Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 10 Issue 1(2018), Pages 55-67.

# $k$-FRACTIONAL INTEGRAL INEQUALITIES FOR HARMONICALLY CONVEX FUNCTIONS VIA CAPUTO $k$-FRACTIONAL DERIVATIVES

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Abstract. Fractional integral and differential inequalities provide the bounds for the uniqueness of solutions of the fractional differential equations. In this paper we compute some new Hadamard and the Fejér-Hadamard fractional inequalities for harmonically convex functions via Caputo $k$-fractional derivatives. Also results for Caputo fractional derivatives have been induced.

## 1. Introduction

In 1695, while corresponding with each other G. W. Leibniz and Marquis de L'Hospital raised the question of semi-derivatives. This question laid the foundation of fractional calculus which is the generalization of the classical calculus. Since 19th century, the theory of fractional calculus developed very fast due to its wide applications in almost all fields of applied sciences. Fractional calculus is as much important as calculus. Fractional integration and fractional differentiation appear as tools in the subject of partial differential equations [14, 16]. Many types of fractional integral as well as differential operators have been defined in literature, the most classical Caputo fractional derivatives are defined as follows:

**Definition 1.1.** Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, ...\}$, $n = [\alpha] + 1$, $f \in AC^n[a, b]$, the space of functions having $nth$ derivatives absolutely continuous. The right-sided and left-sided Caputo fractional derivatives of order $\alpha$ are defined as follows:

$$({}^C D^{\alpha}_{a+} f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, x > a$$

and

$$({}^C D^{\alpha}_{b-} f)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt, x < b.$$

2000 *Mathematics Subject Classification.* 26B15,26A51,26A33.

*Key words and phrases.* Harmonically convex functions, Hadamard inequality, $k$-fractional integrals.



Submitted January 1, 2018. Published February 1, 2018.
Communicated by Mircea Ivan.

If $\alpha = n \in \{1, 2, 3, ...\}$ and usual derivative $f^{(n)}(x)$ of order $n$ exists, then Caputo fractional derivative $({}^C D^n_{a+} f)(x)$ coincides with $f^{(n)}(x)$ whereas $({}^C D^n_{b-} f)(x)$ coincides with $f^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$. In particular we have

$$({}^C D^0_{a+} f)(x) = ({}^C D^0_{b-} f)(x) = f(x)$$

where $n = 1$ and $\alpha = 0$.

In the following we define Caputo $k$-fractional derivatives [9].

**Definition 1.2.** Let $\alpha > 0, k \geq 1$ and $\alpha \notin \{1, 2, 3, ...\}$, $n = [\alpha] + 1$, $f \in AC^n[a, b]$. The Caputo $k$-fractional derivatives of order $\alpha$ are defined as follows:

$${}^C D^{\alpha,k}_{a+} f(x) = \frac{1}{k\Gamma_k(n - \frac{\alpha}{k})} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\frac{\alpha}{k}-n+1}} dt, x > a$$

and

$${}^C D^{\alpha,k}_{b-} f(x) = \frac{(-1)^n}{k\Gamma_k(n - \frac{\alpha}{k})} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\frac{\alpha}{k}-n+1}} dt, x < b$$

where $\Gamma_k(\alpha)$ is the $k$-Gamma function defined as

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{\frac{-t^k}{k}} dt,$$

also

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha).$$

If $\alpha = n \in \{1, 2, 3, ...\}$ and usual derivative $f^{(n)}(x)$ of order $n$ exists, then Caputo $k$-fractional derivative $({}^C D^{\alpha,1}_{a+} f)(x)$ coincides with $f^{(n)}(x)$, whereas $({}^C D^{\alpha,1}_{b-} f)(x)$ coincides with $f^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$.

In particular we have

$$({}^C D^{0,1}_{a+} f)(x) = ({}^C D^{0,1}_{b-} f)(x) = f(x)$$

where $n, k = 1$ and $\alpha = 0$.
For $k = 1$, Caputo $k$-fractional derivatives give the definition of Caputo fractional derivatives.

We use in the whole paper the convolution $f * g$ of functions $f$ and $g$ for Caputo $k$-fractional derivatives as follows

$${}^C D^{\alpha,k}_{a+} (f * g)(x) = \frac{1}{k\Gamma_k(n - \frac{\alpha}{k})} \int_a^x \frac{f^{(n)}(t) g^{(n)}(t)}{(x-t)^{\frac{\alpha}{k}-n+1}} dt, x > a$$

and

$${}^C D^{\alpha,k}_{b-} (f * g)(x) = \frac{(-1)^n}{k\Gamma_k(n - \frac{\alpha}{k})} \int_x^b \frac{f^{(n)}(t) g^{(n)}(t)}{(t-x)^{\frac{\alpha}{k}-n+1}} dt, x < b.$$

In the following we define convex function and functions due to convex function.

**Definition 1.3.** A function $f : [a, b] \to \mathbb{R}$ is said to be convex if for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y).$$

If above inequality is reversed, then $f$ is said to be concave function.

**Theorem 1.1.** *Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a, b \in I, a < b$. Then the following inequality holds*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

If $f$ is concave, then the above inequality holds in the reverse direction. It is well known in the literature as the Hadamard inequality.
Many generalizations and refinements of the Hadamard inequality have been made by many researchers. For details one can see [1, 2, 3, 4, 6, 7, 8, 10, 11, 13, 18, 20, 21] and the references therein.

Many other kinds of convex functions have been defined by mathematicians. For example $p$-convex function, harmonically convex function, $s$-convex function etc. In [22], İscan defined harmonically convex function as follows.

**Definition 1.4.** Let $I \subset \mathbb{R}\backslash\{0\}$ be a real interval. A function $f : I \longrightarrow \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{1.1}$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (1.1) is reversed, then $f$ is said to be harmonically concave.

In [22], İscan proved the following Hadamard type inequality for harmonically convex functions.

**Theorem 1.2.** *Let $f : I \subset \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$, then the following inequality holds*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}.$$

In [15], Latif et al. defined harmonically symmetric function as follows.

**Definition 1.5.** A function $f : [a, b] \subset \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$ is said to be harmonically symmetric with respect to $\frac{2ab}{a+b}$, if

$$f(x) = f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)$$

holds for all $x \in [a, b]$.

In [5], Chen and Wu represented the Fejér-Hadamard inequality for harmonically convex functions as follows.

**Theorem 1.3.** *Let $f : I \subset \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$ be a harmonically convex function and $a, b \in I$, with $a < b$. If $f \in L[a, b]$ and $g : [a, b] \subset \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then*

$$f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx \leq \int_a^b \frac{f(x)g(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx.$$

In [24], İscan and Wu presented the Hadamard inequality for harmonically convex functions via fractional integrals as follows.

**Theorem 1.4.** *Let $f : I \subset (0,\infty) \longrightarrow \mathbb{R}$ be a function such that $f \in L[a,b]$, where $a,b \in I$ with $a < b$. If $f$ is harmonically convex function on $[a,b]$, then the following inequalities for fractional integrals hold*

$$\begin{aligned} & f\left(\frac{2ab}{a+b}\right) \\ & \leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{a}-}(fog)\left(\frac{1}{b}\right)+J^{\alpha}_{\frac{1}{b}+}(fog)\left(\frac{1}{a}\right)\right] \\ & \leq \frac{f(a)+f(b)}{2} \end{aligned}$$

*with $\alpha > 0$ and $g(x) = \frac{1}{x}$, $x \in [\frac{1}{b}, \frac{1}{a}]$.*

In [23], İscan proved the Fejér-Hadamard inequality for harmonically convex functions via fractional integrals as follows.

**Theorem 1.5.** *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a harmonically convex function with $a < b$ and $f \in L[a,b]$. If $g : [a,b] \longrightarrow \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then the following inequalities for fractional integrals hold*

$$\begin{aligned} & f\left(\frac{2ab}{a+b}\right)\left[J^{\alpha}_{\frac{1}{b}+}(goh)\left(\frac{1}{a}\right)+J^{\alpha}_{\frac{1}{a}-}(goh)\left(\frac{1}{b}\right)\right] \\ & \leq \left[J^{\alpha}_{\frac{1}{b}+}(fgoh)\left(\frac{1}{a}\right)+J^{\alpha}_{\frac{1}{a}-}(fgoh)\left(\frac{1}{b}\right)\right] \\ & \leq \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{\frac{1}{b}+}(goh)\left(\frac{1}{a}\right)+J^{\alpha}_{\frac{1}{a}-}(goh)\left(\frac{1}{b}\right)\right] \end{aligned}$$

*with $\alpha > 0$ and $h(x) = \frac{1}{x}$, $x \in [\frac{1}{b}, \frac{1}{a}]$.*

In [5, 22, 23, 24] İscan, Chen and Wu established different versions of fractional inequalities for harmonically convex functions via fractional integrals. In this paper we are interested to establish some new versions of the Hadamard and the Fejér-Hadamard inequalities for harmonically convex functions via Caputo $k$-fractional derivatives. We also obtain fractional inequalities for Caputo fractional derivatives.

## 2. Main Results

In this section we prove the Hadamard and the Fejér-Hadamard inequalities for harmonically convex functions via Caputo $k$-fractional derivatives.

**Theorem 2.1.** *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a differentiable function such that $f^{(n)}$ be harmonically convex with $a < b$ and $f^{(n)} \in L[a,b]$. If $g : [a,b] \longrightarrow \mathbb{R}$ be a function such that $g^{(n)}$ is a nonnegative, integrable and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then the following inequalities for Caputo $k$-fractional derivatives hold*

$$\begin{aligned} & 2(-1)^n f^{(n)}\left(\frac{2ab}{a+b}\right) {}^{C}D^{\alpha,k}_{\frac{1}{a}-}g\left(h\left(\frac{1}{b}\right)\right) \qquad (2.1) \\ & \leq \left[{}^{C}D^{\alpha,k}_{\frac{1}{b}+}((f*g)\circ h)\left(\frac{1}{a}\right)+(-1)^n\,{}^{C}D^{\alpha,k}_{\frac{1}{a}-}((f*g)\circ h)\left(\frac{1}{b}\right)\right] \\ & \leq (-1)^n\left[f^{(n)}(a)+f^{(n)}(b)\right]{}^{C}D^{\alpha,k}_{\frac{1}{a}-}g\left(h\left(\frac{1}{b}\right)\right) \end{aligned}$$

*with* $\alpha > 0$, $k \geq 1$ *and* $h(x) = \frac{1}{x}$, $x \in [\frac{1}{b}, \frac{1}{a}]$.

*Proof.* As $f^{(n)}$ is harmonically convex function. Putting $t = 1/2$ and $x = \frac{ab}{tb+(1-t)a}$, $y = \frac{ab}{ta+(1-t)b}$ in (1.1) we get

$$2f^{(n)}\left(\frac{2ab}{a+b}\right) \leq f^{(n)}\left(\frac{ab}{tb+(1-t)a}\right) + f^{(n)}\left(\frac{ab}{ta+(1-t)b}\right). \tag{2.2}$$

Multiplying above inequality by $\frac{g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}$ and integrating over $[0,1]$ we have

$$\begin{aligned}
&2f^{(n)}\left(\frac{2ab}{a+b}\right)\int_0^1 \frac{g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}dt\\
&\leq \int_0^1 \frac{g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}} f^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)dt\\
&+\int_0^1 \frac{g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}} f^{(n)}\left(\frac{ab}{ta+(1-t)b}\right)dt.
\end{aligned}$$

Setting $\frac{tb+(1-t)a}{ab} = x$ in above inequality we have

$$\begin{aligned}
&2f^{(n)}\left(\frac{2ab}{a+b}\right)\int_{1/b}^{1/a} \frac{g^{(n)}(1/x)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx\\
&\leq \int_{1/b}^{1/a} \frac{f^{(n)}(1/x)\,g^{(n)}(1/x)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx\\
&+\int_{1/b}^{1/a} \frac{f^{(n)}\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)g^{(n)}(1/x)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx.
\end{aligned}$$

On right hand side substituting $\frac{1}{a}+\frac{1}{b}-x = y$ in second integral of above inequality and $g^{(n)}$ is harmonically symmetric with respect to $\frac{2ab}{a+b}$, we have

$$\begin{aligned}
&2f^{(n)}\left(\frac{2ab}{a+b}\right)\int_{1/b}^{1/a} \frac{g^{(n)}(1/x)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx\\
&\leq \int_{1/b}^{1/a} \frac{f^{(n)}(1/x)\,g^{(n)}(1/x)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx + \int_{1/b}^{1/a} \frac{f^{(n)}(1/y)\,g^{(n)}(1/y)}{\left(\frac{1}{a}-y\right)^{\frac{\alpha}{k}-n+1}}dy
\end{aligned}$$

$$\begin{aligned}
&2(-1)^n f^{(n)}\left(\frac{2ab}{a+b}\right)\, {}^{C}D^{\alpha,k}_{\frac{1}{a}-} g\left(h\left(\frac{1}{b}\right)\right)\\
&\leq \left[{}^{C}D^{\alpha,k}_{\frac{1}{b}+}((f*g)\circ h)\left(\frac{1}{a}\right) + (-1)^n\, {}^{C}D^{\alpha,k}_{\frac{1}{a}-}((f*g)\circ h)\left(\frac{1}{b}\right)\right].
\end{aligned}$$

First inequality of (2.1) is proved. To prove the second inequality of (2.1) we use the following inequality as $f^{(n)}$ is harmonically convex function.

$$f^{(n)}\left(\frac{ab}{tb+(1-t)a}\right) + f^{(n)}\left(\frac{ab}{ta+(1-t)b}\right) \leq f^{(n)}(a) + f^{(n)}(b). \tag{2.3}$$

Again multiplying above inequality by $\frac{g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}$ and integrating over $[0,1]$ we have

$$\begin{aligned}&\int_0^1 f^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)\frac{g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}dt\\&+\int_0^1 \frac{g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}f^{(n)}\left(\frac{ab}{ta+(1-t)b}\right)dt\\&\leq\left[f^{(n)}(a)+f^{(n)}(b)\right]\int_0^1 \frac{g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}dt.\end{aligned}$$

Setting $\frac{tb+(1-t)a}{ab}=x$ in above inequality we have

$$\begin{aligned}&\left[{}^CD^{\alpha,k}_{\frac{1}{b}+}((f*g)\circ h)\left(\frac{1}{a}\right)+(-1)^n\,{}^CD^{\alpha,k}_{\frac{1}{a}-}((f*g)\circ h)\left(\frac{1}{b}\right)\right]\\&\leq(-1)^n\left[f^{(n)}(a)+f^{(n)}(b)\right]\,{}^CD^{\alpha,k}_{\frac{1}{a}-}g\left(h\left(\frac{1}{b}\right)\right).\end{aligned}$$

That is the second inequality of (2.1). □

If we put $k=1$ in (2.1) we get the following result for Caputo fractional derivatives.

**Corollary 2.2.** *Let $f:[a,b]\longrightarrow\mathbb{R}$ be a differentiable function such that $f^{(n)}$ be harmonically convex with $a<b$ and $f^{(n)}\in L[a,b]$. If $g:[a,b]\longrightarrow\mathbb{R}$ be a function such that $g^{(n)}$ is a nonnegative, integrable and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then the following inequalities for Caputo fractional derivatives hold*

$$\begin{aligned}&2(-1)^n f^{(n)}\left(\frac{2ab}{a+b}\right)\,{}^CD^{\alpha}_{\frac{1}{a}-}g\left(h\left(\frac{1}{b}\right)\right)\\&\leq\left[{}^CD^{\alpha}_{\frac{1}{b}+}((f*g)\circ h)\left(\frac{1}{a}\right)\right.\\&\left.+(-1)^n\,{}^CD^{\alpha}_{\frac{1}{a}-}((f*g)\circ h)\left(\frac{1}{b}\right)\right]\\&\leq(-1)^n\left[f^{(n)}(a)+f^{(n)}(b)\right]\,{}^CD^{\alpha}_{\frac{1}{a}-}g\left(h\left(\frac{1}{b}\right)\right)\end{aligned}$$

*with $\alpha>0$, $g(x)=\frac{1}{x}$ , $x\in[\frac{1}{b},\frac{1}{a}]$.*

**Theorem 2.3.** *Let $f:I\subset(0,\infty)\longrightarrow\mathbb{R}$ be a function such that $f\in L[a,b]$, where $a,b\in I$ with $a<b$. If $f^{(n)}$ is harmonically convex function on $[a,b]$, then the following inequalities for k-fractional integrals hold*

$$\begin{aligned}&\frac{2}{\left(n-\frac{\alpha}{k}\right)}f^{(n)}\left(\frac{2ab}{a+b}\right)\\&\leq\left(\frac{ab}{b-a}\right)^{n-\frac{\alpha}{k}}k\Gamma_k\left(n-\frac{\alpha}{k}\right)\left[(-1)^n\,{}^CD^{\alpha,k}_{\frac{1}{a}-}f\left(g\left(\frac{1}{b}\right)\right)+{}^CD^{\alpha,k}_{\frac{1}{b}+}f\left(g\left(\frac{1}{a}\right)\right)\right]\\&\leq\left[\frac{f^{(n)}(a)+f^{(n)}(b)}{\left(n-\frac{\alpha}{k}\right)}\right]\end{aligned}\tag{2.4}$$

*with* $\alpha > 0$, $k \geq 1$ *and* $g(x) = \frac{1}{x}$ , $x \in [\frac{1}{b}, \frac{1}{a}]$.

*Proof.* Multiplying inequality (2.2) by $t^{n-\left(1+\frac{\alpha}{k}\right)}$ and integrating over $[0,1]$ we have

$$\begin{aligned} & 2f^{(n)}\left(\frac{2ab}{a+b}\right)\int_0^1 t^{n-\left(1+\frac{\alpha}{k}\right)}dt \\ & \leq \int_0^1 \frac{f^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}dt \\ & + \int_0^1 \frac{f^{(n)}\left(\frac{ab}{ta+(1-t)b}\right)}{t^{\frac{\alpha}{k}-n+1}}dt. \end{aligned}$$

Setting $\frac{tb+(1-t)a}{ab} = x$ and $\frac{ta+(1-t)b}{ab} = y$ in above inequality we have

$$\begin{aligned} & \frac{2}{\left(n-\frac{\alpha}{k}\right)}f^{(n)}\left(\frac{2ab}{a+b}\right) \\ & \leq \left(\frac{ab}{b-a}\right)^{n-\frac{\alpha}{k}}\left[\int_{1/b}^{1/a}\frac{f^{(n)}(1/x)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx \right. \\ & \left. + \int_{1/b}^{1/a}\frac{f^{(n)}(1/y)}{\left(\frac{1}{a}-y\right)^{\frac{\alpha}{k}-n+1}}dy\right] \\ & = \left(\frac{ab}{b-a}\right)^{n-\frac{\alpha}{k}}\left[\int_{1/b}^{1/a}\frac{f^{(n)}(g(x))}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx \right. \\ & \left. + \int_{1/b}^{1/a}\frac{f^{(n)}(g(y))}{\left(\frac{1}{a}-y\right)^{\frac{\alpha}{k}-n+1}}dy\right] \\ & = \left(\frac{ab}{b-a}\right)^{n-\frac{\alpha}{k}} k\Gamma_k\left(n-\frac{\alpha}{k}\right)\left[(-1)^n\, {}^{C}D^{\alpha,k}_{\frac{1}{a}-}f\left(g\left(\frac{1}{b}\right)\right) + {}^{C}D^{\alpha,k}_{\frac{1}{b}+}f\left(g\left(\frac{1}{a}\right)\right)\right]. \end{aligned}$$

That is the left inequality of (2.4). As $f^{(n)}$ is harmonically convex, multiplying (2.3) by $t^{n-\left(1+\frac{\alpha}{k}\right)}$ and integrating over $[0,1]$ we have

$$\begin{aligned} & \left(\frac{ab}{b-a}\right)^{n-\frac{\alpha}{k}} k\Gamma_k\left(n-\frac{\alpha}{k}\right)\left[(-1)^n\, {}^{C}D^{\alpha,k}_{\frac{1}{a}-}f\left(g\left(\frac{1}{b}\right)\right) + {}^{C}D^{\alpha,k}_{\frac{1}{b}+}f\left(g\left(\frac{1}{a}\right)\right)\right] \\ & \leq \left[\frac{f^{(n)}(a)+f^{(n)}(b)}{\left(n-\frac{\alpha}{k}\right)}\right]. \end{aligned}$$

This completes the proof of rigth inequality of (2.4). □

If we put $k = 1$ in (2.4) we get the following result for Caputo fractional derivatives.

**Corollary 2.4.** *Let* $f : I \subset (0, \infty) \longrightarrow \mathbb{R}$ *be a function such that* $f \in L[a, b]$, *where* $a, b \in I$ *with* $a < b$. *If* $f^{(n)}$ *is harmonically convex function on* $[a, b]$, *then* *the following inequalities for fractional integrals hold*

$$
\begin{aligned}
& f^{(n)}\left(\frac{2ab}{a+b}\right) \\
& \leq \left(\frac{ab}{b-a}\right)^{n-\alpha} \frac{(n-\alpha)\,\Gamma\,(n-\alpha)}{2}\left[(-1)^n\,{}^{C}D^{\alpha}_{\frac{1}{a}-}f\left(g\left(\frac{1}{b}\right)\right)+{}^{C}D^{\alpha}_{\frac{1}{b}+}f\left(g\left(\frac{1}{a}\right)\right)\right] \\
& \leq \left[\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\right]
\end{aligned}
$$

*with* $\alpha > 0$, $g(x) = \frac{1}{x}$ , $x \in [\frac{1}{b}, \frac{1}{a}]$.

**Theorem 2.5.** *Let* $f : [a, b] \longrightarrow \mathbb{R}$ *be a fuction such that* $f^{(n)}$ *is a harmonically convex function with* $a < b$ *and* $f^{(n)} \in L[a, b]$. *If* $g : [a, b] \longrightarrow \mathbb{R}$ *be a function such that* $g^{(n)}$ *is nonnegative, integrable and harmonically symmetric with respect to* $\frac{2ab}{a+b}$, *then the following inequalities for Caputo k-fractional integrals hold*

$$
\begin{aligned}
& 2(-1)^n f^{(n)}\left(\frac{2ab}{a+b}\right)\,{}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}-}g\left(h\left(\frac{1}{b}\right)\right) \qquad (2.5)\\
& \leq \left[{}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}+}\left((f*g)\circ h\right)\left(\frac{1}{a}\right)\right. \\
& \left. +(-1)^n\,{}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}-}\left((f*g)\circ h\right)\left(\frac{1}{b}\right)\right] \\
& \leq (-1)^n\,{}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}-}g\left(h\left(\frac{1}{b}\right)\right)\left[f^{(n)}(a)+f^{(n)}(b)\right]
\end{aligned}
$$

*with* $\alpha > 0$, $k \geq 1$ *and* $h(x) = \frac{1}{x}$, $x \in [\frac{1}{b}, \frac{1}{a}]$.

*Proof.* Integrating (2.2) over $[0, 1/2]$ after multiplying with $t^{n-\left(1+\frac{\alpha}{k}\right)}g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)$ we have

$$
\begin{aligned}
& 2f^{(n)}\left(\frac{2ab}{a+b}\right)\int_0^{1/2} t^{n-\left(1+\frac{\alpha}{k}\right)}g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)dt \\
& \leq \int_0^{1/2} t^{n-\left(1+\frac{\alpha}{k}\right)}g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)f^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)dt \\
& + \int_0^{1/2} t^{n-\left(1+\frac{\alpha}{k}\right)}g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)f^{(n)}\left(\frac{ab}{ta+(1-t)b}\right)dt.
\end{aligned}
$$

Setting $\frac{ab}{tb+(1-t)a} = 1/x$ in above inequality and doing some calculation leads to

$$
\begin{aligned}
& 2f^{(n)}\left(\frac{2ab}{a+b}\right)\int_{1/b}^{\frac{a+b}{2ab}} \frac{g^{(n)}\left(1/x\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx \\
& \leq \int_{1/b}^{\frac{a+b}{2ab}} \frac{f^{(n)}\left(1/x\right)g^{(n)}\left(1/x\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx + \int_{1/b}^{\frac{a+b}{2ab}} \frac{f^{(n)}\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)g^{(n)}\left(1/x\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx.
\end{aligned}
$$

Setting $\frac{1}{a}+\frac{1}{b}-x=y$ in above inequality and $g^{(n)}$ is harmonically symmetric with respect to $\frac{2ab}{a+b}$ we have

$$\begin{aligned}
&2f^{(n)}\left(\frac{2ab}{a+b}\right)\int_{1/b}^{\frac{a+b}{2ab}}\frac{g^{(n)}\left(1/x\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx\\
&\leq\int_{1/b}^{\frac{a+b}{2ab}}\frac{f^{(n)}\left(1/x\right)g^{(n)}\left(1/x\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx+\int_{\frac{a+b}{2ab}}^{1/a}\frac{f^{(n)}\left(1/y\right)g^{(n)}\left(1/y\right)}{\left(\frac{1}{a}-y\right)^{\frac{\alpha}{k}-n+1}}dy.
\end{aligned}$$

Using the definition of Caputo $k$-fractional derivatives in above inequality we get

$$\begin{aligned}
&2(-1)^n f^{(n)}\left(\frac{2ab}{a+b}\right)\,{}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}-}g\left(h\left(\frac{1}{b}\right)\right)\\
&\leq\left[{}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}+}\left((f*g)\circ h\right)\left(\frac{1}{a}\right)\right.\\
&\left.+(-1)^n\,{}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}-}\left((f*g)\circ h\right)\left(\frac{1}{b}\right)\right].
\end{aligned}$$

First inequality of (2.5) is proved. For the proof of second inequality in (2.5), multiplying (2.3) with $t^{n-\left(1+\frac{\alpha}{k}\right)}g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)$ and integrating over $[0,1/2]$ we get

$$\begin{aligned}
&\int_0^{1/2}f^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)\frac{g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}dt\\
&+\int_0^{1/2}\frac{g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}f^{(n)}\left(\frac{ab}{ta+(1-t)b}\right)dt\\
&\leq\left[f^{(n)}(a)+f^{(n)}(b)\right]\int_0^{1/2}\frac{g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}dt.
\end{aligned}$$

Setting $\frac{ab}{tb+(1-t)a}=1/x$ in above inequality and doing some calculation we get

$$\begin{aligned}
&{}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}+}\left((f*g)\circ h\right)\left(\frac{1}{a}\right)+(-1)^n\,{}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}-}\left((f*g)\circ h\right)\left(\frac{1}{b}\right)\\
&\leq(-1)^n\,{}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}-}g\left(h\left(\frac{1}{b}\right)\right)\left[f^{(n)}(a)+f^{(n)}(b)\right].
\end{aligned}$$

This completes the proof of second inequality in (2.5). □

If we put $k=1$ in (2.5) we get the following result for Caputo fractional derivatives.

**Corollary 2.6.** *Let $f:[a,b]\longrightarrow\mathbb{R}$ be a fuction such that $f^{(n)}$ is a harmonically convex function with $a<b$ and $f^{(n)}\in L[a,b]$. If $g:[a,b]\longrightarrow\mathbb{R}$ be a function such that $g^{(n)}$ is nonnegative, integrable and harmonically symmetric with respect*

*to $\frac{2ab}{a+b}$, then the following inequalities for Caputo fractional integrals hold*

$$\begin{aligned}
&(-1)^n f^{(n)}\left(\frac{2ab}{a+b}\right) {}^C D^{\alpha}_{\frac{a+b}{2ab}-} g\left(h\left(\frac{1}{b}\right)\right) \\
&\leq \frac{1}{2}\left[{}^C D^{\alpha}_{\frac{a+b}{2ab}+}((f*g)\circ h)\left(\frac{1}{a}\right)\right. \\
&\left. +(-1)^n\, {}^C D^{\alpha}_{\frac{a+b}{2ab}-}((f*g)\circ h)\left(\frac{1}{b}\right)\right] \\
&\leq \frac{(-1)^n}{2}\, {}^C D^{\alpha}_{\frac{a+b}{2ab}-} g\left(h\left(\frac{1}{b}\right)\right)\left[f^{(n)}(a)+f^{(n)}(b)\right]
\end{aligned}$$

*with $\alpha > 0$, $h(x) = \frac{1}{x}$, $x \in [\frac{1}{b}, \frac{1}{a}]$.*

**Theorem 2.7.** *Let $f : I \subset (0, \infty) \longrightarrow \mathbb{R}$ be a function such that $f^{(n)} \in L[a, b]$, where $a, b \in I$ with $a < b$. If $f^{(n)}$ is harmonically convex function on $[a, b]$, then the following inequalities for Caputo k-fractional integrals hold*

$$\begin{aligned}
&\frac{2^{1-n+\frac{\alpha}{k}}}{n-\frac{\alpha}{k}} f^{(n)}\left(\frac{2ab}{a+b}\right) \qquad (2.6)\\
&\leq k\Gamma_k(n-\frac{\alpha}{k})\left(\frac{ab}{b-a}\right)^{n-\frac{\alpha}{k}}\left[{}^C D^{\alpha,k}_{\frac{a+b}{2ab}+} f\left(g\left(\frac{1}{a}\right)\right) + (-1)^n\, {}^C D^{\alpha,k}_{\frac{a+b}{2ab}-} f\left(g\left(\frac{1}{b}\right)\right)\right] \\
&\leq \frac{2^{2-n+\frac{\alpha}{k}}}{n-\frac{\alpha}{k}}\left[\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\right]
\end{aligned}$$

*with $\alpha > 0$, $k \geq 1$ and $g(x) = \frac{1}{x}$ , $x \in [\frac{1}{b}, \frac{1}{a}]$.*

*Proof.* Multiplying (2.2) by $t^{n-\left(1+\frac{\alpha}{k}\right)}$ and integrating over $[0, 1/2]$ we have

$$\begin{aligned}
&2f^{(n)}\left(\frac{2ab}{a+b}\right)\int_0^{1/2} t^{n-\left(1+\frac{\alpha}{k}\right)} dt \\
&\leq \int_0^{1/2} t^{n-\left(1+\frac{\alpha}{k}\right)} f^{(n)}\left(\frac{ab}{ta+(1-t)b}\right) dt + \\
&\int_0^{1/2} t^{n-\left(1+\frac{\alpha}{k}\right)} f^{(n)}\left(\frac{ab}{tb+(1-t)a}\right) dt
\end{aligned}$$

$$\begin{aligned}
&\frac{2^{1-n+\frac{\alpha}{k}}}{n-\frac{\alpha}{k}} f^{(n)}\left(\frac{2ab}{a+b}\right) \\
&\leq \int_0^{1/2} t^{n-\left(1+\frac{\alpha}{k}\right)} f^{(n)}\left(\frac{ab}{ta+(1-t)b}\right) dt + \\
&\int_0^{1/2} t^{n-\left(1+\frac{\alpha}{k}\right)} f^{(n)}\left(\frac{ab}{tb+(1-t)a}\right) dt.
\end{aligned}$$

Setting $\frac{ab}{ta+(1-t)b}=1/x$ we get

$$\begin{aligned}
&\frac{2^{1-n+\frac{\alpha}{k}}}{n-\frac{\alpha}{k}} f^{(n)}\left(\frac{2ab}{a+b}\right)\\
&\leq \left(\frac{ab}{b-a}\right)^{n-\frac{\alpha}{k}} \left[\int_{\frac{a+b}{2ab}}^{1/a} \left(\frac{1}{a}-x\right)^{n-\left(1+\frac{\alpha}{k}\right)} f^{(n)}\left(g(x)\right) dx\right.\\
&\left.+\int_{\frac{a+b}{2ab}}^{1/a} \left(\frac{1}{a}-x\right)^{n-\left(1+\frac{\alpha}{k}\right)} f^{(n)}\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right) dx\right]
\end{aligned}$$

Setting $\frac{1}{a}+\frac{1}{b}-x=y$ in second integral on right hand side of above inequality we get

$$\begin{aligned}
&\frac{2^{1-n+\frac{\alpha}{k}}}{n-\frac{\alpha}{k}} f^{(n)}\left(\frac{2ab}{a+b}\right)\\
&\leq \left(\frac{ab}{b-a}\right)^{n-\frac{\alpha}{k}} \left[\int_{\frac{a+b}{2ab}}^{1/a} \left(\frac{1}{a}-x\right)^{n-\left(1+\frac{\alpha}{k}\right)} f^{(n)}\left(1/x\right) dx\right.\\
&\left.+\int_{1/b}^{\frac{a+b}{2ab}} \left(y-\frac{1}{b}\right)^{n-\left(1+\frac{\alpha}{k}\right)} f^{(n)}\left(\frac{1}{y}\right) dy\right].
\end{aligned}$$

Applying the definition of Caputo $k$-fraction derivatives in above inequality we get

$$\begin{aligned}
&\frac{2^{1-n+\frac{\alpha}{k}}}{n-\frac{\alpha}{k}} f^{(n)}\left(\frac{2ab}{a+b}\right)\\
&\leq \left(\frac{ab}{b-a}\right)^{n-\frac{\alpha}{k}} k\Gamma_k(n-\frac{\alpha}{k}) \left[{}^C D^{\alpha,k}_{\frac{a+b}{2ab}+} f\left(g\left(\frac{1}{a}\right)\right)+(-1)^n\, {}^C D^{\alpha,k}_{\frac{a+b}{2ab}-} f\left(g\left(\frac{1}{b}\right)\right)\right].
\end{aligned}$$

First inequality of (2.6) is proved.
For the proof second inequality in (2.6). As $f^{(n)}$ is harmonically convex function we have (2.3). Multiplying (2.3) by $t^{n-\left(1+\frac{\alpha}{k}\right)}$ and integrating over $[0,1/2]$ we get

$$\begin{aligned}
&\int_0^{1/2} t^{n-\left(1+\frac{\alpha}{k}\right)} f^{(n)}\left(\frac{ab}{tb+(1-t)a}\right) dt+\int_0^{1/2} t^{n-\left(1+\frac{\alpha}{k}\right)} f^{(n)}\left(\frac{ab}{ta+(1-t)b}\right) dt\\
&\leq \left[f^{(n)}(a)+f^{(n)}(b)\right]\int_0^{1/2} t^{n-\left(1+\frac{\alpha}{k}\right)} dt
\end{aligned}$$

Setting $\frac{ab}{ta+(1-t)b}=1/x$ in above inequality and doing simple math we get

$$\begin{aligned}
&k\Gamma_k(n-\frac{\alpha}{k})\left(\frac{ab}{b-a}\right)^{n-\frac{\alpha}{k}} \left[{}^C D^{\alpha,k}_{\frac{a+b}{2ab}+} f\left(g\left(\frac{1}{a}\right)\right)+(-1)^n\, {}^C D^{\alpha,k}_{\frac{a+b}{2ab}-} f\left(g\left(\frac{1}{b}\right)\right)\right]\\
&\qquad\leq \frac{2^{2-n+\frac{\alpha}{k}}}{n-\frac{\alpha}{k}}\left[\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\right].
\end{aligned}$$

This completes the proof of second inequality in (2.6). □

If we put $k=1$ in (2.6) we get the following result for Caputo fractional derivatives.

**Corollary 2.8.** *Let $f : I \subset (0,\infty) \longrightarrow \mathbb{R}$ be a function such that $f^{(n)} \in L[a,b]$, where $a,b \in I$ with $a < b$. If $f^{(n)}$ is harmonically convex function on $[a,b]$, then the following inequalities for Caputo fractional integrals hold*

$$\begin{aligned}
&\frac{2^{1-n+\alpha}}{n-\alpha} f^{(n)}\left(\frac{2ab}{a+b}\right) \\
&\leq \left(\frac{ab}{b-a}\right)^{n-\alpha} \Gamma(n-\alpha)\left[{}^{C}D^{\alpha}_{\frac{a+b}{2ab}+} f\left(g\left(\frac{1}{a}\right)\right) + (-1)^{n}\, {}^{C}D^{\alpha}_{\frac{a+b}{2ab}-} f\left(g\left(\frac{1}{b}\right)\right)\right] \\
&\leq \frac{2^{2-n+\alpha}}{n-\alpha}\left[\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\right]
\end{aligned}$$

*with* $\alpha > 0$, $g(x) = \frac{1}{x}$ , $x \in [\frac{1}{b}, \frac{1}{a}]$.

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