant Lagrangian of matter fields. For a full Lagrangian $L_0(u)$ we determined in Lagrange manner, the equations of motion. In [13] we studied the local gauge invariance of a full Lagrangian, the conserved currents and the corresponding conservation laws.

This paper continues the line of [12] and [13]. Using the structure constants of the Lie group G , the gauge fields and an arbitrary gauge Miron connection we shall define $(k+1)$ gauge – covariant derivatives of strength fields and then obtain four types of gauge Bianchi identities.

2 Preliminaries

Let M be a real n -dimensional C^∞ -differentiable manifold and its k -osculator bundle $(OSC^{(k)}M, \pi, M)$. The local coordinates on the total space $E = OSC^{(k)}M$ are denoted by $u = (x^i, y^{(1)i}, \dots, y^{(k)i})$. Let us consider G a compact subgroup in $GL(m, \mathbb{R})$ and $G^{(k)}$ its prolongation of order k . Let $P_G^{(k)}(M)$ be a principal bundle having the base M and structural group $G^{(k)}$. We consider $F = \mathbb{R}^{km}$. A gauge k -osculator bundle $GOSC^{(k)}M$ is a G -structure of order k of the principal bundle $P_G^{(k)}(M)$. The geometrical theory of the gauge k -osculator bundle $GOSC^{(k)}M$ is the geometrical theory of the k -osculator bundle $OSC^{(k)}M$ restricted to the group $G^{(k)}$.

The notions of gauge transformation in $GOSC^{(k)}M$, nonlinear connection on $GOSC^{(k)}M$, N -linear connection on $GOSC^{(k)}M$, generalized Lagrange metric of order k , are given in [8] and [10]. The transformations of coordinates on $GOSC^{(k)}M$ are given by

$$(2.1) \quad \begin{cases} \tilde{x}^i = \tilde{x}^i(x) \\ \tilde{y}^{(1)i} = \frac{\partial \tilde{x}^i}{\partial x^j} \cdot y^{(1)j} \\ \dots \\ k \tilde{y}^{(k)i} = k \frac{\partial \tilde{y}^{(k-1)i}}{\partial y^{(k-1)j}} y^{(k)j} + \dots + \frac{\partial \tilde{y}^{(k-1)i}}{\partial x^j} y^{(1)j}. \end{cases} \quad \det \left(\frac{\partial \tilde{x}^i}{\partial x^j} \right) \neq 0$$

In local coordinates on manifold $GOSC^{(k)}M$, a gauge transformation can be represented by equations of the form

$$(2.2) \quad \begin{cases} \bar{x}^i = X^i(x) \\ \bar{y}^{(1)i} = Y^{(1)i}(x, y^{(1)}) \\ \dots \\ \bar{y}^{(k)i} = Y^{(k)i}(x, y^{(1)}, \dots, y^{(k)}), \end{cases}$$

where

$$\det \left(\frac{\partial X^i}{\partial x^j} \right) \det \left(\frac{\partial Y^{(1)i}}{\partial y^{(1)j}} \right) \cdot \dots \cdot \det \left(\frac{\partial Y^{(k)i}}{\partial y^{(k)j}} \right) \neq 0.$$

Let a generalized Lagrange metric of order k on $GOSC^{(k)}M$ be given by the components $(g_{ij}(u))$, a gauge nonlinear connection $N = (N_{(1)j}^i, N_{(2)j}^i, \dots, N_{(k)j}^i)$, and a gauge Miron connection $DFN = (L_{jk}^i, C_{(1)jk}^i, \dots, C_{(k)jk}^i)$ on $GOSC^{(k)}M$ and $L_0(u)$,

$u = (x^i, y^{(1)i}, \dots, y^{(k)i})$ a Lagrangian defined on the domain $\Omega \subset \mathbb{R}^{(k+1)n}$. Suppose that there exist p differentiable scalar fields (physical fields) Q^A , $A = \overline{1, p}$ so that the Lagrangian L_0 depends only on the variable $x^i, y^{(1)i}, \dots, y^{(k)i}$ by means of Q^A and their derivatives $\frac{\delta Q^A}{\delta x^i}, \frac{\delta Q^A}{\delta y^{(\alpha)i}}, \alpha = 1, \dots, k$. More accurately, L_0 is a scalar field on $GOSC^{(k)}M$ given by

$$(2.3) \quad L_0(x, y^{(1)}, \dots, y^{(k)}) = L\left(Q^A, \frac{\delta Q^A}{\delta x^i}, \frac{\delta Q^A}{\delta y^{(1)i}}, \dots, \frac{\delta Q^A}{\delta y^{(k)i}}\right).$$

In order to obtain the gauge-invariant Lagrangians with respect to the local gauge invariance of the Lie group G , we considered a new Lagrangian

$$(2.4) \quad L_0(u) = L'\left(Q^A, \frac{\delta Q^A}{\delta x^i}, \frac{\delta Q^A}{\delta y^{(\alpha)i}}, H_i^a(u), V_i^{(\alpha)a}(u)\right)$$

in which $H_i^a(u)$ and $V_i^{(\alpha)a}(u)$, $\alpha = 1, \dots, k$ are the components of some gauge d-covectors, called local gauge fields, satisfying the following nonhomogeneous conditions of variations

$$(2.5) \quad \begin{cases} \delta^*(H_i^a(u)) = \varepsilon^b(u) \cdot f_{bc}^a \cdot H_i^c + \frac{\delta \varepsilon^a}{\delta x^i} \\ \delta^*\left(V_i^{(\alpha)a}(u)\right) = \varepsilon^b(u) \cdot f_{bc}^a \cdot V_i^{(\alpha)c} + \frac{\delta \varepsilon^a}{\delta y^{(\alpha)i}}, \end{cases}$$

where f_{bc}^a are the structure constants of the Lie group G and $\varepsilon^a(u)$ are differentiable functions.

The study of Lagrangians involving only gauge fields is given in [12]. These Lagrangians can be generated by means of the following functions

$$(2.6) \quad F_{ij}^{(h)a} = A_{ij}^{(h)a} - f_{bc}^a \cdot H_i^c \cdot H_j^b + \sum_{\alpha=1}^k R_{(0\alpha)ij}^m \cdot V_m^{(\alpha)a}$$

$$(2.7) \quad F_{ij}^{(h,v_\alpha)a} = A_{ij}^{(h,v_\alpha)a} - f_{bc}^a \cdot H_i^c \cdot V_j^{(\alpha)b} + \sum_{\beta=1}^k B_{(\alpha\beta)ij}^m \cdot V_m^{(\beta)a}$$

$$(2.8) \quad F_{ij}^{(v_\alpha, v_\beta)a} = A_{ij}^{(v_\alpha, v_\beta)a} - f_{bc}^a \cdot V_i^{(\alpha)c} \cdot V_j^{(\beta)b} + \sum_{\gamma=1}^k C_{(\alpha\beta)ij}^m \cdot V_m^{(\gamma)a},$$

where

$$(2.9) \quad A_{ij}^{(h)a} = \frac{\delta H_i^a}{\delta x^j} - \frac{\delta H_j^a}{\delta x^i}, \quad A_{ij}^{(h,v_\alpha)a} = \frac{\delta H_i^a}{\delta y^{(\alpha)j}} - \frac{\delta V_j^{(\alpha)a}}{\delta x^i}, \quad A_{ij}^{(v_\alpha, v_\beta)a} = \frac{\delta V_i^{(\alpha)a}}{\delta y^{(\beta)j}} - \frac{\delta V_j^{(\beta)a}}{\delta y^{(\alpha)i}}.$$

We call $F_{ij}^{(h)a}$, $F_{ij}^{(h,v_\alpha)a}$ and $F_{ij}^{(v_\alpha, v_\beta)a}$ the horizontal strength fields, the mixed strength fields and v_α -vertical strength fields, respectively.

3 Gauge Bianchi identities with respect to covariant derivatives of strength fields

In this section, using the structure constants of the Lie group G , the gauge fields and an arbitrary gauge Miron connection we shall define $(k + 1)$ gauge-covariant derivatives of strength fields and then obtain four types of Bianchi identities.

Denote by K_{ij}^a the local components of one of the strength fields $F_{ij}^{(h)^a}$, $F_{ij}^{(h,v_\alpha)^a}$, $F_{ij}^{(v_\alpha,v_\beta)^a}$ given by (2.6), (2.7) and (2.8) respectively. Then we define the *horizontal gauge covariant derivative* of K_{ij}^a as follows

$$(3.1) \quad K_{ij|k}^a = \frac{\delta K_{ij}^a}{\delta x^k} + f_{bc}^a \cdot K_{ij}^b \cdot H_k^c - K_{hj}^a \cdot L_{ik}^h - K_{ih}^a \cdot L_{jk}^h.$$

In a similar way, we define the v_α - *vertical gauge covariant derivative* of K_{ij}^a by

$$(3.2) \quad K_{ij|k}^{a(\alpha)} = \frac{\delta K_{ij}^a}{\delta y^{(\alpha)k}} + f_{bc}^a \cdot K_{ij}^b \cdot V_k^{(\alpha)c} - K_{hj}^a \cdot C_{(\alpha)ik}^h - K_{ih}^a \cdot C_{(\alpha)jk}^h.$$

By a direct but very long calculation we can prove the following three results

Proposition 3.1. *Both covariant derivatives define d -gauge tensor fields of type $(0,3)$. More precisely, with respect to (2.1) and (2.2) we have*

$$(3.3) \quad K_{ij|k}^a = \sigma_i^h \cdot \sigma_j^l \cdot \sigma_k^r \cdot \tilde{K}_{hl|r}^a$$

$$(3.4) \quad K_{ij|k}^{a(\alpha)} = \sigma_i^h \cdot \sigma_j^l \cdot \sigma_k^r \cdot \tilde{K}_{hl|r}^{a(\alpha)},$$

and respectively

$$(3.5) \quad K_{ij|k}^a = X_i^h \cdot X_j^l \cdot X_k^r \cdot \bar{K}_{hl|r}^a$$

$$(3.6) \quad K_{ij|k}^{a(\alpha)} = X_i^h \cdot X_j^l \cdot X_k^r \cdot \bar{K}_{hl|r}^{a(\alpha)},$$

where $\sigma_i^j = \frac{\partial \tilde{x}^j}{\partial x^i}$ and $X_i^h = \frac{\partial X^h}{\partial x^i}$.

Proposition 3.2. *With respect to the local gauge action of Lie group G , the gauge covariant derivatives verify the following homogeneous laws of transformation:*

$$(3.7) \quad \delta^* \left(K_{ij|k}^a \right) = \varepsilon^b \cdot f_{bc}^a \cdot K_{ij|k}^c$$

$$(3.8) \quad \delta^* \left(K_{ij|k}^{a(\alpha)} \right) = \varepsilon^b \cdot f_{bc}^a \cdot K_{ij|k}^c.$$

Proposition 3.3. *With respect to d -vector fields of adapted basis of $T_uGOSC^{(k)}M$, the following Jacobi identities hold:*

$$(3.9) \quad \sum_{(i,j,k)} \left(\frac{\delta R^{(0\alpha)m}}{\delta x^k} + \sum_{\beta=1}^k R^{(0\beta)h} \cdot B^{(\beta\alpha)m}_{kh} \right) = 0$$

$$(3.10) \quad \begin{aligned} & \sum_{\beta=1}^k \left(R^{(0\beta)h} \cdot C^{(\gamma)m}_{(\beta\alpha)hk} - B^{(\alpha\beta)p} \cdot B^{(\beta\gamma)m}_{ip} + B^{(\alpha\beta)p} \cdot B^{(\beta\gamma)m}_{ik} - B^{(\beta\gamma)p} \cdot B^{(\alpha\beta)m}_{jp} \right) = \\ & = \frac{\delta R^{(0\gamma)m}}{\delta y^{(\alpha)k}} + \frac{\delta B^{(\alpha\gamma)m}}{\delta x^i} - \frac{\delta B^{(\alpha\gamma)m}}{\delta x^j} \end{aligned}$$

$$(3.11) \quad \begin{aligned} & \sum_{\gamma=1}^k \left(B^{(\alpha\gamma)p} \cdot C^{(\zeta)m}_{(\gamma\beta)pk} - B^{(\beta\gamma)p} \cdot C^{(\zeta)m}_{(\gamma\alpha)pj} - B^{(\gamma\zeta)m} \cdot C^{(\gamma)p}_{(\alpha\beta)jk} \right) \\ & = \frac{\delta B^{(\alpha\zeta)m}}{\delta y^{(\beta)k}} - \frac{\delta B^{(\beta\zeta)m}}{\delta y^{(\alpha)j}} + \frac{\delta C^{(\zeta)m}}{\delta x^i} \end{aligned}$$

$$(3.12) \quad \begin{aligned} & \sum_{\zeta=1}^k \left(C^{(\zeta)p}_{(\alpha\beta)ij} \cdot C^{(\eta)m}_{(\zeta\gamma)pk} + C^{(\zeta)p}_{(\beta\gamma)jk} \cdot C^{(\eta)m}_{(\zeta\alpha)pi} + C^{(\zeta)p}_{(\gamma\alpha)ki} \cdot C^{(\eta)m}_{(\zeta\beta)pj} \right) \\ & = \frac{\delta C^{(\eta)m}}{\delta y^{(\alpha\beta)ij}} + \frac{\delta C^{(\eta)m}}{\delta y^{(\beta\gamma)jk}} + \frac{\delta C^{(\eta)m}}{\delta y^{(\gamma\alpha)ki}} \end{aligned}$$

Here and in sequel, by $\sum_{(i,j,k)}$ we mean the cyclic sum with respect to (i, j, k) . The

above propositions are useful for proving the following main results

Theorem 3.1. *The following gauge Bianchi identity with respect to the horizontal gauge-covariant derivative of the horizontal strength fields holds:*

$$(3.13) \quad \sum_{(i,j,k)} \left\{ F^{(h)a}_{ij|k} + \sum_{\alpha=1}^k R^{(h,v_\alpha)a}_{(0\alpha)ij} + T^{(h)a}_{(00)ij} \right\} = 0.$$

Proof. First, we have

$$(3.14) \quad \begin{aligned} \frac{\delta F^{(h)a}_{ij}}{\delta x^k} & = \frac{\delta^2 H^a_i}{\delta x^k \delta x^j} - \frac{\delta^2 H^a_j}{\delta x^k \delta x^i} - f^a_{bc} \cdot \left(\frac{\delta H^c_i}{\delta x^k} \cdot H^b_j + \frac{\delta H^b_j}{\delta x^k} \cdot H^c_i \right) + \\ & + \sum_{\alpha=1}^k \frac{\delta R^{(0\alpha)m}}{\delta x^k} \cdot V^a_m + \sum_{\alpha=1}^k R^{(0\alpha)m}_{ij} \cdot \frac{\delta V^a_m}{\delta x^k}. \end{aligned}$$

Using (3.14) we can easily obtain

$$(3.15) \quad \sum_{(i,j,k)} \left(\frac{\delta F_{ij}^{(h)^a}}{\delta x^k} - f_{bc}^a \cdot H_i^b \cdot A_{jk}^{(h)^c} + \sum_{\alpha=1}^k R_{(0\alpha)ij}^m \cdot \left(\frac{\delta H_k^a}{\delta y^{(\alpha)^m}} - \frac{\delta V_m^{(\alpha)^a}}{\delta x^k} \right) - \sum_{\alpha=1}^k \frac{\delta R_{(0\alpha)ij}^m}{\delta x^k} \cdot V_m^{(\alpha)^a} \right) = 0.$$

Now, using (3.9) and (3.15) we have

$$(3.16) \quad \sum_{(i,j,k)} \left(\frac{\delta F_{ij}^{(h)^a}}{\delta x^k} - f_{bc}^a \cdot H_i^b \cdot A_{jk}^{(h)^c} + \sum_{\alpha=1}^k R_{(0\alpha)ij}^m \cdot \left(\frac{\delta H_k^a}{\delta y^{(\alpha)^m}} - \frac{\delta V_m^{(\alpha)^a}}{\delta x^k} + \sum_{\beta=1}^k B_{(\alpha\beta)km}^h \cdot V_h^{(\beta)^a} \right) \right) = 0.$$

Taking into account (2.6) and the Jacobi identity from the general theory of Lie groups ($f_{bc}^a \cdot f_{de}^c + f_{dc}^a \cdot f_{eb}^c + f_{ec}^a \cdot f_{bd}^c = 0$) we obtain the following relation

$$(3.17) \quad f_{bc}^a \cdot \sum_{(i,j,k)} \left\{ \left(F_{ij}^{(h)^b} - A_{ij}^{(h)^b} - \sum_{\alpha=1}^k R_{(0\alpha)ij}^m \cdot V_m^{(\alpha)^b} \right) \cdot H_k^c \right\} = 0.$$

From (3.16) and (3.17) we have

$$(3.18) \quad \sum_{(i,j,k)} \left\{ \frac{\delta F_{ij}^{(h)^a}}{\delta x^k} + f_{bc}^a \cdot F_{ij}^{(h)^b} \cdot H_k^c + \sum_{\alpha=1}^k R_{(0\alpha)ij}^m \cdot \left(\frac{\delta H_k^a}{\delta y^{(\alpha)^m}} - \frac{\delta V_m^{(\alpha)^a}}{\delta x^k} + \sum_{\beta=1}^k B_{(\alpha\beta)km}^h \cdot V_h^{(\beta)^a} - f_{bc}^a \cdot H_k^c \cdot V_m^{(\alpha)^b} \right) \right\} = 0 \iff \sum_{(i,j,k)} \left\{ F_{ij|k}^{(h)^a} + F_{hj}^{(h)^a} \cdot L_{ik}^h + F_{ih}^{(h)^a} \cdot L_{jk}^h + \sum_{\alpha=1}^k R_{(0\alpha)ij}^m \cdot F_{km}^{(h,v_\alpha)^a} \right\} = 0.$$

Because $T_{(00)ij}^h = L_{ij}^h - L_{ji}^h$ we have

$$(3.19) \quad \sum_{(i,j,k)} \left(F_{hj}^{(h)^a} \cdot L_{ik}^h + F_{ih}^{(h)^a} \cdot L_{jk}^h \right) = \sum_{(i,j,k)} F_{kh}^{(h)^a} \cdot T_{(00)ij}^h.$$

From (3.18) and (3.19) we can easily obtain the conclusion of theorem. \square

Next, by $\sum_{\substack{(i,j,k) \\ (\alpha,\beta,\gamma)}}$ we mean the cyclic sum with respect to (i, j, k) and (α, β, γ) when

we move simultaneously the indices of these triples. In the same manner we can prove the following two results

Theorem 3.2. *The following gauge Bianchi identity with respect to the v_α -gauge-covariant derivatives of the v_β -vertical strength fields holds:*

$$(3.20) \quad \sum_{\substack{(i,j,k) \\ (\alpha,\beta,\gamma)}} \left\{ F_{ij|k}^{(v_\alpha, v_\beta)^a} \cdot F_{ij}^{(\gamma)} + \sum_{\zeta=1}^k C_{(\alpha\beta)ij}^{(\zeta)m} \cdot F_{km}^{(v_\gamma, v_\zeta)^a} + D_{(\gamma)ijk}^{hp} \cdot F_{hp}^{(v_\alpha, v_\beta)^a} \right\} = 0,$$

where

$$(3.21) \quad D_{(\gamma)ijk}^{hp} = C_{(\gamma)ik}^h \cdot \delta_j^p + C_{(\gamma)jk}^p \cdot \delta_i^h.$$

Theorem 3.3. *The following mixed gauge Bianchi identities with respect to the gauge v_α -covariant derivatives of the mixed strength fields, and the horizontal covariant derivative of v_β -vertical strength fields, respectively hold:*

$$(3.22) \quad \begin{aligned} & F_{ij|k}^{(h, v_\alpha)^a} - F_{ik|j}^{(h, v_\beta)^a} + F_{jk|i}^{(v_\alpha, v_\beta)^a} + \sum_{\gamma=1}^k \left\{ C_{(\alpha\beta)jk}^{(\gamma)m} \cdot F_{im}^{(h, v_\gamma)^a} + \right. \\ & \left. + B_{(\beta\gamma)ik}^m \cdot F_{mj}^{(v_\gamma, v_\alpha)^a} - B_{(\alpha\gamma)ij}^m \cdot F_{mk}^{(v_\gamma, v_\beta)^a} \right\} + D_{(\beta)ijk}^{hp} \cdot F_{hp}^{(h, v_\alpha)^a} - \\ & - D_{(\alpha)ikj}^{hp} \cdot F_{hp}^{(h, v_\beta)^a} + E_{jki}^{hp} \cdot F_{hp}^{(v_\alpha, v_\beta)^a} = 0 \end{aligned}$$

$$(3.23) \quad \begin{aligned} & F_{ij|k}^{(h, v_\alpha)^a} - F_{kj|i}^{(h, v_\alpha)^a} + F_{ki|j}^{(h)^a} + \\ & + F_{hj}^{(h, v_\alpha)^a} \cdot T_{(00)ik}^h + F_{ph}^{(h, v_\alpha)^a} \cdot F_{jki}^{hp} + F_{hp}^{(h)^a} \cdot D_{(\alpha)kij}^{hp} + \\ & + \sum_{\beta=1}^k \left\{ B_{(\alpha\beta)ij}^m \cdot F_{km}^{(h, v_\beta)^a} - B_{(\alpha\beta)kj}^m \cdot F_{im}^{(h, v_\beta)^a} - R_{(0\beta)ik}^m \cdot F_{jm}^{(v_\alpha, v_\beta)^a} \right\} = 0, \end{aligned}$$

where

$$(3.24) \quad E_{jki}^{hp} = L_{ji}^h \cdot \delta_k^p + L_{ki}^p \cdot \delta_j^h$$

$$(3.25) \quad F_{jki}^{hp} = L_{jk}^h \cdot \delta_i^p - L_{ji}^h \cdot \delta_k^p,$$

and the functions $D_{(\alpha)kij}^{hp}$ are given by (3.21).

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Adrian Sandovici
High School "Gh. Cartianu"
5600, Piatra Neamt