

Mean Curvature and Shape Operator of Slant Immersions in a Sasakian Space Form

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Abstract

For submanifolds, in a Sasakian space form, which are tangential to the structure vector field, we establish a basic inequality between squared mean curvature and Ricci curvature. Equality cases are also discussed. Some applications of these results are given for slant, invariant, anti-invariant and CR -submanifolds. We also establish an inequality between the shape operator and the sectional curvature for slant submanifolds in a Sasakian space form. In particular, we give similar results for invariant and anti-invariant submanifolds.

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Key words: Sasakian space form, invariant submanifold, anti-invariant submanifold, slant submanifold, CR -submanifold, totally geodesic submanifold, Ricci curvature, sectional curvature and squared mean curvature.

1 Introduction

According to B.-Y. Chen, one of the basic problems in submanifold theory is to find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold. In [5], he establishes a relationship between sectional curvature function K and the shape operator for submanifolds in real space forms. In [6], he also gives a relationship between Ricci curvature and squared mean curvature.

A contact version of B.-Y. Chen's inequality and its applications to slant immersions in a Sasakian space form $\tilde{M}(c)$ are given in [4]. In the present paper, we continue the study of submanifolds in a Sasakian space form, which are tangent to the structure vector field. Necessary details about Sasakian space forms and slant submanifolds are reviewed in section 2. In section 3, for those submanifolds in Sasakian space forms which are tangential to the structure vector field, we establish a basic inequality between Ricci curvature and squared mean curvature function. We also discuss equality cases. As applications, we state similar results for slant, invariant, anti-invariant and CR -submanifolds. In the last section, we establish an inequality between the shape operator and the sectional curvature for slant submanifolds in a Sasakian space form. In particular, we give similar results for invariant and anti-invariant submanifolds.

2 Preliminaries

Let \tilde{M} be a $(2m+1)$ -dimensional almost contact manifold endowed with an almost contact structure (φ, ξ, η) , that is, φ is a $(1, 1)$ tensor field, ξ is a vector field and η is 1-form such that $\varphi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1$. Then, $\varphi(\xi) = 0$ and $\eta \circ \varphi = 0$. The almost contact structure is said to be *normal* if in the product manifold $\tilde{M} \times R$ the induced almost complex structure J defined by $J(X, \lambda d/dt) = (\varphi X - \lambda \xi, \eta(X) d/dt)$ is integrable, where X is tangent to \tilde{M} , t is the coordinate of R and λ is a smooth function on $\tilde{M} \times R$. The condition for almost contact structure to be *normal* is equivalent to vanishing of the torsion tensor $[\varphi, \varphi] + 2d\eta \otimes \xi$, where $[\varphi, \varphi]$ is the Nijenhuis tensor of φ .

Let g be a compatible Riemannian metric with the structure (φ, ξ, η) , that is, $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ or equivalently, $g(X, \varphi Y) = -g(\varphi X, Y)$ and $g(X, \xi) = \eta(X)$ for all $X, Y \in T\tilde{M}$. Then, \tilde{M} becomes an almost contact metric manifold equipped with the almost contact metric structure (φ, ξ, η, g) . Moreover, if $g(X, \varphi Y) = d\eta(X, Y)$, then \tilde{M} is said to have a *contact metric structure* (φ, ξ, η, g) , and \tilde{M} is called a *contact metric manifold*. A normal contact metric structure in \tilde{M} is a *Sasakian structure* and \tilde{M} is a *Sasakian manifold*. A necessary and sufficient condition for an almost contact metric structure to be a Sasakian structure is

$$(1) \quad (\tilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in T\tilde{M},$$

where $\tilde{\nabla}$ is the Levi-Civita connection of the Riemannian metric g . R^{2m+1} and S^{2m+1} are equipped with standard Sasakian structures. For more details, we refer to [2].

The sectional curvature $\tilde{K}(X \wedge \varphi X)$ of a plane section spanned by a unit vector X orthogonal to ξ is called a φ -*sectional curvature*. If \tilde{M} has constant φ -sectional curvature c then it is called a *Sasakian space form* and is denoted by $\tilde{M}(c)$. The curvature tensor \tilde{R} of a Sasakian space form $\tilde{M}(c)$ is given by

$$(2) \quad \begin{aligned} 4\tilde{R}(X, Y)Z &= (c+3)\{g(Y, Z)X - g(X, Z)Y\} + \\ &+ (c-1)\{g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y - \\ &- 2g(\varphi X, Y)\varphi Z + \eta(X)\eta(Z)Y - \\ &- \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}. \end{aligned}$$

Let M be an $(n+1)$ -dimensional submanifold immersed in an almost contact metric manifold $\tilde{M}(\varphi, \xi, \eta, g)$. Let g denote the induced metric on M also. We denote by σ the second fundamental form of M and by A_N the shape operator associated to any vector N in the normal bundle $T^\perp M$. Then $g(\sigma(X, Y), N) = g(A_N X, Y)$ for all $X, Y \in TM$ and $N \in T^\perp M$. The Gauss equation is

$$(3) \quad \begin{aligned} \tilde{R}(X, Y, Z, W) &= R(X, Y, Z, W) - g(\sigma(X, W), \sigma(Y, Z)) \\ &+ g(\sigma(X, Z), \sigma(Y, W)) \end{aligned}$$

for all $X, Y, Z, W \in TM$, where R is the induced curvature tensor of M . The relative null space of M at a point $p \in M$ is defined by

$$\mathcal{N}_p = \{X \in T_p M \mid \sigma(X, Y) = 0, \text{ for all } Y \in T_p M\}.$$

Let $\{e_1, \dots, e_{n+1}\}$ be an orthonormal basis of the tangent space T_pM . The mean curvature vector $H(p)$ at $p \in M$ is

$$(4) \quad H(p) \equiv \frac{1}{n+1} \sum_{i=1}^{n+1} \sigma(e_i, e_i).$$

The submanifold M is *totally geodesic* in \tilde{M} if $\sigma = 0$; *minimal* if $H = 0$; and *totally umbilical* if $\sigma(X, Y) = g(X, Y)H$ for all $X, Y \in TM$. We put

$$\sigma_{ij}^r = g(\sigma(e_i, e_j), e_r) \quad \text{and} \quad \|\sigma\|^2 = \sum_{i,j=1}^{n+1} g(\sigma(e_i, e_j), \sigma(e_i, e_j)),$$

where e_r belongs to an orthonormal basis $\{e_{n+2}, \dots, e_{2m+1}\}$ of the normal space $T_p^\perp M$. The scalar curvature $\tau(p)$ at $p \in M$ is given by

$$(5) \quad \tau(p) = \sum_{i < j} K(e_i \wedge e_j),$$

where $K(e_i \wedge e_j)$ is the sectional curvature of the plane section spanned by e_i and e_j .

For a vector field X in M , we put

$$\varphi X = PX + FX, \quad PX \in TM, \quad FX \in T^\perp M.$$

Thus, P is an endomorphism of the tangent bundle of M and satisfies $g(X, PY) = -g(PX, Y)$ for all $X, Y \in TM$. The squared norm of P is given by

$$\|P\|^2 = \sum_{i,j=1}^{n+1} g(e_i, Pe_j)^2$$

for any local orthonormal basis $\{e_1, e_2, \dots, e_{n+1}\}$ for T_pM .

A submanifold M of an almost contact metric manifold with $\xi \in TM$ is called a *semi-invariant submanifold* ([1]) or a *contact CR submanifold* ([8]) if there exists two differentiable distributions \mathcal{D} and \mathcal{D}^\perp on M such that **(i)** $TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \mathcal{E}$, **(ii)** the distribution \mathcal{D} is invariant by φ , i.e., $\varphi(\mathcal{D}) = \mathcal{D}$, and **(iii)** the distribution \mathcal{D}^\perp is anti-invariant by φ , i.e., $\varphi(\mathcal{D}^\perp) \subseteq T^\perp M$.

The submanifold M tangent to ξ is said to be *invariant* or *anti-invariant* ([8]) according as $F = 0$ or $P = 0$. Thus, a *CR*-submanifold is invariant or anti-invariant according as $\mathcal{D}^\perp = \{0\}$ or $\mathcal{D} = \{0\}$. A proper *CR*-submanifold is neither invariant nor anti-invariant.

For each non zero vector $X \in T_pM$, such that X is not proportional to ξ_p , we denote the angle between φX and T_pM by $\theta(X)$. Then M is said to be *slant* ([7],[3]) if the angle $\theta(X)$ is constant, that is, it is independent of the choice of $p \in M$ and $X \in T_pM - \{\xi\}$. The angle θ of a slant immersion is called the *slant angle* of the immersion. Invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \pi/2$ respectively. A *proper* slant immersion is neither invariant nor anti-invariant.

3 Mean curvature and Ricci curvature

Let M be an $(n+1)$ -dimensional submanifold in a $(2m+1)$ -dimensional Sasakian space form $\tilde{M}(c)$ tangential to the structure vector field ξ . In view of (2) and (3), it implies that

$$\begin{aligned}
 R(X, Y, Z, W) &= \frac{c+3}{4} \{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} + \\
 &+ \frac{c-1}{4} \{g(X, \varphi W)g(Y, \varphi Z) - g(X, \varphi Z)g(Y, \varphi W) - \\
 (6) \quad &- 2g(X, \varphi Y)g(Z, \varphi W) - \\
 &- g(X, W)\eta(Y)\eta(Z) + g(X, Z)\eta(Y)\eta(W) - \\
 &- g(Y, Z)\eta(X)\eta(W) + g(Y, W)\eta(X)\eta(Z)\} + \\
 &+ g(\sigma(X, W), \sigma(Y, Z)) - g(\sigma(X, Z), \sigma(Y, W))
 \end{aligned}$$

for all $X, Y, Z, W \in TM$, where R is the induced curvature tensor of M . Thus, we have

$$(7) \quad (n+1)^2 \|H\|^2 = 2\tau + \|\sigma\|^2 - \frac{1}{4}n(n+1)(c+3) - \frac{1}{4}(3\|P\|^2 - 2n)(c-1).$$

In [6], B.-Y. Chen established a relationship between Ricci curvature and the squared mean curvature for a submanifold in a real space form as follows.

Theorem 3.1 *Let M be an n -dimensional submanifold in a real space form $R^m(c)$. Then,*

1. *For each unit vector $X \in T_pM$, we have*

$$(8) \quad \|H\|^2 \geq \frac{4}{n^2} \{Ric(X) - (n-1)c\}.$$

2. *If $H(p) = 0$, then a unit vector $X \in T_pM$ satisfies the equality case of (8) if and only if X lies in the relative null space N_p at p .*

3. *The equality case of (8) holds for all unit vectors $X \in T_pM$, if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

In this section, we find similar results for different kind of submanifolds in a Sasakian space form.

Theorem 3.2 *Let M be an $(n+1)$ -dimensional submanifold in a $(2m+1)$ -dimensional Sasakian space form $\tilde{M}(c)$ tangential to the structure vector field ξ . Then,*

(i) *For each unit vector $U \in T_pM$, we have*

$$(9) \quad 4Ric(U) \leq (n+1)^2 \|H\|^2 + n(c+3) + \{3\|PU\|^2 - (n-1)\eta(U)^2 - 1\}(c-1).$$

(ii) *If $H(p) = 0$, a unit vector $U \in T_pM$ satisfies the equality case of (9) if and only if U belongs to the relative null space \mathcal{N}_p .*

(iii) *The equality case of (9) holds for all unit vectors $U \in T_pM$ if and only if M is a totally geodesic submanifold.*

Proof. We choose an orthonormal basis $\{e_1, \dots, e_{n+1}, e_{n+2}, \dots, e_{2m+1}\}$ such that $e_1, \dots, e_{n+1} \in T_p M$. The squared second fundamental form and the squared mean curvature vector also satisfy

$$\begin{aligned} \|\sigma\|^2 &= \frac{1}{2} (n+1)^2 \|H\|^2 + \frac{1}{2} \sum_{r=n+2}^{2m+1} (\sigma_{11}^r - \sigma_{22}^r - \dots - \sigma_{n+1 \ n+1}^r)^2 + \\ (10) \quad &+ 2 \sum_{r=n+2}^{2m+1} \sum_{j=2}^{2m+1} (\sigma_{1j}^r)^2 - 2 \sum_{r=n+2}^{2m+1} \sum_{2 \leq i < j \leq n+1} (\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2). \end{aligned}$$

From (7) and (10), we get

$$\begin{aligned} \frac{1}{4} (n+1)^2 \|H\|^2 &= \tau - \frac{1}{8} n(n+1) (c+3) - \frac{1}{8} (3\|P\|^2 - 2n) (c-1) + \\ (11) \quad &+ \frac{1}{4} \sum_{r=n+2}^{2m+1} (\sigma_{11}^r - \sigma_{22}^r - \dots - \sigma_{n+1 \ n+1}^r)^2 + \sum_{r=n+2}^{2m+1} \sum_{j=2}^{n+1} (\sigma_{1j}^r)^2 - \\ &- \sum_{r=n+2}^{2m+1} \sum_{2 \leq i < j \leq n+1} (\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2). \end{aligned}$$

From the equation of Gauss we also have

$$\begin{aligned} K(e_i \wedge e_j) &= \sum_{r=n+2}^{2m+1} (\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2) + \frac{c+3}{4} + \\ &+ \frac{c-1}{4} (3g(e_i, Pe_j)^2 - \eta(e_i)^2 - \eta(e_j)^2), \end{aligned}$$

which gives

$$\begin{aligned} \sum_{2 \leq i < j \leq n+1} K(e_i \wedge e_j) &= \sum_{r=n+2}^{2m} \sum_{2 \leq i < j \leq n+1} (\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2) + \frac{1}{8} n(n-1) (c+3) + \\ (12) \quad &+ \frac{1}{8} \{3\|P\|^2 - 6\|Pe_1\|^2 - 2(n-1)(1-\eta(e_1)^2)\} (c-1). \end{aligned}$$

From (11) and (12), we get

$$\begin{aligned} \frac{1}{4} (n+1)^2 \|H\|^2 &= \tau - \sum_{2 \leq i < j \leq n+1} K(e_i \wedge e_j) - \\ &- \frac{1}{4} n(c+3) - \frac{1}{4} (3\|Pe_1\|^2 - (n-1)\eta(e_1)^2 - 1) (c-1) + \\ &+ \frac{1}{4} \sum_{r=n+2}^{2m+1} (\sigma_{11}^r - \sigma_{22}^r - \dots - \sigma_{n+1 \ n+1}^r)^2 + \sum_{r=n+2}^{2m+1} \sum_{j=2}^{2m+1} (\sigma_{1j}^r)^2. \end{aligned}$$

or

$$\text{Ric}(e_1) = \frac{1}{4} \left\{ (n+1)^2 \|H\|^2 + n(c+3) + \right.$$

$$(13) \quad + \left(3 \|Pe_1\|^2 - (n-1)\eta(e_1)^2 - 1 \right) (c-1) \Big\} - \\ - \frac{1}{4} \sum_{r=n+2}^{2m+1} (\sigma_{11}^r - \sigma_{22}^r - \cdots - \sigma_{n+1 \ n+1}^r)^2 - \sum_{r=n+2}^{2m+1} \sum_{j=2}^{2m+1} (\sigma_{1j}^r)^2.$$

Since $e_1 = X$ can be chosen to be any arbitrary unit vector in T_pM , the above equation implies (9).

In view of (13), the equality case of (9) is valid if and only if

$$(14) \quad \begin{aligned} \sigma_{11}^r &= \sigma_{22}^r + \cdots + \sigma_{n+1 \ n+1}^r, \\ \sigma_{12}^r &= \cdots = \sigma_{1 \ n+1}^r = 0, \quad r \in \{n+2, \dots, 2m+1\}. \end{aligned}$$

If $H(p) = 0$, (14) implies that $e_1 = X$ belongs to the relative null space N_p at p . Conversely, if $e_1 = X$ lies in the relative null space, then (14) holds because $H(p) = 0$ is assumed. This proves statement (ii).

Now, we prove (iii). Assume that the equality case of (9) for all unit tangent vectors to M at $p \in M$ is true. Then, in view of (13), for each $r \in \{n+2, \dots, 2m+1\}$, we have

$$(15) \quad \begin{aligned} 2\sigma_{ii}^r &= \sigma_{11}^r + \cdots + \sigma_{n+1 \ n+1}^r, \quad i \in \{1, \dots, n+1\}, \\ \sigma_{ij}^r &= 0, \quad i \neq j. \end{aligned}$$

Thus, we have two cases, namely either $n = 1$ or $n \neq 1$. In the first case p is a totally umbilical point, while in the second case p is a totally geodesic point. Since $\xi \in TM$, therefore each totally umbilical point is totally geodesic. Thus in both the cases, p is a totally geodesic point. The proof of converse part is straightforward. \square

The above theorem implies the following three results for slant, invariant and anti-invariant submanifolds isometrically immersed in a Sasakian space form.

Theorem 3.3 *Let M be an $(n+1)$ -dimensional θ -slant submanifold isometrically immersed in a $(2m+1)$ -dimensional Sasakian space form $\check{M}(c)$ such that $\xi \in TM$. Then*

(i) *For each unit vector $U \in T_pM$, we have*

$$(16) \quad \begin{aligned} 4Ric(U) &\leq (n+1)^2 \|H\|^2 + n(c+3) \\ &+ \{3 \cos^2 \theta - (n-1 + 3 \cos^2 \theta) \eta(U)^2 - 1\} (c-1). \end{aligned}$$

(ii) *If $H(p) = 0$, a unit vector $U \in T_pM$ satisfies the equality case of (16) if and only if $U \in N_p$.*

(iii) *The equality case of (16) holds for all unit vectors $U \in T_pM$ if and only if M is a totally geodesic submanifold.*

Proof. A θ -slant submanifold M of an almost contact metric manifold satisfies

$$(17) \quad g(PX, PY) = \cos^2 \theta g(\varphi X, \varphi Y), \quad g(FX, FY) = \sin^2 \theta g(\varphi X, \varphi Y)$$

for all $X, Y \in TM$. In view of (17), for a unit vector $U \in T_pM$, we get

$$\|PU\|^2 = g(PU, PU) = \cos^2 \theta (1 - \eta(U)^2).$$

Using this in (9), we get (16). Rest of the proof is similar to that of Theorem 3.2. \square

Theorem 3.4 *Let M be an $(n + 1)$ -dimensional invariant submanifold isometrically immersed in a $(2m + 1)$ -dimensional Sasakian space form $\tilde{M}(c)$ such that $\xi \in TM$. Then,*

(i) *For each unit vector $U \in T_pM$, we have*

$$(18) \quad 4Ric(U) \leq (n + 1)^2 \|H\|^2 + n(c + 3) + \{2 - (n + 2)\eta(U)^2\}(c - 1).$$

(ii) *If $H(p) = 0$, a unit vector $U \in T_pM$ satisfies the equality case of (18) if and only if $U \in \mathcal{N}_p$.*

(iii) *The equality case of (18) holds for all unit vectors $U \in T_pM$ if and only if M is a totally geodesic submanifold.*

Theorem 3.5 *Let M be an $(n + 1)$ -dimensional anti-invariant submanifold isometrically immersed in a $(2m + 1)$ -dimensional Sasakian space form $\tilde{M}(c)$ such that $\xi \in TM$. Then,*

(i) *For each unit vector $U \in T_pM$, we have*

$$(19) \quad 4Ric(U) \leq (n + 1)^2 \|H\|^2 + n(c + 3) - \{(n - 1)\eta(U)^2 + 1\}(c - 1).$$

(ii) *If $H(p) = 0$, a unit vector $U \in T_pM$ satisfies the equality case of (19) if and only if $U \in \mathcal{N}_p$.*

(iii) *The equality case of (19) holds for all unit vectors $U \in T_pM$ if and only if M is a totally geodesic submanifold.*

We also have the following

Theorem 3.6 *Let M be an $(n + 1)$ -dimensional CR-submanifold in a Sasakian space form $\tilde{M}(c)$. Then, the following statements are true.*

1. *For each unit vector $U \in \mathcal{D}$, we have*

$$(20) \quad 4Ric(U) \leq (n + 1)^2 \|H\|^2 + (n + 2)c + 3n - 2.$$

2. *For each unit vector $U \in \mathcal{D}^\perp$, we have*

$$(21) \quad 4Ric(U) \leq (n + 1)^2 \|H\|^2 + (n - 1)c + 3n + 1.$$

3. *If $H(p) = 0$, a unit vector $U \in \mathcal{D}$ (resp. \mathcal{D}^\perp) satisfies the equality case of (20) (resp. (21)) if and only if $U \in \mathcal{N}_p$.*

4 Shape operator for slant immersion

Let M be an $(n + 1)$ -dimensional θ -slant submanifold in a $(2m + 1)$ -dimensional Sasakian space form $\tilde{M}(c)$ such that $\xi \in TM$. Let $p \in M$ and a number

$$b > \frac{c + 3}{4} + \frac{3(c - 1)}{4} \cos^2 \theta$$

such that the sectional curvature $K \geq b$ at p . We choose an orthonormal basis $\{e_1, \dots, e_{n+1} = \xi, e_{n+2}, \dots, e_{2m+1}\}$ at p such that e_{n+2} is parallel to the mean curvature vector H , and e_1, \dots, e_{n+1} diagonalize the shape operator A_{n+2} . Then we have

$$(22) \quad A_{n+2} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n+1} \end{pmatrix},$$

$$(23) \quad A_r = (\sigma_{ij}^r), \quad \text{trace } A_r = \sum_{i=1}^{n+1} \sigma_{ii}^r = 0, \quad i, j = 1, \dots, n+1; r = n+3, \dots, 2m+1.$$

For $i \neq j$, we put

$$(24) \quad u_{ij} \equiv a_i a_j = u_{ji}.$$

In view of Gauss equation (6), for $X = Z = e_i$, $Y = W = e_j$, we have

$$(25) \quad u_{ij} \geq b - \frac{c+3}{4} - \frac{3(c-1)}{4} g(e_i, P e_j)^2 - \sum_{r=n+3}^{2m+1} (\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2).$$

Now, we prove the following Lemma.

Lemma 4.1 For u_{ij} we have

(1) For any fixed $i \in \{1, \dots, n+1\}$, we find

$$\sum_{i \neq j} u_{ij} \geq n \left(b - \frac{c+3}{4} - \frac{3(c-1)}{4} \cos^2 \theta \right).$$

(2) For distinct $i, j, k \in \{1, \dots, n+1\}$ it follows that $a_i^2 = u_{ij} u_{ik} / u_{jk}$.

(3) For a fixed k , $1 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$ and for each $B \in S_k \equiv \{B \subset \{1, \dots, n+1\} : |B| = k\}$, we have

$$\sum_{j \in B} \sum_{t \in \bar{B}} u_{jt} \geq k(n-k+1) \left(b - \frac{c+3}{4} - \frac{3(c-1)}{4} \cos^2 \theta \right),$$

where \bar{B} is the complement of B in $\{1, \dots, n+1\}$.

(4) For distinct $i, j \in \{1, \dots, n+1\}$, it follows that $u_{ij} > 0$.

Proof. (1) From (23), (24) and (25), we obtain

$$\begin{aligned} \sum_{i \neq j} u_{ij} &\geq n \left(b - \frac{c+3}{4} - \frac{3(c-1)}{4} \cos^2 \theta \right) - \sum_{r=n+3}^{2m+1} \left(\sigma_{ii}^r \left(\sum_{j \neq i} \sigma_{jj}^r \right) - \sum_{j \neq i} (\sigma_{ij}^r)^2 \right) = \\ &= n \left(b - \frac{c+3}{4} - \frac{3(c-1)}{4} \cos^2 \theta \right) - \sum_{r=n+3}^{2m+1} \left(\sigma_{ii}^r (-\sigma_{ii}^r) - \sum_{j \neq i} (\sigma_{ij}^r)^2 \right) = \\ &= n \left(b - \frac{c+3}{4} - \frac{3(c-1)}{4} \cos^2 \theta \right) + \sum_{r=n+3}^{2m+1} \sum_{j=1}^{n+1} (\sigma_{ij}^r)^2 \geq \\ &\geq n \left(b - \frac{c+3}{4} - \frac{3(c-1)}{4} \cos^2 \theta \right) > 0. \end{aligned}$$

(2) We have $u_{ij}u_{ik}/u_{jk} = a_i a_j a_i a_k / a_j a_k = a_i^2$.

(3) Let $B = \{1, \dots, k\}$ and $\bar{B} = \{k + 1, \dots, n + 1\}$. Then

$$\begin{aligned} \sum_{j \in B} \sum_{t \in \bar{B}} u_{jt} &\geq k(n - k + 1) \left(b - \frac{c + 3}{4} - \frac{3(c - 1)}{4} \cos^2 \theta \right) - \\ &\quad - \sum_{r=n+3}^{2m+1} \left(\sum_{j=1}^k \sum_{t=k+1}^{n+1} [\sigma_{jj}^r \sigma_{tt}^r - (\sigma_{jt}^r)^2] \right) = \\ &= k(n - k + 1) \left(b - \frac{c + 3}{4} - \frac{3(c - 1)}{4} \cos^2 \theta \right) + \\ &\quad + \sum_{r=n+3}^{2m+1} \left(\sum_{j=1}^k \sum_{t=k+1}^{n+1} (\sigma_{jt}^r)^2 + \sum_{j=1}^k (\sigma_{jj}^r) \right) \geq \\ &\geq k(n - k + 1) \left(b - \frac{c + 3}{4} - \frac{3(c - 1)}{4} \cos^2 \theta \right). \end{aligned}$$

(4) For $i \neq j$, if $u_{ij} = 0$ then $a_i = 0$ or $a_j = 0$. The statement $a_i = 0$ implies that $u_{il} = a_i a_l = 0$ for all $l \in \{1, \dots, n + 1\}$, $l \neq i$. Then, we get

$$\sum_{j \neq i} u_{ij} = 0,$$

which is a contradiction with (1). Thus, for $i \neq j$, it follows that $u_{ij} \neq 0$. We assume that $u_{1 \ n+1} < 0$. From (2), for $1 < i < n + 1$, we get $u_{1i}u_{i \ n+1} < 0$. Without loss of generality, we may assume

$$(26) \quad \begin{aligned} &u_{12}, \dots, u_{1l}, u_{l+1 \ n+1}, \dots, u_{n \ n+1} > 0, \\ &u_{1 \ l+1}, \dots, u_{1 \ n+1}, u_{2 \ n+1}, \dots, u_{l \ n+1} < 0, \end{aligned}$$

for some $\lfloor \frac{n}{2} + 1 \rfloor \leq l \leq n$. If $l = n$, then $u_{1 \ n+1} + u_{2 \ n+1} + \dots + u_{n \ n+1} < 0$, which contradicts to (1). Thus, $l < n$. From (2), we get:

$$(27) \quad a_{n+1}^2 = \frac{u_{i \ n+1} u_{t \ n+1}}{u_{i \ t}} > 0,$$

where $2 \leq i \leq l$, $l + 1 \leq t \leq n$. By (26) and (27), we obtain $u_{it} < 0$, which implies that

$$\sum_{i=1}^l \sum_{t=l+1}^{n+1} u_{it} = \sum_{i=2}^l \sum_{t=l+1}^n u_{it} + \sum_{i=1}^l u_{i \ n+1} + \sum_{t=l+1}^{n+1} u_{1t} < 0,$$

which is a contradiction to (3). Thus (4) is proved. □

B.-Y. Chen establishes a sharp relationship between the shape operator and the sectional curvature for submanifolds in real space forms [5]. In the following theorem, we establish a similar inequality between the shape operator and the sectional curvature for slant submanifolds in a Sasakian space form.

Theorem 4.2 *Let M be an $(n + 1)$ -dimensional slant submanifold isometrically immersed in a $(2m + 1)$ -dimensional Sasakian space form $\tilde{M}(c)$. If at a point $p \in M$ there exists a number $b > (c + 3)/4 + (3/4)(c - 1)\cos^2\theta$ such that the sectional curvature $K \geq b$ at p , then the shape operator A_H at the mean curvature vector satisfies*

$$(28) \quad A_H > \frac{n}{n+1} \left(b - \frac{c+3}{4} - \frac{3(c-1)}{4} \cos^2\theta \right) I_n, \quad \text{at } p,$$

where I_n is the identity map.

Proof. Let $p \in M$ and a number $b > (c + 3)/4 + (3/4)(c - 1)\cos^2\theta$ such that the sectional curvature $K \geq b$ at p . We choose an orthonormal basis $\{e_1, \dots, e_{n+1}, e_{n+2}, \dots, e_{2m+1}\}$ at p such that e_{n+2} is parallel to the mean curvature vector H , and e_1, \dots, e_{n+1} diagonalize the shape operator A_{n+2} . Now, from Lemma 4.1 it follows that a_1, \dots, a_{n+1} have the same sign. We assume that $a_j > 0$ for all $j \in \{1, \dots, n+1\}$. Then

$$(29) \quad \sum_{j \neq i} u_{ij} = a_i(a_1 + \dots + a_{n+1}) - a_i^2 \geq n \left(b - \frac{c+3}{4} - \frac{3(c-1)}{4} \cos^2\theta \right).$$

From (29) and (22), we obtain

$$\begin{aligned} a_i(n+1)\|H\| &\geq n \left(b - \frac{c+3}{4} - \frac{3(c-1)}{4} \cos^2\theta \right) + a_i^2 \\ &> n \left(b - \frac{c+3}{4} - \frac{3(c-1)}{4} \cos^2\theta \right), \end{aligned}$$

which implies that

$$a_i\|H\| > \frac{n}{n+1} \left(b - \frac{c+3}{4} - \frac{3(c-1)}{4} \cos^2\theta \right).$$

Hence, we get (28). \square

In particular, the above theorem implies the following two theorems.

Theorem 4.3 *Let M be an $(n + 1)$ -dimensional anti-invariant submanifold isometrically immersed in a $(2m + 1)$ -dimensional Sasakian space form $\tilde{M}(c)$ such that $\xi \in TM$. If at a point $p \in M$ there exists a number $b > (c + 3)/4$ such that the sectional curvature $K \geq b$ at p , then the shape operator A_H at the mean curvature vector satisfies*

$$(30) \quad A_H > \frac{n}{n+1} \left(b - \frac{c+3}{4} \right) I_n, \quad \text{at } p.$$

Theorem 4.4 *Let M be an $(n + 1)$ -dimensional invariant submanifold isometrically immersed in a $(2m + 1)$ -dimensional Sasakian space form $\tilde{M}(c)$ such that $\xi \in TM$. If at a point $p \in M$ there exists a number $b > c$ such that the sectional curvature $K \geq b$ at p , then the shape operator A_H at the mean curvature vector satisfies*

$$(31) \quad A_H > \frac{n}{n+1} (b - c) I_n, \quad \text{at } p.$$

The above equation is same as equation (5.1) of Theorem 4.1 in the paper of B.-Y. Chen [5].

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