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ON A UNIQUENESS CONDITION FOR CR FUNCTIONS ON HYPERSURFACES

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Abstract. Let f be a smooth CR function on a smooth hypersurface $M \subset \mathbb{C}^n$, such that f vanishes to infinite order along a C^∞-smooth curve $\gamma \subset M$. Assume that for each $q \in \gamma$ there exists a truncated double cone C at q in M, such that at least one of the following three conditions holds true: (a) There is a constant $\theta \in \mathbb{R}$, such that $\mathsf{C} \subset \{|\mathrm{Re}(e^{i\theta} f)| \leq |\mathrm{Im}(e^{i\theta} f)|\}$. (b) $\mathsf{C} \subset \{\mathrm{Re}\, f \geq 0\}$. (c) $|f(z)|^{|z-q|} \to 0$, $z \to q$, $z \in \mathsf{C}$. Then f vanishes on an M-open neighborhood of γ.

Contents

1. Introduction and statement of the main result

Our starting point is the following definition of vanishing to infinite order on a submanifold.

Definition 1.1 (See Baouendi & Zachmanoglou [9], p.9)**.** Let $\Omega \subset \mathbb{R}^N$ be an open set and let M and $\gamma \subset M$, be two differentiable submanifolds of Ω. We say that a continuous complex-valued function f, defined on M, *vanishes to infinite order on* γ, if for every $\alpha \in \mathbb{R}$, the function,

$$z \mapsto f(z)(\mathrm{dist}(z, \gamma))^\alpha, \tag{1.1}$$

is bounded in any compact set of M.

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Remark 1.2. Let f, M, γ be as in Definition 1.1 and let $0 \in \gamma$. We automatically know that for $\alpha \geq 0$, $f(z)(\mathrm{dist}(z,\gamma))^\alpha$, is bounded on any compact subset of M, and that for any pair of $\alpha < 0$, $c > 0$, it is bounded on the intersection, of any compact subset of M with $\{z \in M\colon \mathrm{dist}(z,\gamma) \geq c\}$. On the set $\{z \in M\colon \mathrm{dist}(z,\gamma) < 1\}$ it is obvious that $f(z)(\mathrm{dist}(z,\gamma))^\alpha$, is bounded on every compact subset if and only if, for every $k \in \mathbb{N}$, $f(z)(\mathrm{dist}(z,\gamma))^{-k}$, is bounded locally near each point of γ. Hence, vanishing to infinite order on γ, is equivalent to the requirement that $f(z)(\mathrm{dist}(z,\gamma))^{-k}$ is bounded locally near each point of γ, for each fix $k \in \mathbb{N}$. We shall be interested in the version of Definition 1.1, where $\Omega \subset \mathbb{R}^{2n}$ and where we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$. Let $M \subset \mathbb{C}^n$ be a CR submanifold and let $\gamma \subset M$ be a submanifold. We say that a continuous CR function[1] $f : M \to \mathbb{C}$, *vanishes to infinite order on* γ if for every $a \in \gamma$, and $\forall k \in \mathbb{N}$, there exists a constant $C_k > 0$, and $U \subset M$ an open neighborhood of a satisfying,

$$|f(z)| \leq C_k (\mathrm{dist}(z,\gamma))^k, \quad z \in U. \tag{1.2}$$

Note that for any $p \in \gamma$, we have (sufficiently near p), $|f(z)|\cdot|z-p|^{-(k+1)} \leq C_{k+1} \Rightarrow |f(z)| \cdot |z-p|^{-k} \leq C_{k+1}|z-p|$, thus letting $z \to p$, we see that,

$$\lim_{z \to p} \frac{f(z)}{|z-p|^k} = 0, \quad k \in \mathbb{N}, \tag{1.3}$$

(where the case $k = 0$ is due to the fact that $|f(z)| \leq C_1|z-p| \to 0$ as $z \to p$).

In the case of a generic embedded CR submanifolds $M \subset \mathbb{C}^n$, there exists (contrary to the case of complex manifolds) choices of M allowing for smooth CR functions which vanish to infinite order at a point $p \in M$, but not identically, see e.g. Schmalz [25]. In our main result we shall use so-called truncated double cones at a point in a hypersurface.

Definition 1.3. Let M be a C^1-smooth N-dimensional real manifold. We define a set $\mathsf{C}(q) \subset M$ to be a *truncated double cone in* M *at* $q \in M$ if there exists a parametrization of M by local Euclidean coordinates $(x_1, \ldots, x_N)$ centered at q, such that $\mathsf{C}(q)$ is parametrized, in the variables $(x_1, \ldots, x_N)$, by an open nonempty truncated double cone at q in $\mathbb{R}^N$.

For a smooth hypersurface $M \subset \mathbb{C}^n$, where $n \geq 2$, we denote $T^cM := T_pM \cap J_pT_pM$, where J is the complex structure map on $T\mathbb{C}^n$ defined by J_p on each $T_p\mathbb{C}^n$. It is still an open problem, to determine necessary and sufficient conditions, under which a CR^∞ function (by which we mean a C^∞-smooth CR function) on a C^∞-smooth hypersurface, such that the function vanishes to infinite order along a curve, is forced to vanish identically. The work of Nirenberg [22] initiated the following question on unique continuation, see Fornaess & Sibony [12]: Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a domain with smooth boundary. Let γ be a smooth curve in $\partial\Omega$, transverse to $T_p^c(\partial\Omega)$, for every $p \in \gamma$. Does it hold true that if $f \in C^\infty(\overline{\Omega})$, holomorphic on Ω and vanishes to infinite order on γ, then $f \equiv 0$?
In our main result we provide some additional conditions under which we have an affirmative answer to the question.

[1]By which we mean that $Xf = 0$, for all sections X, of $H^{0,1}M$.

Theorem 1.4 (Main result)**.** *Let $M \subset \mathbb{C}^n$, be a C^∞-smooth real hypersurface. Let $\gamma \subset M$ be a real C^∞-smooth curve such that,*

$$T_z^c M + T_z\gamma = T_z M, \quad z \in \gamma. \tag{1.4}$$

Let $f \in CR^\infty(M)$ such that f vanishes to infinite order along γ. Assume that for each $q \in \gamma$, there exists a truncated double cone C at q in M, such that at least one of the following holds true:

(a) *There is a constant $\theta \in \mathbb{R}$, such that $\mathsf{C} \subset \{|\mathrm{Re}(e^{i\theta} f)| \leq |\mathrm{Im}(e^{i\theta} f)|\}$.*
(b) $\mathsf{C} \subset \{\mathrm{Re}\, f \geq 0\}$.
(c) $|f(z)|^{|z-q|} \to 0$, $z \to q$, $z \in \mathsf{C}$.

Then f vanishes on an M-open neighborhood of γ.

2. Preliminary definitions and remarks

Remark 2.1. For a smooth vector field X on an open $\Omega \subset \mathbb{R}^n$ and any point $p \in \Omega$ there exists a unique integral curve, κ, satisfying $\kappa : [0, T) \to \Omega$, (for a maximal T) $\dot{\kappa}(t) = X(\kappa)(t)$, of X which passes through p when $t = 0$ i.e. $\kappa(0) = p$. We shall denote this integral curve by $t \mapsto \Phi_{X,t}(p)$. It is further known that if $X = X(\vartheta) =: X_\vartheta$, i.e., X depends upon a parameter ϑ then $T = T(p, \vartheta)$ is a lower semi-continuous function of (p, ϑ) and $t \mapsto \Phi_t(p)$ is continuous on the set $0 < t < T(p, \vartheta)$, as (p, ϑ) vary on an open neighborhood of $(p, 0)$.[2]

Definition 2.2. Let H be a collection of smooth vector fields on Ω. By a *polygonal path of a finite number of integral curves, of vector fields in H* joining $q' \in \Omega$ to $q \in \Omega$ we mean a piecewise smooth curve $\kappa : [0, 1] \to \Omega$ such that $\kappa(0) = q$, $\kappa(1) = q'$ and $0 = s_0 < s_1 < \cdots < s_k = 1$ such that,

$$\kappa(s) = \Phi_{X^j, t_j(s)}(\kappa(s_{j-1})), \quad s_{j-1} \leq s \leq s_j, \quad 1 \leq j \leq k, \tag{2.1}$$

where $X^j \in H$ and $t_j(s)$ is a smooth diffeomorphism of $[s_{j-1}, s_j]$ onto some closed interval of $\mathbb{R}$ with $t_j(s_{j-1}) = 0$. For $t = (t_1, \ldots, t_k) \in \mathbb{R}^k$ one may use the notation,

$$q' = \Phi_{X^1, t_1 1}(\Phi_{X^2, t_2}(\cdots \Phi_{X^k, t_k}(q) \cdots)), \tag{2.2}$$

for expressing that q' can be reached from q by a polygonal path of integral curves of the vector fields X^j (in the given order). This gives a mapping $\mathbb{R}^k \times \Omega \ni (t, q) \mapsto \Omega$, which for fixed choice of $X^1, \ldots, X^k$ and for t near 0 in $\mathbb{R}^k$, is given by,

$$(t, q) \mapsto \Phi_{X^1, t_1}(\Phi_{X^2, t_2}(\cdots \Phi_{X^k, t_k}(q) \cdots)) =: \Phi_{X,t}(q), \tag{2.3}$$

(where we are using the notation $X = (X^1, \ldots, X^k)$) for more details on this map, see Baouendi et al. [5], p.69.

Definition 2.3. Let $M \subset \mathbb{R}^n$, for a positive integer n, be a submanifold and let $p \in M$. We say that a submanifold $M' \subset \mathbb{R}^n$ is equivalent to M at p, denoted $M \sim_p M'$, if: $p \in M'$ and there exists an open neighborhood $V \subset \mathbb{R}^n$, of p, such that $V \cap M = V \cap M'$. The equivalence class of M, under the equivalence relation $\sim_p$, is called the *germ of the submanifold M at p*. If $N \subset M$ is a submanifold and $p \in N$, then a submanifold $N' \subset \mathbb{R}^n$, is said to belong to the germ of N at p *in* M,

[2]This is a consequence of the fundamental theorem of ODE, see e.g. Hartmann [16], p.94, which is usually stated in terms of a unique solution $\gamma(t) = \eta(t, t_0, \gamma_0, \xi)$, (defined for a maximal interval, which may depend on t_0, γ_0 and the parameters ξ, i.e. $t \in (a(t_0, \gamma_0, \xi), b(t_0, \gamma_0, \xi))$) to the initial value problem $\gamma'(t) = f(t, \gamma, \xi)$, $\gamma(t_0) = \gamma_0$. In our case $f(t, \gamma, \vartheta) = (X_\vartheta \gamma)(t)$, where X is a vector field.

if $N' \subset M$ and belongs to the germ of the submanifold N at p. Any submanifold of $\mathbb{R}^n$ that belongs to the germ of a submanifold M at a point $p \in M$, will be called a *representative* (or *member*) of the germ of the submanifold M at p.

Definition 2.4 (See Baouendi et al. [5], p.94)**.** Let M be a smooth CR manifold and let $p \in M$. By a known theorem (see Baouendi et al. [5], p.68) there exists a C^∞-smooth submanifold $W \subset M$, $p \in W$, satisfying (i) if $p \in W'$, where W' is another C^∞-smooth submanifold to which all vector fields of T^cM are tangent at every point then there is an open $V \subset M$, $p \in V$, with $W \cap V \subset W' \cap V$, (ii) for every open $U \subset M$, $p \in U$, there exists $N \in \mathbb{Z}_+$, and open $V_1 \subset V_2 \subset U$, with $p \in V_1$, such that any $q \in V_1 \cap W$ can be reached by a polygonal path of N integral curves, of vector fields in T^cM, contained in $W \cap V_2$.
We denote by $\mathfrak{o}(p)$, the members of the germ of W at p, in M, such that the tangent space at each point of the member contains T^c_qM. We call $\mathfrak{o}(p)$ the *local CR-orbit at p.*

Any representative of $\mathfrak{o}(p)$ contains a CR submanifold of M which passes through p and whose CR dimension equals the CR dimension of M.

Definition 2.5 (See e.g. Baouendi et al. [5], p.20)**.** Let $M \subset \mathbb{C}^n$ be an embedded CR submanifold and $p_0 \in M$. M is said to be *minimal* at p_0, if there is no real submanifold $S \subset M$, $p_0 \in S$, such that the following two conditions hold true simultaneously: (1) T^c_pM is tangent to S at every $p \in S$. (2) $\dim_{\mathbb{R}} S < \dim_{\mathbb{R}} M$.

If a CR submanifold $M \subset \mathbb{C}^n$, is not minimal at a point $p_0 \in M$, then we shall say that p_0 is a *non-minimal point of M*.

3. Some known results used in the proof of Theorem 1.4.

The problem of unique continuation for CR functions has been studied by many authors, see e.g. Rosay [23], Airapetyan & Khenkin [1], Hunt et al. [18], Baouendi & Treves [7], Alinhac et al. [3], Grachev [14], Schmalz [25], Berhanu & Mendoza [8], Huang et al. [17] and Baouendi & Rothschild [6], Alexander [2] and very recently (in relation to growth conditions) Della Sala & Lamel [11]. Here we mention just a few, which we shall make use of.

Theorem 3.1 (Alinhac et al. [3], p.635)**.** *Let $W \subset \mathbb{C}$ be an open neighborhood of* 0*, let $W^+ := W \cap \{\operatorname{Im} z > 0\}$, and let $A \subset \mathbb{C}^n$ be a totally real C^2-smooth submanifold. Let $F \in \mathscr{O}(W^+)$ and continuous up to the boundary such that F maps $W \cap \{\operatorname{Im} z = 0\}$ into A. If F vanishes to infinite order at the origin then F vanishes identically in the connected component of the origin in $\overline{W^+}$.*

Theorem 3.2 (Huang et al. [17] and Baouendi & Rothschild [6])**.** *If $f(z)$ is a holomorphic function in the intersection, with the upper half plane, of a domain containing* 0*, f continuous up to the boundary, vanishing to infinite order at* 0 *(in the sense that $f(z) = O(|z|^N)$ for every $\mathbb{N} \in \mathbb{N}$) and $Ref(x) \geq 0$, $x :=Rez$, then f must vanish identically.*

Theorem 3.3 (Huang et al. [17], See Remark 5.6 regarding stronger version)**.** *If $f = u + iv$ is holomorphic in $H^+ := \{z \in \mathbb{C} : Imz > 0\}$, and continuous up to $(-1, 1) \subset \partial H^+$, such that $|v(t)| \leq |u(t)|$ for $t \in (-1, 1)$, and if f vanishes to infinite order at* 0*, then $f \equiv 0$.*

Given any of the conditions in the last two theorems, there is a certain technique of reducing to one-variable, to be applied for obtaining a uniqueness result (see Lemma 4.1). The following is a version of a uniqueness theorem for hypersurfaces due to Shafin [26], where the author originally requires that the hypersurface has a positive eigenvalue of the Levi form at 0 and that the growth condition in the theorem is independent of direction, but the proof reveals that the result holds true given one-sided holomorphic extension and that the growth conditions are only required with respect to an open non-empty double cone at 0.

Theorem 3.4. *Let $M \subset \mathbb{C}^n$ be a C^∞-smooth hypersurface, $0 \in M$. Let f be a C^∞-smooth CR function near 0 such that f has holomorphic extension to one side at 0 and there is a double cone, C, at 0 in M, $\lim_{z\to 0} |f(z)|^{|z|} = 0$, $z \in \mathsf{C}$. Then $f \equiv 0$ on an M-neighborhood of the origin.*

We shall use the following results, in the proof of Theorem 1.4 (see the proof of Claim 4.3).

Theorem 3.5 (See Treves [30], proof of Theorem II.3.3, p.91)**.** *Let M be a C^∞-smooth real manifold equipped with a locally integrable structure L, let $\mathcal{V} \subset M$ be an open subset and X a C^1-smooth section of L over $\mathcal{V}$ (denoted $X \in \Gamma^1(\mathcal{V}, L)$). Let $\gamma : [0,1] \to \Omega$ be an integral curve of $\operatorname{Re} X$ and let f be a distribution solution to the system of equations induced by L (i.e. $Xf = 0$ on $\mathcal{V}$, for each $X \in \Gamma^1(\mathcal{V}, L)$). If $f \equiv 0$ on an open neighborhood of $\gamma(0)$, then $f \equiv 0$ on an open neighborhood of $\gamma(1)$.*

We have the following special case, where $M \subset \mathbb{C}^n$ is a C^∞-smooth hypersurface, $L = H^{0,1}M$, i.e. tangential CR vector fields[3] (where we identify[4] $\operatorname{Re} L$ with T^cM).

Corollary 3.6 (to Theorem 3.5)**.** *Let $M \subset \mathbb{C}^n$ be a C^∞-smooth hypersurface and let $p_0' \in M$. Assume there is an integral curve of a CR vector field, such that the curve originates at p_0' and whose end point is p_0. If f is a continuous CR function on M which vanishes on an open M-neighborhood of p_0', then f vanishes on an open M-neighborhood of p_0.*

4. Proof of Theorem 1.4.

We begin with the following lemma.

Lemma 4.1. *Let $M \subset \mathbb{C}^n$, be a C^∞-smooth real hypersurface, $0 \in M$. Let $\gamma \subset M$ be a real C^∞-smooth curve, $0 \in \gamma$, such that,*

$$T_z^cM + T_z\gamma = T_zM, z \in \gamma. \tag{4.1}$$

Let $f \in CR^\infty(M)$ such that f vanishes to infinite order on γ. Assume f has holomorphic extension to at least one side of M, near 0. Assume that there is a truncated double cone, C, at 0 in M such that at least one of the following holds true:

(a) *There is a constant $\theta \in \mathbb{R}$, such that* $\mathsf{C} \subset \{|\operatorname{Re}(e^{i\theta} f)| \le |\operatorname{Im}(e^{i\theta} f)|\}$.
(b) $\mathsf{C} \subset \{\operatorname{Re} f \ge 0\}$.

[3] For the fact that $H^{0,1}M$ is integrable see e.g. Baouendi et al. [5], p.36 (a short proof of the, in itself not sufficient, involutivity can be found in e.g. Boggess [10]).

[4] This is done via the identification $X \mapsto \frac{X+iJX}{2}$, $X \in T^cM$, with inverse $Y \mapsto Y + \overline{Y}$, $Y \in T^{0,1}M$, see e.g. Zampieri [36], p.112 and p.116.

(c) $|f(z)|^{|z|} \to 0$, $z \to 0$, $z \in \mathsf{C}$.

Then $f \equiv 0$ *on an* M*-open neighborhood of* 0.

Proof. Let $\mathcal{U}$ be an open subset of $\mathbb{C}^n$, such that $V := \partial\mathcal{U} \cap M$ is open in M, $0 \in M$, and there exists a function $F \in \mathscr{O}(\mathcal{U}) \cap C^0(\mathcal{U} \cup V)$, $F|_V = f|_V$. Let $D \subset \mathbb{C}^n$ be a complex line passing through 0 such that $T_0D + T_0M = T_0\mathbb{C}^n$, and, for any sufficiently small open $B \subset \mathbb{C}^n$, $0 \in B$, such that $B \cap D \cap M$ is a connected C^∞-smooth curve. Since we know that F has C^∞-extension to the boundary, then for each $\alpha \in \mathbb{N}^n$, the function $\frac{\partial^\alpha F}{\partial z^\alpha}$ has continuous extension to the boundary, thus vanishing to infinite order on γ of f, implies,

$$\lim_{p \in \mathcal{U}, p \to 0} \frac{\partial^\alpha F}{\partial z^\alpha}(p) = 0 \tag{4.2}$$

Next, pick a sufficiently small open subset $B \subset \mathbb{C}^n$, $0 \in B$, such that $U := B \cap \mathcal{U}$, satisfies that $M \cap \partial U \cap D$ is a C^∞-smooth curve. Assuming further that $\mathcal{U}$ is a bounded domain and that $U \cap D$ is a bounded simply connected domain, bounded by a finite union of smooth curves, there is a bijective holomorphic map $\mathscr{R}$ from the (non-empty, bounded and simply connected one-dimensional complex) domain $U^+ := U \cap D$, onto an open half-set, $W^+ \subset \mathbb{C}$ as in Theorem 3.1, and furthermore $\mathscr{R}$ extends to a homeomorphism up to the boundary, see e.g. Taylor [28], p.342 (for a short proof of the fact that a bijective holomorphic map of a domain necessarily has holomorphic inverse, see e.g. Rudin [24], p.217). So we can assume $\mathscr{R}$ is a biholomorphism of $U \cap D$ and a homeomorphism of $\overline{U \cap D}$. Since $\mathscr{R}$ is an open mapping of $U \cap D$ with open inverse we can assume $\mathscr{R}(0)$ is the origin, belonging to the boundary of W^+ in $\mathbb{C}$. Now given a holomorphic coordinate z centered at 0, for D near 0, and setting $\mathscr{R}(z) =: \zeta$, we can (by the chain rule) for any $j \in \mathbb{N}$, and any $q \in W^+$, express $\frac{\partial^j F}{\partial \zeta^j}(q)$ as finite sum of multiples of $\frac{\partial^k F}{\partial z^k}(\mathscr{R}^{-1}(q))$, and $\frac{\partial^l \mathscr{R}^{-1}}{\partial \zeta^l}(q)$, $k, l \in \{1, \dots, j\}$ Here we are considering the restriction of F to $D \cap \mathcal{U}$, so there is only one complex coordinate z (which explains why we write j instead of a multi-index α). By (4.2), we obtain that $(F \circ \mathscr{R}^{-1})$ is a continuous map of $\overline{U \cap D}$ (holomorphic on $U \cap D$) which vanishes to infinite order at $\mathscr{R}(0) \in \partial W^+$. Since M is smooth, D a complex one-dimensional manifold We can (by appropriate choice of B, if necessary we replace $U \cap D$ but keep the same notation) assume that both $\mathcal{U} \cap D$ and $\mathscr{R}(\mathcal{U} \cap D)$ have C^∞-smooth boundary. In particular we can assume $(F \circ \mathscr{R}^{-1})$ is smooth up to the boundary. If condition (c) holds true then, by Theorem 3.4, f vanishes on an open M-neighborhood of 0. If condition (c) does not hold true, then by assumption one of (a) or (b) must hold true. Then we are able to choose D such that $D \cap V$ (for sufficiently small V) belongs to the intersection with M of a double cone as in (a) or (b). We obtain that $(F \circ \mathscr{R}^{-1})$ maps an interval containing 0 into (a) $\{|\mathrm{Re}\, z| \leq |\mathrm{Im}\, z|\}$ (if necessary after a fixed rotation of its image, by some $\theta \in \mathbb{R}$), or (b) $\{\mathrm{Re}\, z \geq 0\}$. In the case (a) Theorem 3.3 applies and in the case (b) Theorem 3.2 applies, in each case implying that the function $(F \circ \mathscr{R}^{-1})$ vanishes on the connected component of the origin in $\overline{W^+}$, which implies that it vanishes on an open subset of W^+ so using that $\mathscr{R}$ has open inverse we obtain (by the identity theorem) $F \equiv 0$ on $\mathcal{U} \cap D$, and by continuity $f = 0$ on $D \cap V$. Now D was an arbitrary complex one-dimensional manifold which sufficiently near 0 had intersection with M belonging to a certain double cone near 0. This can be repeated for all one-dimensional complex $\tilde{D}$ which are perturbations of D, each passing through 0, and whose intersection with M belong, near 0, to the given double cone. So F vanishes on the union of intersections $\tilde{D} \cap \mathcal{U}$, as $\tilde{D}$ varies over such complex one dimensional manifolds. The union of all such $\tilde{D}$ covers an open subset of $\mathcal{U}$, so again by the identity theorem $f \equiv 0$ near 0. $\square$

We shall use the following observation.

Observation 4.2. Any representative, W, of $\mathfrak{o}(p_0)$, $p_0 \in M$, is an embedded CR submanifold (see e.g. Baouendi et al. [5], p.95). This implies $T^c_z W \subset T^c_z M$ for each $z \in W$. Assume p_0 (for the remainder of this observation) is non-minimal. By definition of non-minimality at p_0, $CR\dim(M) \leq \dim_{\mathbb{R}} \mathfrak{o}(p_0) < \dim_{\mathbb{R}} M$, and since the real codimension of M is one, $CR - \dim(M) = \dim_{\mathbb{R}} \mathfrak{o}(p_0)$, thus $T_z W = T^c_z W = T^c_z M$ for each $z \in W$, which implies that W is a complex $(n-1)$-dimensional manifold containing p_0. By the transversality condition (4.1) for γ, it must be transversal to any member of the local CR orbit at a point of γ. If $\mathscr{W} \subset M$ is a small open neighborhood of p_0, then every point in $\mathscr{W}$ which also belongs to γ is associated to a family of complex $(n-1)$-dimensional manifolds (each a member of a different local CR orbit). In the case of real codimension one, the restriction of the CR function f to any complex submanifold $\mathfrak{W}_{p_0} \subset M$, passing through $p_0 \in \gamma$ is a holomorphic function, hence f vanishes within $\mathfrak{W}_{p_0}$, as soon as $\mathfrak{W}_{p_0}$ is a member of the local orbit at $p_0 \in \gamma$. This concludes the observation.

Given a reference point $p_0 \in \gamma$, we parametrize γ, locally near a sufficiently small neighborhood $\mathscr{W} \subset M$ of p_0, by introducing smooth local coordinates,

$$\gamma \cap \mathscr{W} = \{(\phi_1, \ldots, \phi_{2n-2}, \varphi) : \phi_1 = \cdots = \phi_{2n-2} = 0\}, \tag{4.3}$$

where $p_0 = (0, \hat{\varphi})$ is a point of $\gamma \cap \mathscr{W}$.

The strategy of the proof is as follows:

- We construct a *open* M-neighborhood, denoted $\mathcal{C}_{\hat{\varphi}}$ (see (4.8)), of $(0, \hat{\varphi})$, such that every point of $\mathcal{C}_{\hat{\varphi}}$ belongs to the global Sussmann orbit, $\mathcal{S}_{(0,\varphi)}$, of some point $(0, \varphi)$ (with φ near $\hat{\varphi}$).
- We then proceed to prove that $f \equiv 0$ on $\mathcal{C}_{\hat{\varphi}}$, see Claim 4.3, and the proof of the latter claim is divided into two main cases based upon minimality.
- In the first case (denoted (i)) appearing in the proof of Claim 4.3, Lemma 4.1 is invoked.
- The second case (denoted (ii)) is divided into two subcases (based upon existence and non-existence respectively, of minimal points in a given global orbit passing γ near the reference point). Observation 4.2 is used to handle the easy subcase when all points of a given global orbit are non-minimal. The second subcase requires more work in terms of invoking known propagation results in fusion with the properties of $\mathcal{C}_{\hat{\varphi}}$.

Remark 2.1 shows that if we pick a nonzero vector field $Z \in \Gamma(\mathscr{W}, T^c M)$, and we introduce the parameter ϑ, (to be further specified later) on which Z depends, i.e. $Z = Z_\vartheta$, then there is a unique integral curve, $\eta(t, (0, \varphi), \vartheta) =: \Phi_{Z_\vartheta, t}((0, \varphi))$, of Z originating at $(0, \varphi)$ defined for $t \in [0, T((0, \varphi), \vartheta))$, where $T((0, \varphi), \vartheta))$ is a lower semi-continuous function near $((0, \varphi), 0)$, (i.e T is the maximal time parameter as in Remark 2.1), specifically, given any $\epsilon > 0$, $(0, \hat{\varphi}) \in \gamma \cap \mathscr{W}$, and any ϑ_0, there exists a $\delta(\hat{\varphi}, \vartheta_0, \epsilon)$ such that,

$$(|(\phi,\varphi)-(0,\hat{\varphi})| < \delta(\hat{\varphi},\vartheta_0,\epsilon)) \wedge \left(\left|\vartheta - \hat{\vartheta}\right| < \delta(\hat{\varphi},\vartheta_0,\epsilon)\right) \Rightarrow \\ T((\phi,\varphi),\vartheta) \geq T((0,\hat{\varphi}),\vartheta_0) - \epsilon, \quad (4.4)$$

hence T is bounded from below as $((\phi,\varphi),\vartheta)$ varies on an open box-neighborhood of $(\hat{\varphi},\vartheta_0)$. This in turn implies that $\Phi_{Z_\vartheta,t}((0,\varphi))$, which defines the end point of the integral curve of the vector field Z_ϑ passing through $(0,\varphi)$, varies smoothly with respect to the base point, near $(0,\hat{\varphi})$. For each $z \in \gamma$, we let $\mathcal{S}_z$ denote the set of points of M which can be reached from q by a polygonal path (see the definition in the preliminaries) of integral curves to sections of T^cM ($\mathcal{S}$ is called the global Sussmann orbit at q). Let $\mathscr{W}$ be sufficiently small such that there is a basis, of vector fields $v_1,\ldots,v_{2n-2}$ (we assume each v_k is normalized), for the set of sections of T^cM over $\mathscr{W}$. Let,

$$Z_\vartheta := Z_0 + \sum_{k=1}^{2n-2} \vartheta_k v_k. \quad (4.5)$$

We shall use $Z_0 = 0$, in which case we already know that each Z_ϑ is a section of T^cM, and we shall use $\vartheta_0 = 0$.

Given $\vartheta_0 = 0$, we set $T(\hat{\varphi}) := T((\hat{\varphi},0),0)$. We complement $v_1,\ldots,v_{2n-2}$ to full basis by adjoining a vector field v_{2n-1} which along γ coincides with $\frac{\partial}{\partial\varphi}$.

Next we consider the map,

$$\Psi : (\vartheta,\varphi) \mapsto \Phi_{(Z_\vartheta+(\varphi-\hat{\varphi})v_{2n-1}),1}((0,\hat{\varphi})). \quad (4.6)$$

Since $v_1,\ldots,v_{2n-1}$ form a basis for $T\mathscr{W}$, the map Ψ has nonzero determinant at $(0,\hat{\varphi})$ (see e.g. Baouendi et al. [5], p.65) so let $\{\varphi v_{2n-1} + \sum_{j=1}^{2n-2}\vartheta_j v_j\colon\ |\varphi-\hat{\varphi}| < \upsilon,\ |\vartheta| < \upsilon\} =: B_\upsilon \subset TM$ be such that the image of any subdomain of B_υ containing $(0,\hat{\varphi})$, under Ψ, is an open subset of M, and such that the maximal time parameter T above is bounded from below on B_υ by $7T(\hat{\varphi})/8$. In particular we must chose $\epsilon < T(\hat{\varphi})/8$ above and $\upsilon < \delta(\hat{\varphi},0,\epsilon)$. Let $a = \min\{1/8, T(\hat{\varphi})/8, \upsilon/8\}$. Define the following sets,

$$\mathcal{C}(\varphi) = \bigcup_{|\vartheta|<a} \Phi_{Z_\vartheta,a}((0,\varphi)), \quad (4.7)$$

$$\mathcal{C}_{\hat{\varphi}} = \bigcup_{\varphi\in\{s\colon\ |s-\hat{\varphi}|<a\}} \mathcal{C}(\varphi). \quad (4.8)$$

Now for fixed φ (sufficiently near $\hat{\varphi}$ as above) and $|\vartheta| < a$, we have,

$$\Phi_{Z_\vartheta,a}((0,\varphi)) = \Phi_{Z_\vartheta,a}\left(\Phi_{v_{2n-1},\varphi-\hat{\varphi}}((0,\hat{\varphi}))\right) = \Psi((a\vartheta,\varphi)). \quad (4.9)$$

Also for fixed φ such that $|\varphi-\hat{\varphi}| < a$, the union, $\bigcup_{|\vartheta|<a}\Phi_{Z_\vartheta,a}(0,\varphi)$ belongs to $\mathcal{S}_{(0,\varphi)}$ (the global Sussmann orbit). Since we already know that its image under Ψ is an open subset of M containing $(0,\hat{\varphi})$ we obtain that the union, $\bigcup_{|\varphi-\hat{\varphi}|<\upsilon}\mathcal{S}_\varphi$ contains an M-open neighborhood of $(0,\hat{\varphi})$.

Claim 4.3. $f \equiv 0$ on $\mathcal{C}_{\hat{\varphi}}$.

Proof. Indeed, there are two cases which can occur given a $\varphi \in \{|\varphi-\hat{\varphi}| < a\}$:

(i) $(0,\varphi)$ is a minimal point of M. It is a known result (due to Trepreau [31] and generalized by Tumanov [34], for our precise formulation, see Trepreau [32], p.409) that minimality at a point implies holomorphic extension of f to one side of M near that point i.e. we assume that f has holomorphic extension to one side of M, near $(0,\varphi)$. By Lemma 4.1 we obtain that $f \equiv 0$ on an M-neighborhood of $(0,\varphi)$. This however, by definition implies that $(0,\varphi)$ does not belong to supp$f$, which in turn by the known result of Treves [30], p.91, this implies that $\mathcal{S}_{(0,\varphi)} \cap \mathrm{supp} f = \emptyset$, so f vanishes on $\mathcal{C}(\varphi)$ because the latter set is a subset of $\mathcal{S}_{(0,\varphi)}$.

(ii) $(0,\varphi)$ is a non-minimal point of M. If all points of $S_{(0,\varphi)}$ are non-minimal then there passes through each, a complex $(n-1)$- dimensional manifold and the vanishing of f near $(0,\varphi)$ (in the sense of Remark 4.2) propagates along each such manifold, from $(0,\varphi)$, so f must vanish on $\mathcal{S}_{(0,\varphi)}$. Assume instead that there is a minimal point, q, belonging to $\mathcal{S}_{(0,\varphi)}$. By definition q can be reached from $q_0 = (0,\varphi)$ by a polygonal path of CR curves in $\mathcal{S}_{(0,\varphi)}$. For a C^∞-smooth hypersuface $M \subset \mathbb{C}^n$, it is a known result that holomorphic extension to one side of M, at a given point $q \in M$, of continuous CR functions, holds true iff there does not pass a germ of a complex $(n-1)$-dimensional submanifold of M through q, and the last condition is equivalent to minimality at q (see e.g. Baouendi et al. [5], Theorem 1.5.15, p.20). Also, in the case of C^∞-smooth hypersurfaces holomorphic extension to one side of M coincides with holomorphic wedge-extension, which in turn propagates along a given CR curve (the direct consequence of the latter result, stated in the terms we shall use it, can be found in Trepreau [32], Theorem 2, p.409; the more detailed cause of propagation can be found in Trepreau [32], p.418, and information about directionality in Tumanov [33]-[35]). Hence, we can assume f has holomorphic extension to one side of M, near each point of $\mathcal{S}_{(0,\varphi)}$. Let $\mathcal{U}$ be an open subset of $\mathbb{C}^n$, such that $V := \partial\mathcal{U} \cap M$ contains q_0, is open in M, and such that there exists a function $F \in \mathscr{O}(\mathcal{U}) \cap C^0(\mathcal{U} \cup V)$, $F|_V = f|_V$. By Lemma 4.1 $f \equiv 0$ on an M-neighborhood of q_0. Theorem 3.5 implies that f vanishes at all points of $\mathcal{S}_{(0,\varphi)}$. This completes the proof of Claim 4.3. □

By Claim 4.3, $f \equiv 0$ on an open M-neighborhood of p_0 and since the latter point was an arbitrary point of γ this also completes the proof of Theorem 1.4. □

5. Some examples on geometric conditions on M with reduced growth conditions

Example 5.1 (The Levi flat case)**.** Let $M \subset \mathbb{C}^n$ be a C^∞-smooth hypersurface and let $\gamma \subset M$ be a C^∞-smooth curve which is *not* locally the intersection with a complex line, but satisfies the condition of (4.1). Assume M is Levi flat on an M-open neighborhood, U, of γ.

Then any smooth CR function which vanishes to infinite order along γ must vanish on an open M-neighborhood of γ: Let $p_0 \in \gamma$, and assume w.l.o.g., p_0 coincides with the origin, in γ and in M. It is a known consequence of the complex version of Frobenius theorem (see Freeman [13]) that there passes through each point of $U \cap \gamma$, a complex manifold of complex dimension $n-1$ (i.e. the CR dimension of M). In particular every point of U is a non-minimal point of M. Hence we can apply the proof of (i), to the C^∞-smooth CR function f, vanishing to infinite order along γ, in the C^∞-smooth hypersurface U (the reason being that in Claim 4.3, the requirements (a)-(c) are not invoked). This will yield that f vanishes on the Sussmann orbit of each point of $\gamma \cap U$, in U. As the proof of our main result shows, the union of such Sussmann orbits cover an M-open neighborhood of p_0.

Example 5.2. When $M \subset \mathbb{C}^n$ is a C^∞-smooth hypersurface then it was proved by Rosay [23] (the proof uses a result of Andreotti & Hill [4]), that if $U \cap \gamma = U \cap D$ for a complex ambient line D, transversal to $U \cap M$, for some small open U, then any $f \in CR^\infty(M)$ which vanishes to infinite order along $U \cap \gamma$, must vanish on an open M-neighborhood of $U \cap \gamma$. This example does not require that the origin is a minimal point, and does not have additional growth conditions compared to Theorem 1.4.

Example 5.3 (The real-analytic case)**.** When $M \subset \mathbb{C}^n$ is a C^∞-smooth hypersurface and $\gamma \subset M$ a real-analytic curve, then any Lipschitz continuous CR function that vanishes to infinite order along γ, vanishes on an M-open neighborhood of γ. This result is due to Baouendi & Treves [7] (see Treves [30], Theorem II.8.1, together with Corollary II.8.1, p.118, for a textbook version). Let $(z_1, \dots, z_n)$ denote holomorphic coordinates, $z_n = x_n + iy_n$, let $0 \in \gamma$ and U be open in M such that, $0 \in U$, $U \cap M = U \cap \{y_n = h(z_1, \dots, z_{n-1}, x_n)\}$ for a real-analytic graphing function h. The unique continuation result of Baouendi & Treves [7] in this real-analytic case, is a consequence of the so called compact cocycle property for $\{y_n = h(0, x_n)\}$.

Definition 5.4 (see Treves [30], p.115)**.** Let Λ be a maximally real submanifold of $\mathbb{C}^n$, $p \in \Lambda$. Λ is said to have the *compact cocycle property* at p if there is a basis of neighborhoods of p such that, if N is any one of these neighborhoods, then there is $F \in \mathscr{O}(N)$ with $F(p) \neq 0$ and $\{w \in \Lambda \cap N : F(w) \neq 0\} \Subset \Lambda \cap N$.

Here is an example, covered by Theorem 1.4 of this paper, where $\Sigma \neq \emptyset$.

Example 5.5. The following function on $\mathbb{R}$ is known to be C^∞-smooth but nowhere real-analytic (see e.g. Kim & Kwon [21]),

$$\rho(x) := \sum_{k=1}^{\infty} \frac{1}{k!}\theta\left(2^k(x - \lceil x \rceil)\right), \tag{5.1}$$

where $\lceil \cdot \rceil$ denotes the least upper integer, $\theta(x) := \exp(-\frac{1}{x^2})\exp\left(-\frac{1}{(x-1)^2}\right)$, $0 < x < 1$, and $\theta(x) = 0$, $x \notin (0,1)$. Let $(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2) \in \mathbb{C}^2$ be holomorphic coordinates and define for each $j \in \mathbb{Z}_+$, $\chi_j(z_1, x_2) \in C_c^\infty(B_j)$ (where $B_j := B_{\frac{1}{j}}\left(0, \frac{1}{j}\right)$, and $B_r(p)$ denotes the ball in $\mathbb{C} \times \mathbb{R}$, of radius r, and center p) such that $\chi_j = 1$ on $C_j := B_{\frac{1}{2j}}\left(0, \frac{1}{j}\right)$, (see e.g. HÃűrmander [19], Theorem 1.4.1, p.25, for the existence of such χ_j). Let $\mathcal{B} := \bigcup_{j \in \mathbb{Z}_+} B_j$, $\mathcal{C} := \bigcup_{j \in \mathbb{Z}_+} C_j$, and define $M := \{(z_1, z_2) \in \mathbb{C}^2 \colon y_2 = h(z_1, x_2)\}$, where,

$$h(z_1, x_2) := \begin{cases} \rho(x_2) + \chi(x_2, z_1)\left(|z_1|^2 - \rho(x_2)\right) & \text{, on } \mathcal{B} \\ \rho(x_2) & \text{, otherwise.} \end{cases} \tag{5.2}$$

By construction $M \subset \mathbb{C}^2$ is: (i) a smooth hypersurface ($M = \{(z_1, z_2) \in \mathbb{C}^2 : \psi = 0\}$ where $\psi(z_1, z_2) := y_2 - h(z_1, x_2)$, with $\frac{\partial \psi}{\partial y_2} \equiv 1$, so $d\psi|_0 \neq 0$) near 0, (ii) not real-analytic on any open subset, (iii) strictly pseudoconvex on the subset $\mathcal{C} \subset M$, and (iv) Levi flat at all points of $M \setminus \overline{\mathcal{B}}$.[5] Then, for any C^∞-smooth curve $\gamma \subset \{0\} \cup (M \setminus \overline{\mathcal{B}})$ with $0 \in \gamma$ we have a decomposition $\gamma = \Sigma \cup (\gamma \setminus \Sigma)$, where Σ

[5] $H^{1,0}M$ spanned by (see Boggess [10], p.144), $\overline{L} = -2i\left(\frac{1}{1+i\frac{\partial h}{\partial x_2}}\right)\frac{\partial h}{\partial \bar z_1}\frac{\partial}{\partial \bar z_2} + \frac{\partial}{\partial \bar z_1} = \frac{\partial}{\partial \bar z_1}$, since h is (recall that we are speaking of the set $M \setminus \overline{\mathcal{B}}$) independent of $\operatorname{Re} z_1, \operatorname{Im} z_1$. Thus $[L, \overline{L}] = 0$.

denotes be the set of points $z \in \gamma$. such that every M-neighborhood of z contains a point where the Levi form[6] of M is nonzero.

Remark 5.6. After the completion of this paper we were informed that Alexander [2], Theorem 1, p.2, proved a stronger version of Theorem 3.3, indeed the condition that f map the part of the real axis as in Theorem 3.3 into a non-spiraling set will imply that either $f \equiv 0$ or f cannot vanish to infinite order at 0. In fact by Alexander's result we may replace the condition ($|v(t)| \leq |u(t)|$ for $t \in (-1,1)$) with the condition ($|v(t)| \leq C\,|u(t)|$ for $t \in (-1,1)$, and a non-negative constant C). We have chosen to state our results using the weaker version found in Theorem 3.3.

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[6]This paper only deals with the case of a smooth hypersurface M in which case the Levi form at $p_0 \in M$, is particularly easy to describe. Let U be an open subset containing p_0, such that $M \cap U = \{\rho = 0\}$, for some $\rho : U \to \mathbb{R}$ with $|\nabla\rho(p_0)| = 1$. The Levi form at $p_0 \in M$, is (a constant multiple of), $\mathcal{L}_{M,p_0}(W) = -\left(\sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}(p_0)\zeta_j\bar{\zeta}_k\right)\nabla\rho(p_0), \quad W = \sum_{k=1}^{n} \zeta_k \frac{\partial}{\partial z_k} \in H^{1,0}_{p_0}M.$

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