Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://bmathaa.org
Volume 10 Issue 4(2018), Pages 14-30.

ON BERTRAND SUPERCURVES IN SUPER-EUCLIDEAN SPACE

HATICE TOZAK, CUMALI EKICI, CANSEL YORMAZ

Abstract. Using Banach Grassmann algebra, given by Rogers, a new scalar product, a new definition of the orthogonality and of Frenet frame associated to supersmooth supercurve are introduced on the (m, n)-dimensional total super-Euclidean space. It is well known that a characteristic property of Bertrand curve is the existence of a linear relation between its curvature and torsion. In this study, definition of Bertrand supercurve in B_L^{m+n} is given and also some theorems for Bertrand supercurve in B_L^{4+4} are obtained.

1. Introduction

In recent years, much conventional differential geometry has been extended to include anticommuting variables; objects in this extended field of study are distinguishable by the prefix "super" which derives from the same prefix in supersymmetry, the fermi-base symmetry which is under such intense study by elementary particle physicist. Historically, the consideration of supermanifolds has a dual origin. Due to the first origin the earliest work is that Berezin and Leites [6] and Konstant [19] arose from the study of the mathematics of fermi field quantisation, their approach was sheaf theoretic, extending the sheaf of C^∞ functions on a manifold, rather than the manifold itself. Afterwards, a supermanifold was developed with a lot of study such as [7], [20]. Secondly, a more geometric approach grew directly from the physicists' superspace [18] as a space with points labelled by even elements (x^μ) and odd elements (θ^α) of a Grassmann algebra; a supermanifold is a topological space with local coordinates (x^μ, θ^α) of this nature [11], [23]. Alternatively, the the best relationships between them have been made by Rogers [23], Bartocci et al. [4] and Batchelor [5]. Then, (m, n)- dimensional total super-Euclidean space B_L^{m+n} is studied by Rogers [23]. Using Banach Grassmann algebra B_L, a new superscalar product, a new definition of the ortogonality and Frenet frame associated to a supersmooth supercurve in general position are given by Cristea [11]. Also, Inoque and Maeda define super-Euclidean space with a different algebra, called a Frechet-Grassmann algebra [17].

2000 *Mathematics Subject Classification.* 58A50, 53A04, 57R25.
Key words and phrases. Super-Euclidean space, Supercurve, Bertrand curve, Frenet frame.

Submitted August 25, 2018. Published October 21, 2018.
C. E. was supported by the grant 2013-291 from ESOGU-BAP Turkey.
Communicated by U. C. De.

French mathematician Saint-Venant proposed in 1845 [26] the question whether upon the ruled surface generated by the principal normal of a curve in the three-dimensional Euclidean space R^3 and a second curve can exist which has for this principal normals of the given curve. The second question was answered by Bertrand [8] in a paper in which he showed that a necessary and sufficient condition for the existence of such a second curve is that a linear relationship with constant coefficients shall exist between the curvature and torsion of the given original curve. Since the publication of Bertrand's paper, a pair of curves of this kind has been called Conjugate Bertrand curves or, more commonly, Bertrand curves. Bertrand curves have attracted many mathematicians since the beginning. Later, the relations between Frenet frames of Bertrand couple in the space R^n were given in [16]. Also, Bertrand couple is studied by many researchers in Euclidean 3-space R^3 [10], [12], [27], [28]. In [22], Pears extended the well-known properties of Bertrand curves in Euclidean 3-space R^3 to the curves in the n-dimensional Euclidean space R^n, $n > 3$. However, in the last case, he found that either k_2 or k_3 must be zero; in other words, Bertrand curves in R^n $(n > 3)$ are degenerate, i.e. a Bertrand curve in R^n must belong to a three-dimensional subspace $R^3 \subset R^n$. The same result has been obtained recently in [1] and [21]. As a natural consequence, some extensions of that concept have been proposed [16], [25], and more recently have been generalized in [9]. Many authors have studied Bertrand curves in other ambient spaces: in the three-dimensional Lorentz-Minkowski space R^3_1 [2], [3], [14], in semi-Euclidean spaces R^{n+1}_v [15], etc.

In this paper, we firstly define Bertrand supercurve, a one dimensional supermanifold, couples in definition of Bertrand supercurve on total super-Euclidean space B_L^{m+n}. Later, using the methods expressed in [16] and Frenet frame we calculate some theorems for Bertrand supercurve in B_L^{4+4}.

2. Preliminary Notes

In this section, we refer to a few basic definitions for the so-called geometric theory of supernumbers, supermanifolds, total super-Euclidean space, supervector space and operators, initialized by Dewitt, Rogers and Cristea. For further developments of the theory, which eliminated some drawbacks of research topic, the reader may utilize [11], [13], [23], [24].

Definition 2.1. *For each positive integer L, B_L will denote Grassmann algebra over the real numbers with generators $1^{(L)}, \beta_1^{(L)}, ..., \beta_L^{(L)}$ and relations*

$$\begin{array}{ll} 1^{(L)}\beta_i^{(L)} = \beta_i^{(L)}1^{(L)} = \beta_i^{(L)} & i = 1, 2, ..., L \\ \beta_i^{(L)}\beta_j^{(L)} = -\beta_j^{(L)}\beta_i^{(L)} & i, j = 1, 2, ...L. \end{array} \quad (2.1)$$

B_L is a gradded algebra and can be written as

$$B_L = (B_L)_0 \oplus (B_L)_1 \quad (2.2)$$

where $\oplus$ be the direct sum and $(B_L)_0$ and $(B_L)_1$ be the even and odd part of B_L, respectively [23].

Definition 2.2. *Let M_L denote the set of finite sequences of positive integers $\mu = (\mu_1, \mu_2, ..., \mu_k)$ with $1 \leq \mu_1 < \mu_2 < ... < \mu_k \leq L$* [19]. *$M_L$ includes the sequence* *with no elements, denoted ϕ . As it follows in [24] for each μ in M_L,*

$$\begin{aligned} \beta_{\mu}^{(L)} &= \beta_{\mu_1}^{(L)} ... \beta_{\mu_k}^{(L)} \\ \beta_{\phi}^{(L)} &= 1^{(L)} \end{aligned} \tag{2.3}$$

and typical element b of B_L may be expressed as

$$b = \sum_{\mu \in M_L} b^{\mu} \beta_{\mu}^{(L)} \tag{2.4}$$

where the coefficients b^{μ} are real numbers. We consider the body map [12]

$$\varepsilon^{(L)}(b) = b^{\phi} \tag{2.5}$$

is given by

$$\varepsilon : B_L \to R. \tag{2.6}$$

with the norm of B_L is defined by

$$\|b\| = \sum_{\mu \in M_L} |b^{\mu}| \,. \tag{2.7}$$

Definition 2.3. *B_L is Banach algebra, considering $L^{'}$ also a positive integer, with $L \geq L^{'}$, there is a natural projection*

$$i_{L^{'},L} : B_{L^{'}} \to B_L \tag{2.8}$$

which is the unique algebra homomorphism satisfying

$$\begin{aligned} i_{L^{'},L}\left(\beta_i^{(L^{'})}\right) &= \beta_i^{(L)} \qquad i = 1, 2, ...L \\ i_{L^{'},L}\left(1^{(L^{'})}\right) &= 1^{(L)}. \end{aligned} \tag{2.9}$$

B_L naturally has a $B_{L^{'}}$ module structure with

$$ab = i_{L^{'},L}(a)b \qquad a \in B_{L^{'}}, \quad b \in B_L \tag{2.10}$$

[24].

Definition 2.4. *The (m, n)-dimensional total super-Euclidean space B_L^{m+n} as the space, which is the cartesian product of $m + n$ copies of B_L, is defined by*

$$B_L^{m+n} = \left(B_L^{m+n}\right)_0 \oplus \left(B_L^{m+n}\right)_1 . \tag{2.11}$$

A typical element of B_L^{m+n} is written $\left(x^1, x^2, ..., x^m, \theta^1, \theta^2, ..., \theta^n\right)$ or simply (x, θ), an element of $\left(B_L^{m+n}\right)_0$ is called c-type or even element and is written in the form $\left(x'^1, x'^2, ..., x'^m, \theta'^1, \theta'^2, ..., \theta'^n\right)$ with $x'^1, x'^2, ..., x'^m \in (B_L)_0$. Also, $\theta'^1, \theta'^2, ...,$ $\theta'^n \in (B_L)_1$, an element of $\left(B_L^{m+n}\right)_1$ is called a-type or odd element is written in the form

$$\left(x''^1, x''^2, ..., x''^m, \theta''^1, \theta''^2, ..., \theta''^n\right) \text{ with } x''^1, x''^2, ..., x''^m \in (B_L)_1 \tag{2.12}$$

and $\theta''^1, \theta''^2, ..., \theta''^n \in (B_L)_0$. An even element has the parity 0 and an odd element has the parity 1 [13].

Definition 2.5. *The body map ε is defined by* [11]

$$\begin{array}{llll} \varepsilon_{(m,n)}^{(L)}: & \left(B_L^{m+n}\right)_0 & \to & \mathbb{R}^m \\ & (x^{'}, \theta^{'}) & \mapsto & \varepsilon_{(m,n)}^{(L)}(x^{'}, \theta^{'}) = \left(\varepsilon^{(L)}(x'^1), ...\varepsilon^{(L)}(x'^m)\right) \end{array}$$

where $(x^{'}, \theta^{'}) = \left(x'^1, x'^2, ..., x'^m, \theta'^1, \theta'^2, ..., \theta'^n\right) \in \left(B_L^{m+n}\right)_0$ *and the body map* ε'

$$\begin{array}{lccl} \varepsilon'^{(L)}_{(m,n)}: & \left(B_L^{m+n}\right)_1 & \to & \mathbb{R}^n \\ & (x'', \theta'') & \mapsto & \varepsilon'^{(L)}_{(m,n)}(x'', \theta'') = \left(\varepsilon'^{(L)}(\theta''^1), ...\varepsilon'^{(L)}(\theta''^n)\right) \end{array}$$

where $(x'', \theta'') = \left(x''^1, x''^2, ..., x''^m, \theta''^1, \theta''^2, ..., \theta''^n\right) \in \left(B_L^{m+n}\right)_1$.

Definition 2.6. *Suppose that* $V \subset B_L^{m+n}$ *is open and that* $U = \varepsilon^{(L)}_{(m,n)}(V)$. *Let* $L > 2n$ *and* $L' = \left[\frac{1}{2}L\right]$ *be the least integer not less than* $\frac{1}{2}L$. $GH^{\infty}(V)$ *denotes the set of functions,*

$$f : V \to B_L$$

for which there exists $f_m \in C^{\infty}\left(U, B_{L'}\right)$ *such that*

$$f(x,\theta) = \sum_{\mu \in M_n} Z_{L',L}(\partial_i f_m)(x)\theta^{\mu}$$

that the map $Z_{L',L} : C^{\infty}\left(U, B_{L'}\right) \to \left[\varepsilon^{(L)}_{(m,0)}(U)\right]^{B_L}$ *is defined by*

$$Z_{L',L}(f)\left(X\right) = \sum_{i_1 = 0 ... i_m = 0} \begin{array}{c} \left[\frac{1}{i_1! ... i_m!}\left(\partial_1^{i_1} ... \partial_m^{i_m} f\left(\varepsilon^{(L)}(x^1), ..., \varepsilon^{(L)}(x^m)\right)\right) \\ \times s(x^1)^{i_1} ... s(x^m)^{i_m} \cdot i_{L',L}\right] \end{array} \tag{2.13}$$

where $(X) = \left(x^1, ..., x^m\right)$ *and* $s(x^i) = x^i - \varepsilon^{(L)}(x^i)1$ *for* $i = 1, 2, ...m$. *Here* $\theta^{\mu} = \theta^{\mu_1} ... \theta^{\mu_k}$ *and* $\theta^{\phi} = 1^L$ [24].

Definition 2.7. *Suppose* $n = 2r$ *and the supervectors*

$$v = \left(x^1, x^2, ..., x^m, \theta^1, \theta^2, ..., \theta^n\right), w = \left(y^1, y^2, ..., y^m, \theta_1^1, \theta_1^2, ..., \theta_1^n\right) \tag{2.14}$$

are the elements of B_L^{m+n}. *Superscalar product is defined by*

$$\langle v, w\rangle = x^1y^1 + ... + x^ny^n + \theta^1\theta_1^{r+1} + ... + \theta^r\theta_1^n - \theta^{r+1}\theta_1^1 - ... - \theta^n\theta_1^r \tag{2.15}$$

$$\begin{aligned} \langle v, w\rangle_f &= \sum_{k=1}^{m} x^k y^k + \sum_{j_1=1}^{r}\left(\theta^{j_1}\theta_1^{f(j_1)} - \theta^{f(j_1)}\theta_1^{j_1}\right) \\ &= x^1y^1 + ... + x^ny^n + \theta^1\theta_1^{r+1} + ... + \theta^r\theta_1^n - \theta^{r+1}\theta_1^1 - ... - \theta^n\theta_1^r \end{aligned} \tag{2.16}$$

where $f : \{1, ..., r\} \to \{r+1, ..., 2r\}$ *is one-to-one function* [11].

Definition 2.8. *Supervector* $v \in B_L^{m+n}$ *is orthogonal to supervector* $w \in B_L^{m+n}$ *if and only if* $\varepsilon^{(L)}\left(\langle v, w\rangle\right) = 0$. *The standart base vectors on* $\left(B_L^{m+n}\right)_0$ *form as*

$$\begin{array}{lll} E_1 = (1, 0, ..., 0)\,, \ E_2 = (0, 1, ..., 0) & , ..., & E_m = (0, ..., 1, ..., 0) \\ E_{m+1} = (0, ..., 0, -1, 0, ..., 0) & , ..., & E_{m+r} = (0, 0, ..., -1) \\ E_{m+r+1} = (0, ..., 0, 1, 0, ..., 0) & , ..., & E_{m+n} = (0, ..., 0, 1, 0, ..., 0) \end{array} \tag{2.17}$$

where the first m *supervectors are even or c-type and the last* n *supervectors are odd or a-type* [11].

Definition 2.9. *Let* f *be an element of* $GH^{\infty}(V)$. *Then, for* $i = 1, 2, ..., m$,

$$\begin{array}{llll} G_i f: & V & \to & B_L \\ & (x, \theta) & \mapsto & G_i f\left(x, \theta\right) = \sum_{\mu \in M_n} Z_{L',L}(\partial_i f_{\mu})(x)\ \theta^{\mu} \end{array} \tag{2.18}$$

is defined. Also, for $j = 1, 2, ..., n$

$$\begin{array}{rcl} G_{j+m}f : V & \rightarrow & B_L \\ (x,\theta) & \mapsto & G_{j+m}f\,(x,\theta) = \sum\limits_{\mu\in M_n} Z_{L',L}(f_\mu)(x)\ \theta^{\mu/j} \times (\text{-}1)^{|f_\mu(x)|} \end{array} \tag{2.19}$$

is defined where $|f_\mu(x)|$ *is parity of* $f_\mu(x)$ *and* $\theta^{\mu/j} = \theta^{\mu_1}...\theta^{\mu_k}\,(-1)^{i-1}$ *, if* $j = \mu_i$ *for some* i*,* $1 \le i \le k$ *and* $\theta^{\mu/j} = 0$*, otherwise* [24].

Definition 2.10. *Let* B_L^{m+n} *be an* (m, n)*-dimensional total super-Euclidean space for* $L > 2n$ *and* $V \subset B_L^{1,1}$ *be an open set. Assume that*

$$c : V \subset B_L^{1,1} \rightarrow B_L^{m+n}$$

is a function and for $\forall\theta \in V \cap (B_L)_1$ *and* $\forall t \in V \cap (B_L)_0$

$$\begin{array}{rlcl} c_{\theta,0}: & V \cap (B_L)_0 & \rightarrow & \left(B_L^{m+n}\right)_0 \\ & t & \mapsto & c_{\theta,0}(t) = (c\,(t,\theta))_0 \\ c_{\theta,B}: & V \cap (B_L)_0 & \rightarrow & R^m \\ & t & \mapsto & c_{\theta,B}(t) = \varepsilon_{m,n}^{(L)} \circ c_{\theta,0}(t) \end{array} \tag{2.20}$$

are given where $(c\,(t,\theta))_0$ *is the even part of the supervector* $c\,(t,\theta)$*. The function* c *is to be supercurve if and only if* $c_{\theta,B}\,|_{V\cap R}$ *is a curve. The function* c *is called supersmooth supercurve if and only if*

$$\begin{array}{ll} c^i \in GH^\infty(V) & i \in \{1, 2, ..., m\} \\ c^{j+m} \in GH^\infty(V) & j \in \{1, 2, ..., n\} \end{array} \tag{2.21}$$

where

$$\begin{array}{ll} c^i = x^i \circ c & \forall i \in \{1, 2, ..., m\} \\ c^{j+m} = \theta^j \circ c & \forall j \in \{1, 2, ..., n\} \end{array} \tag{2.22}$$

[24].

Definition 2.11. *Let* c *be a regular smooth curve in Euclidean 4-space* E^4 *defined by*

$$x : s \in L \rightarrow x(s) \in E^4$$

where L *denotes a subset of the set* R *of all real numbers, and* s *is the arc-length parameter of* c*. The curve* c *is called a special Frenet curve if there exist three smooth functions* k_1*,* k_2*,* k_3 *on* c *and smooth frame field* $\{e_1, e_2, e_3, e_4\}$ *along the curve* c*. The formulas of Frenet-Serret hold:*

$$\begin{bmatrix} e_1' \\ e_2' \\ e_3' \\ e_4' \end{bmatrix} = \begin{bmatrix} 0 & k_1e_2 & 0 & 0 \\ -k_1e_2 & 0 & k_2e_3 & 0 \\ 0 & -k_2e_3 & 0 & k_3e_4 \\ 0 & 0 & -k_3e_4 & \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \tag{2.23}$$

for $s \in L$*, where the prime* $(')$ *denotes differentiation with respect to* s*. The frame field* $\{e_1, e_2, e_3, e_4\}$ *is of orthonormal positive orientation. The functions* k_1 *and* k_2 *are of positive, and the function* k_3 *doesn't vanish. Also, the functions* k_1*,* k_2*,* k_3 *are called the first, the second, and the third curvature function of* c*, respectively. The frame field* $\{e_1, e_2, e_3, e_4\}$ *is called Frenet frame field on* c [16]*. We refer this notion to* [28].

3. Frenet frame associated to a supersmooth supercurve in general position

In this part, using definition of supersmooth supercurve and Frenet frame associated to a supersmooth supercurve in general position [11], Frenet frame associated to a supersmooth supercurve in general position of even and odd part of super-Euclidean space B_L^{m+n} is given.

Definition 3.1. *Let $\left(B_L^{m+n}\right)_0$ be an even part of (m,n)-dimensional total super-Euclidean space for $L>2n$ and $V\subset B_L^{1,1}$ be an open set and $c: V\subset B_L^{1,1}\to B_L^{m+n}$ be supersmooth supercurve. The supercurve c is in general position if and only if*

$$\left\{G_1c(t,\theta),...,G_1^{(m-1)}c(t,\theta),G_2c(t,\theta),G_1G_2c(t,\theta),...,G_1^{(n-1)}G_2c(t,\theta)\right\}$$

are linearly independent where $G_1c(t,\theta)$ is a supervector which is expressed by

$$\left(G_1c^1(t,\theta),...,G_1c^m(t,\theta),G_1c^{m+1}(t,\theta),...,G_1c^{m+n}(t,\theta)\right)$$

and same as $G_2c(t,\theta)$ is a supervector which is expressed by

$$\left(G_2c^1(t,\theta),...,G_2c^m(t,\theta),G_2c^{m+1}(t,\theta),...,G_2c^{m+n}(t,\theta)\right) \tag{3.1}$$

with

$$G_1^{(0)}c(t,\theta)=c(t,\theta),G_1^{(1)}c(t,\theta)=G_1c(t,\theta),...,G_1^{(s)}c(t,\theta)=\underbrace{G_1...G_1}_{s-times}c(t,\theta)$$

where $\forall(t,\theta)\in V\subset B_L^{1,1}$ [11].

Definition 3.2. *Let $\left(B_L^{m+n}\right)_0$ be an even part of (m,n)-dimensional total super-Euclidean space for $L>2n$ and $V\subset B_L^{1,1}$ be an open set. Consider that*

$$c: V\subset B_L^{1,1}\to\left(B_L^{m+n}\right)_0$$

is a supersmooth supercurve. By a Frenet frame associated to a supersmooth supercurve c we shall mean a system of $m+n$ supervector fields $\{e_1,...,e_{m+n}\}$ along to the supersmooth supercurve c for $\forall(t,\theta)\in V\subset B_L^{1,1}$, we have the following properties:

$$\begin{array}{ll}
\langle e_k(t,\theta),e_h(t,\theta)\rangle=\delta_{kh} & \forall k,h\in\{1,2,...,m\}\\
\left\langle e_{m+\bar{j}}(t,\theta),e_{m+j}(t,\theta)\right\rangle=-\delta_{j\bar{j}} & \begin{array}{l}\forall j\in\{1,2,...,r\},\\ \bar{j}\in\{r+1,r+2,...,2r=n\}\end{array}\\
\left\langle e_{m+j}(t,\theta),e_{m+\bar{j}}(t,\theta)\right\rangle=\delta_{j\bar{j}} & \begin{array}{l}\forall j\in\{1,2,...,r-1\},\\ \bar{j}\in\{r+1,r+2,...,2r=n\}\end{array}\\
\langle e_{m+j_1}(t,\theta),e_{m+j_2}(t,\theta)\rangle=0 & \forall j_1,j_2\in\{1,2,...,r\}\\
\left\langle e_{m+\bar{j}_1}(t,\theta),e_{m+\bar{j}_2}(t,\theta)\right\rangle=0 & \forall\bar{j}_1,\bar{j}_2\in\{r+1,r+2,...,2r=n\}\\
\langle e_i(t,\theta),e_{m+j}(t,\theta)\rangle=0 & \forall i\in\{1,2,...,m\},\forall j\in\{1,2,...,n\}
\end{array} \tag{3.2}$$

where

$$Sp\left(G_1c(t,\theta),...,G_1^{(k)}c(t,\theta)\right)=Sp\left(e_1(t,\theta),...,e_k(t,\theta)\right)\qquad\forall k\in\{1,2,...,m-1\}, \tag{3.3}$$

and

$$Sp\left(G_2c(t,\theta),G_1G_2c(t,\theta),...,G_1^{(j-1)}G_2c(t,\theta)\right)=Sp\left(e_{m+1}(t,\theta),...,e_{m+j}(t,\theta)\right) \tag{3.4}$$

$\forall i\in\{1,2,...,m\}$ *and* $\forall j\in\{1,2,...,n\}$ [11].

Theorem 3.3. *Let $\left(B_L^{m+n}\right)_0$ be an even part of (m,n)-dimensional total super-Euclidean space B_L^{m+n} for $L>2n$ and $V\subset B_L^{1,1}$ be an open set and*

$$c: V\subset B_L^{1,1}\to \left(B_L^{m+n}\right)_0$$

be a supersmooth supercurve in general position which is satisfied following relation: For $\forall(t,\theta)\in V\subset B_L^{1,1}$,

$$\begin{array}{ll}
\varepsilon^{(L)}\left(\left\langle G_1c(t,\theta), G_1^{(r)}G_2c(t,\theta)\right\rangle\right)>0 & \\
\varepsilon^{(L)}\left(\left\langle G_1^{j_1}c(t,\theta), G_1^{(r+j_1)}G_2c(t,\theta)\right\rangle\right)>0 & \forall j_1,j_2\in\{1,2,...,r-1\}\\
\varepsilon^{(L)}\left(\left\langle G_2c(t,\theta), G_1^{(j)}G_2c(t,\theta)\right\rangle\right)=0 & \forall j\in\{1,2,...,n-1\}\\
\varepsilon^{(L)}(\left\langle G_1^{j\prime}G_2c(t,\theta), G_1^{(r+j_1)}G_2c(t,\theta)\right\rangle)=0 & \forall j\prime,\ j\in\{1,2,...,r-1\}\\
 & j\prime\neq\ j, j\prime<\ j.
\end{array}\tag{3.5}$$

Then there exists a unique Frenet frame $\{e_1,...,e_{m+n}\}$ associated to the supercurve c and for $\forall(t,\theta)\in V\subset B_L^{1,1}$

$$\begin{array}{ll}
G_1e_k(t,\theta)=\sum\limits_{h=1}^{m}a_{kh}(t,\theta)e_h(t,\theta) & \forall k\in\{1,2,...,m\}\\
G_1e_{m+j}(t,\theta)=\sum\limits_{l=1}^{m}a_{m+j\quad m+l}(t,\theta)e_{m+l}(t,\theta) & \forall j\in\{1,2,...,n\}
\end{array}\tag{3.6}$$

where

$$\begin{array}{ll}
a_{kh}(t,\theta)+a_{hk}(t,\theta)=0 & \forall k,h\in\{1,2,...,m\}\\
a_{kh}(t,\theta)=0 & h>k,\ \forall k,h\in\{1,2,...,m\}\\
a_{m+j_1\quad m+j_2}(t,\theta)+a_{m+r+j_2\quad m+r+j_1}(t,\theta)=0 & \forall j_1,j_2\in\{1,2,...,r\}\\
a_{m+r+j_1\quad m+j_2}(t,\theta)-a_{m+j_2\quad m+r+j_1}(t,\theta)=0 & \forall j_1,j_2\in\{1,2,...,r\}\\
a_{i\quad m+j}(t,\theta)=0 & \forall i\in\{1,2,...,m\}\\
a_{m+j\quad i}(t,\theta)=0 & j\in\{1,2,...,n\}\\
a_{m+j\quad m+l}(t,\theta)=0 & l\neq j+1
\end{array}\tag{3.7}$$

and

$$\begin{array}{ll}
a_{m+j_1\quad m+j_2}(t,\theta)=\langle G_1e_{m+j_1}(t,\theta), e_{m+r+j_2}(t,\theta)\rangle & \forall j_1,j_2\in\{1,2,...,r\}\\
a_{m+j_1\quad m+r+j_2}(t,\theta)=-\langle G_1e_{m+j_1}(t,\theta), e_{m+j_2}(t,\theta)\rangle & \forall j_1,j_2\in\{1,2,...,r\}\\
a_{m+r+j_1\quad m+j_2}(t,\theta)=\langle G_1e_{m+r+j_1}(t,\theta), e_{m+r+j_2}(t,\theta)\rangle & \forall j_1,j_2\in\{1,2,...,r\}\\
a_{m+r+j_1\quad m+r+j_2}(t,\theta)=-\langle G_1e_{m+r+j_1}(t,\theta), e_{m+j_2}(t,\theta)\rangle & \forall j_1,j_2\in\{1,2,...,r\}\\
a_{kh}(t,\theta)=\langle G_1e_k(t,\theta), e_h(t,\theta)\rangle & \forall k,h\in\{1,2,...,m\}\\
a_{k\quad m+j}(t,\theta)=\langle G_1e_k(t,\theta), e_{m+j}(t,\theta)\rangle & [\forall k\in\{1,2,...,m\},\\
a_{m+j\quad k}(t,\theta)=\langle G_1e_{m+j}(t,\theta), e_k(t,\theta)\rangle & \forall j\in\{1,2,...,n\}]
\end{array}\tag{3.8}$$

are obtained [11].

Definition 3.4. *Let $\left(B_L^{m+n}\right)_1$ be an odd part of (m,n)-dimensional total super-Euclidean space for $L>2n$ and $V\subset B_L^{1,1}$ be an open set and*

$$c: V\subset B_L^{1,1}\to \left(B_L^{m+n}\right)_1$$

be supersmooth supercurve. The supercurve c is in general position if and only if

$$\left\{G_2c(t,\theta), G_1G_2c(t,\theta), ..., G_1^{(m-1)}G_2c(t,\theta), G_1c(t,\theta), ..., G_1^{(n-1)}c(t,\theta)\right\}$$

are linearly independent where $G_1c(t,\theta)$ *is a supervector which is expressed by*

$$\left(G_1c^1(t,\theta),...,G_1c^m(t,\theta),G_1c^{m+1}(t,\theta),...,G_1c^{m+n}(t,\theta)\right)$$

and same as $G_2c(t,\theta)$ *is a supervector which is expressed by*

$$\left(G_2c^1(t,\theta),...,G_2c^m(t,\theta),G_2c^{m+1}(t,\theta),...,G_2c^{m+n}(t,\theta)\right)$$

with

$$G_1^{(0)}c(t,\theta)=c(t,\theta),G_1^{(1)}c(t,\theta)=G_1c(t,\theta),...,G_1^{(s)}c(t,\theta)=\underbrace{G_1...G_1}_{s-times}c(t,\theta).$$

where for $\forall(t,\theta)\in V\subset B_L^{1,1}$.

Definition 3.5. *Let* $\left(B_L^{m+n}\right)_1$ *be an odd part of* (m,n)*-dimensional total super-Euclidean space for* $L>2n$ *and* $V\subset B_L^{1,1}$ *be an open set. Consider*

$$c:V\subset B_L^{1,1}\to\left(B_L^{m+n}\right)_1$$

supersmooth supercurve. By a Frenet frame associated to a supersmooth supercurve $c:V\subset B_L^{1,1}\to\left(B_L^{m+n}\right)_1$ *we shall mean a system of* $m+n$ *supervector fields* $\{e_1,...,e_{m+n}\}$ *along to the supersmooth supercurve* c *such that for* $\forall(t,\theta)\in V\subset B_L^{1,1}$ *we have the following properties:*

$$\begin{array}{ll}
\langle e_k(t,\theta),e_h(t,\theta)\rangle=0 & \forall k,h\in\{1,2,...,r\}\\
\langle e_{r+j_1}(t,\theta),e_{r+j_2}(t,\theta)\rangle=0 & \forall j_1,j_2\in\{1,2,...,r\}\\
\langle e_{j_1}(t,\theta),e_{r+j_2}(t,\theta)\rangle=\delta_{j_1j_2} & \forall j_1,j_2\in\{1,2,...,r\}\\
\langle e_{r+j_1}(t,\theta),e_{j_2}(t,\theta)\rangle=-\delta_{j_1j_2} & \forall j_1,j_2\in\{1,2,...,r\}\\
\langle e_{m+j_1}(t,\theta),e_{m+j_2}(t,\theta)\rangle=\delta_{j_1j_2} & \forall j_1,j_2\in\{1,2,...,n\}\\
\langle e_{j_1}(t,\theta),e_{m+j_2}(t,\theta)\rangle=0 & [\forall j_1\in\{1,2,...,r\},\\
\langle e_{r+j_1}(t,\theta),e_{m+j_2}(t,\theta)\rangle=0 & \forall j_2\in\{1,2,...,n\}]
\end{array}\tag{3.9}$$

where

$$Sp\left(G_2c(t,\theta),G_1G_2c(t,\theta),...,G_1^{(j-1)}G_2c(t,\theta)\right)=Sp\left(e_1(t,\theta),...,e_j(t,\theta)\right)\tag{3.10}$$

$\forall i\in\{1,2,...,m\}$, $\forall j\in\{1,2,...,n\}$ *and*

$$Sp\left(G_1c(t,\theta),...,G_1^{(k)}c(t,\theta)\right)=Sp\left(e_{m+1}(t,\theta),...,e_{m+k}(t,\theta)\right)\tag{3.11}$$

$\forall k\in\{1,2,...,m-1\}$.

Theorem 3.6. *Let* $\left(B_L^{m+n}\right)_1$ *be an odd part of* (m,n)*-dimensional total super-Euclidean space* B_L^{m+n} *for* $L>2n$ *and* $V\subset B_L^{1,1}$ *be an open set and*

$$c:V\subset B_L^{1,1}\to\left(B_L^{m+n}\right)_1$$

be a supersmooth supercurve in general position which satisfies following relations: For $\forall(t,\theta)\in V\subset B_L^{1,1}$,

$$\begin{array}{ll}
\varepsilon'^L\left(\left\langle G_2c(t,\theta),G_1^{(r)}G_2c(t,\theta)\right\rangle\right)>0 & \\
\varepsilon'^L(\left\langle G_1^{j_1}G_2c(t,\theta),G_1^{(r+j_1)}G_2c(t,\theta)\right\rangle)>0 & \forall j_1,j_2\in\{1,2,...,r-1\}\\
\varepsilon'^L\left(\left\langle G_2c(t,\theta),G_1^{(j)}G_2c(t,\theta)\right\rangle\right)=0 & \forall j\in\{1,2,...,n-1\}\\
\varepsilon'^L(\left\langle G_1^{j\prime}G_2c(t,\theta),G_1^{(r+j_1)}G_2c(t,\theta)\right\rangle)=0 & \forall j\prime,\ j\in\{1,2,...,r-1\}\\
 & j\prime\neq\ j,j\prime<\ j.
\end{array}\tag{3.12}$$

Then there exists a unique Frenet frame $\{e_1, ..., e_{m+n}\}$ *associated to the supercurve* c *and for* $\forall(t,\theta) \in V \subset B_L^{1,1}$

$$\begin{aligned} G_1 e_k(t,\theta) &= \sum_{h=1}^{m} a_{kh}(t,\theta) e_h(t,\theta) & \forall k \in \{1,2,...,m\} \\ G_1 e_{m+j}(t,\theta) &= \sum_{l=1}^{m} a_{m+j \quad m+l}(t,\theta) e_{m+l}(t,\theta) & \forall j \in \{1,2,...,n\} \end{aligned} \tag{3.13}$$

are obtained where

$$\begin{array}{ll} a_{j_1\ j_2}(t,\theta) + a_{r+j_2\ r+j_1}(t,\theta) = 0 & \forall j_1, j_2 \in \{1,2,...,r\} \\ a_{r+j_1 \quad j_2}(t,\theta) - a_{r+j_2 \quad j_1}(t,\theta) = 0 & \forall j_1, j_2 \in \{1,2,...,r\} \\ a_{j_1 \quad r+j_2}(t,\theta) - a_{j_2 \quad r+j_1}(t,\theta) = 0 & \forall j_1, j_2 \in \{1,2,...,r\} \\ a_{j_1, \quad m+j_2}(t,\theta) = a_{j_1, \quad m+j_2}(t,\theta) = 0 & \forall j_1 \in \{1,2,...,m\}\,, \ \forall j_2 \in \{1,2,...,n\} \\ a_{kh}(t,\theta) = 0 & h \neq k+1, \ \forall k, h \in \{1,2,...,n\} \end{array} \tag{3.14}$$

and

$$\begin{array}{ll} a_{j_1 j_2}(t,\theta) = \langle G_1 e_{j_1}(t,\theta), e_{r+j_2}(t,\theta)\rangle & \forall j_1, j_2 \in \{1,2,...,r\} \\ a_{j_1 \quad r+j_2}(t,\theta) = -\langle G_1 e_{j_1}(t,\theta), e_{j_2}(t,\theta)\rangle & \forall j_1, j_2 \in \{1,2,...,r\} \\ a_{r+j_1 \quad j_2}(t,\theta) = \langle G_1 e_{r+j_1}(t,\theta), e_{r+j_2}(t,\theta)\rangle & \forall j_1, j_2 \in \{1,2,...,r\} \\ a_{r+j_1 \quad r+j_2}(t,\theta) = -\langle G_1 e_{r+j_1}(t,\theta), e_{j_2}(t,\theta)\rangle & \forall j_1, j_2 \in \{1,2,...,r\} \\ a_{m+j_1 \quad m+j_2}(t,\theta) = \langle G_1 e_{m+j_1}(t,\theta), e_{m+j_2}(t,\theta)\rangle & \forall j_1, \ j_2 \in \{1,2,...,n\} \\ a_{j_1 \quad m+j_2}(t,\theta) = \langle G_1 e_{j_1}(t,\theta), e_{m+j_2}(t,\theta)\rangle & [\forall j_1 \in \{1,2,...,m\}\,, \\ a_{m+j_1 \quad j_2}(t,\theta) = \langle G_1 e_{m+j}(t,\theta), e_k(t,\theta)\rangle & \forall j_2 \in \{1,2,...,m\}]. \end{array} \tag{3.15}$$

4. On Bertrand Supercurves in Super-Euclidean Space

In this section, we introduce Bertrand supercurve couple and give some theorems in super-Euclidean space.

Let $M_1, M_2 \subset B_L^{m+n}$ be two supersmooth supercurves given by (V, c) and (V, c^*), respectively. For $(t,\theta) \in V$, c^* is called Bertrand of the supercurve c or (M_1, M_2) is called Bertrand supercurve couple, if principal normals of body parts at the point $c(t,\theta)$ and $c^*(t,\theta)$ are linearly dependent where $V \subset B_L^{(1,1)}$ is an open subset.

Theorem 4.1. *Let* (M_1, M_2) *be Bertrand supercurve couple which are given by coordinate neighbourhoods* (V, c) *and* (V, c^*) *in* $\left(B_L^{m+n}\right)_0$*, respectively. The distance between the points* $c(t,\theta) \in M_1$ *and* $c^*(t,\theta) \in M_2$ *is given by*

$$d(c(t,\theta), c^*(t,\theta)) = b$$

where b *is a superconstant.*

Proof. If (M_1, M_2) is Bertrand supercurve couple, we have

$$c^*(t,\theta) = c(t,\theta) + A(t,\theta) e_2(t,\theta) \tag{4.1}$$

where $A(t,\theta)$ is supervariable. Differentiating both sides of the expression (4.1) with respect to t:

$$G_1^* c^*(t,\theta) \frac{dt^*}{dt} = G_1 c(t,\theta) + G_1 A(t,\theta) e_2(t,\theta) + (-1)^{|t||A(t,\theta)|} A(t,\theta) G_1 e_2(t,\theta). \tag{4.2}$$

From the equation (3.6), we get

$$G_1^* c^*(t,\theta)\frac{dt^*}{dt} = G_1 c(t,\theta) + G_1 A(t,\theta)e_2(t,\theta) + (-1)^{|t||A(t,\theta)|}A(t,\theta)a_{23}(t,\theta)e_3(t,\theta) \tag{4.3}$$

where t and t^* are arc–parameters of M_1 and M_2, respectively.

Thus we have

$$\begin{array}{rl} e_1^*(t,\theta)\frac{dt^*}{dt} = & (1+(-1)^{|t||A(t,\theta)|}A(t,\theta)a_{21}(t,\theta))e_1(t,\theta) + G_1 A(t,\theta)e_2(t,\theta) \\ & +A(t,\theta)a_{23}(t,\theta)e_3(t,\theta). \end{array} \tag{4.4}$$

Multiplying the the equation (4.4) with $e_2(t,\theta)$ by superscalar product, we have

$$\begin{array}{rl} \langle e_1^*(t,\theta), e_2(t,\theta)\rangle \frac{dt^*}{dt} = & (1+(-1)^{|t||A(t,\theta)|}A(t,\theta)a_{21}(t,\theta))\,\langle e_1(t,\theta), e_2(t,\theta)\rangle \\ & +G_1 A(t,\theta)\,\langle e_2(t,\theta), e_2(t,\theta)\rangle \\ & +(-1)^{|t||A(t,\theta)|}A(t,\theta)a_{23}(t,\theta)\,\langle e_3(t,\theta), e_2(t,\theta)\rangle\,. \end{array} \tag{4.5}$$

From the definition of Bertrand supercurve couple $\varepsilon^{(L)}\,\langle e_1^*(t,\theta), e_2(t,\theta)\rangle = 0$. Thus we obtain

$$\begin{array}{ll} G_1\varepsilon^{(L)}\,(A(t,\theta)) & = 0 \\ \varepsilon^{(L)}\,(A(t,\theta)) & = b \end{array} \tag{4.6}$$

and

$$\begin{array}{rl} 0 & = \varepsilon^{(L)}\,(a_{21}(t,\theta))\,s\,\langle e_1(t,\theta), e_2(t,\theta)\rangle + s\,(a_{21}(t,\theta))\,s\,\langle e_1(t,\theta), e_2(t,\theta)\rangle \\ & +(-1)^{|t||A(t,\theta)|}\varepsilon^{(L)}\,(A(t,\theta))\,\varepsilon^{(L)}\,(a_{21}(t,\theta))\,s\,\langle e_1(t,\theta), e_2(t,\theta)\rangle \\ & +(-1)^{|t||A(t,\theta)|}\varepsilon^{(L)}\,(A(t,\theta))\,s\,(a_{21}(t,\theta))\,s\,\langle e_1(t,\theta), e_2(t,\theta)\rangle \\ & +(-1)^{|t||A(t,\theta)|}s\,(A(t,\theta))\,\varepsilon^{(L)}\,(a_{21}(t,\theta))\,s\,\langle e_1(t,\theta), e_2(t,\theta)\rangle \\ & +(-1)^{|t||A(t,\theta)|}s\,(A(t,\theta))\,s\,(a_{21}(t,\theta))\,s\,\langle e_1(t,\theta), e_2(t,\theta)\rangle \\ & +G_1\varepsilon^{(L)}\,(A(t,\theta))\,(s\,\langle e_2(t,\theta), e_2(t,\theta)\rangle) \\ & +G_1 s\,(A(t,\theta))\,(s\,\langle e_2(t,\theta), e_2(t,\theta)\rangle) + G_1 s\,(A(t,\theta)) \\ & +(-1)^{|t||A(t,\theta)|}\varepsilon^{(L)}\,(A(t,\theta))\,\varepsilon^{(L)}\,(a_{23}(t,\theta))\,s\,\langle e_3(t,\theta), e_2(t,\theta)\rangle \\ & +(-1)^{|t||A(t,\theta)|}\varepsilon^{(L)}\,(A(t,\theta))\,s\,(a_{23}(t,\theta))\,s\,\langle e_3(t,\theta), e_2(t,\theta)\rangle \\ & +(-1)^{|t||A(t,\theta)|}s\,(A(t,\theta))\,\varepsilon^{(L)}\,(a_{23}(t,\theta))\,(s\,\langle e_3(t,\theta), e_2(t,\theta)\rangle) \\ & +(-1)^{|t||A(t,\theta)|}s\,(A(t,\theta))\,s\,(a_{23}(t,\theta))\,s\,\langle e_3(t,\theta), e_2(t,\theta)\rangle \end{array} \tag{4.7}$$

where b is a superconstant. From the definition of the distance on total super-Euclidean space, we can easily find

$$d(c(t,\theta), c^*(t,\theta)) = b \tag{4.8}$$

where b is the superconstant. □

Theorem 4.2. *Let (M_1, M_2) be Bertrand supercurve couple which are given by coordinate neighbourhoods (V, c) and (V, c^*) in $\left(B_L^{m+n}\right)_1$, respectively. The distance between the points $c(t,\theta) \in M_1$ and $c^*(t,\theta) \in M_2$ is given by*

$$d(c(t,\theta), c^*(t,\theta)) = b$$

where b is a superconstant.

Proof. If (M_1, M_2) is Bertrand supercurve couple, we have

$$c^*(t,\theta) = c(t,\theta) + A(t,\theta)e_6(t,\theta) \tag{4.9}$$

where $A(t,\theta)$ is supervariable. Differentiating both sides of the expression (4.9) with respect to t:

$$G_1^* c^*(t,\theta)\frac{dt^*}{dt} = G_1 c(t,\theta)+G_1 A(t,\theta)e_6(t,\theta)+(-1)^{|t||A(t,\theta)|}A(t,\theta)G_1 e_6(t,\theta). \quad (4.10)$$

From the equation (3.13), we get

$$G_1^* c^*(t,\theta)\frac{dt^*}{dt} = G_1 c(t,\theta) + G_1 A(t,\theta)e_6(t,\theta) + (-1)^{|t||A(t,\theta)|}A(t,\theta)a_{67}(t,\theta)e_7(t,\theta) \quad (4.11)$$

where t and t^* are arc-parameters of M_1 and M_2, respectively.

Thus, we have

$$\begin{aligned} e_5^*(t,\theta)\tfrac{dt^*}{dt} = \quad & (1+(-1)^{|t||A(t,\theta)|}A(t,\theta)a_{65}(t,\theta))e_7(t,\theta) \\ & +G_1 A(t,\theta)e_6(t,\theta) + A(t,\theta)a_{67}(t,\theta)e_7(t,\theta). \end{aligned} \quad (4.12)$$

Multiplying the the equation (4.12) with $e_6(t,\theta)$ by superscalar product, we get

$$\begin{aligned} \langle e_5^*(t,\theta), e_6(t,\theta)\rangle \tfrac{dt^*}{dt} = \quad & (1+(-1)^{|t||A(t,\theta)|}A(t,\theta)a_{65}(t,\theta)) \\ & \cdot \langle e_5(t,\theta), e_6(t,\theta)\rangle + G_1 A(t,\theta)\langle e_6(t,\theta), e_6(t,\theta)\rangle \\ & +(-1)^{|t||A(t,\theta)|}A(t,\theta)a_{67}(t,\theta)\langle e_7(t,\theta), e_6(t,\theta)\rangle . \end{aligned} \quad (4.13)$$

From the definition of Bertrand supercurve couple $\varepsilon'^{(L)}\langle e_5^*(t,\theta), e_6(t,\theta)\rangle = 0$. Thus we obtain

$$\begin{aligned} G_1\varepsilon'^{(L)}(A(t,\theta)) &= 0 \\ \varepsilon'^{(L)}(A(t,\theta)) &= b \end{aligned} \quad (4.14)$$

and

$$\begin{aligned} 0 \quad &= \varepsilon'^{(L)}(a_{65}(t,\theta))\, s\langle e_1(t,\theta), e_6(t,\theta)\rangle + s(a_{65}(t,\theta))\, s\langle e_5(t,\theta), e_6(t,\theta)\rangle \\ &+(-1)^{|t||A(t,\theta)|}\varepsilon'^{(L)}(A(t,\theta))\,\varepsilon'^{(L)}(a_{65}(t,\theta))\, s\langle e_5(t,\theta), e_6(t,\theta)\rangle \\ &+(-1)^{|t||A(t,\theta)|}\varepsilon'^{(L)}(A(t,\theta))\, s(a_{65}(t,\theta))\, s\langle e_5(t,\theta), e_6(t,\theta)\rangle \\ &+(-1)^{|t||A(t,\theta)|}s(A(t,\theta))\,\varepsilon'^{(L)}(a_{65}(t,\theta))\, s\langle e_5(t,\theta), e_6(t,\theta)\rangle \\ &+(-1)^{|t||A(t,\theta)|}s(A(t,\theta))\, s(a_{65}(t,\theta))\, s\langle e_5(t,\theta), e_6(t,\theta)\rangle \\ &+G_1\varepsilon'^{(L)}(A(t,\theta))\,(s\langle e_6(t,\theta), e_6(t,\theta)\rangle) \\ &+G_1 s(A(t,\theta))\,(s\langle e_6(t,\theta), e_6(t,\theta)\rangle) + G_1 s(A(t,\theta)) \\ &+(-1)^{|t||A(t,\theta)|}\varepsilon'^{(L)}(A(t,\theta))\,\varepsilon'^{(L)}(a_{67}(t,\theta))\, s\langle e_{53}(t,\theta), e_6(t,\theta)\rangle \\ &+(-1)^{|t||A(t,\theta)|}\varepsilon'^{(L)}(A(t,\theta))\, s(a_{67}(t,\theta))\, s\langle e_7(t,\theta), e_6(t,\theta)\rangle \\ &+(-1)^{|t||A(t,\theta)|}s(A(t,\theta))\,\varepsilon'^{(L)}(a_{67}(t,\theta))\,(s\langle e_7(t,\theta), e_6(t,\theta)\rangle) \\ &+(-1)^{|t||A(t,\theta)|}s(A(t,\theta))\, s(a_{67}(t,\theta))\, s\langle e_7(t,\theta), e_6(t,\theta)\rangle \end{aligned} \quad (4.15)$$

where b is a superconstant. From the definition of the distance on total super-Euclidean space, we can easily find

$$d(c(t,\theta), c^*(t,\theta)) = b \quad (4.16)$$

where b is the superconstant. □

Theorem 4.3. *Let M_1, M_2 be supersmooth supercurves which are given by coordinate neighbourhoods (V,c) and (V,c^*) in $\left(B_L^{m+n}\right)_0$, respectively. Then, M_1, M_2 are Bertrand supercurves if and only if*

$$\lambda\varepsilon^{(L)}(a_{21}(t,\theta)) + \mu\varepsilon^{(L)}(a_{23}(t,\theta)) = 1$$

where λ, μ are superconstants and $a_{21}(t,\theta)$, $a_{23}(t,\theta)$ are supercurvatures in M_1.

Proof. If (M_1, M_2) is Bertrand supercurve couple from *Theorem* 4.1, we have

$$\begin{aligned} e_1^*(t,\theta)\tfrac{dt^*}{dt} = \;& (1-(-1)^{|t||b+s(A(t,\theta))|}\,(b+s(A(t,\theta))))a_{21}(t,\theta))e_1(t,\theta) \\ & +(-1)^{|t||b+s(A(t,\theta))|}\,(b+s(A(t,\theta))))a_{23}(t,\theta)e_3(t,\theta) \end{aligned} \tag{4.17}$$

where $s(A(t,\theta))$ is a odd part of supervariable $A(t,\theta)$ and b is a superconstant. Differentiating both sides of the expression (4.17) with respect to t and from the equation (3.6), an equation

$$\begin{aligned} a_{12}^*(t,\theta)e_2^*(t,\theta)\tfrac{dt^*}{dt} = \;& G_1A_1(t,\theta)e_1(t,\theta)+G_1B_1(t,\theta)e_3(t,\theta) \\ & +[(-1)^{|t||A_1(t,\theta)|}a_{12}(t,\theta) \\ & -(-1)^{|t||B_1(t,\theta)|}B_1(t,\theta)a_{23}(t,\theta)]e_2(t,\theta) \end{aligned} \tag{4.18}$$

is obtained that $A_1(t,\theta) = (1-(-1)^{|t||b+s(A(t,\theta))|}\,(b+s(A(t,\theta)))a_{21}(t,\theta))$ and $B_1(t,\theta) = (-1)^{|t||b+s(A(t,\theta))|}\,(b+s(A(t,\theta)))a_{23}(t,\theta)$. Since (M_1, M_2) is Bertrand supercurve couple, we have

$$e_1^*(t,\theta) = A_1(t,\theta)e_1(t,\theta)+B_1(t,\theta)e_3(t,\theta) \tag{4.19}$$

where $A_1(t,\theta)$ and $B_1(t,\theta)$ are supervariables. Let us differentiate the equation (4.19) with respect to t and use the equation (3.5)

$$\begin{aligned} a_{12}^*(t,\theta)e_2^*(t,\theta)\tfrac{dt^*}{dt} = \;& G_1A_1(t,\theta)e_1(t,\theta)+G_1B_1(t,\theta)e_3(t,\theta) \\ & +[(-1)^{|t||A_1(t,\theta)|}A_1(t,\theta)a_{12}(t,\theta) \\ & -(-1)^{|t||B_1(t,\theta)|}B_1(t,\theta)a_{23}(t,\theta)]e_3(t,\theta) \end{aligned} \tag{4.20}$$

is found. Since $\{e_2(t,\theta), e_2^*(t,\theta)\}$ is a linearly dependent set and using the equation (4.18), we get

$$\varepsilon^{(L)}\,(G_1A_1(t,\theta)) = 0 \text{ and } \varepsilon^{(L)}\,(G_1B_1(t,\theta)) = 0. \tag{4.21}$$

Then, using $\varepsilon^{(L)}\,(A_1(t,\theta))$ =superconstant and $\varepsilon^{(L)}\,(B_1(t,\theta))$ =superconstant

$$\varepsilon^{(L)}\left(\frac{A_1(t,\theta)}{B_1(t,\theta)}\right) = a \tag{4.22}$$

is written where a is a superconstant. From the equation (4.18) and (4.19),

$$\begin{aligned} B_1(t,\theta) = \;& (-1)^{|t||A(t,\theta)|}\,(b+s(A(t,\theta)))a_{21}(t,\theta)B_1(t,\theta) \\ & +(-1)^{|t||A(t,\theta)|}A_1(t,\theta)\,(b+s(A(t,\theta)))a_{23}(t,\theta). \end{aligned} \tag{4.23}$$

If we divide the equation (4.23) with $B_1(t,\theta)$ and seperate into the even and odd parts, then we get

$$\begin{aligned} (-1)^{|t||A(t,\theta)|} = \;& (b+s(A(t,\theta)))\left[\varepsilon^{(L)}\,(a_{21}(t,\theta))+s\,(a_{21}(t,\theta))\right] \\ & +\left[\varepsilon^{(L)}\left(\tfrac{A_1(t,\theta)}{B_1(t,\theta)}\right)+s\left(\tfrac{A_1(t,\theta)}{B_1(t,\theta)}\right)\right](b+s(A(t,\theta))) \\ & \cdot\left[\varepsilon^{(L)}\,(a_{23}(t,\theta))+s\,(a_{23}(t,\theta))\right]. \end{aligned} \tag{4.24}$$

From even and odd parts of the equation (4.24), we get

$$\lambda\varepsilon^{(L)}\,(a_{21}(t,\theta))+\mu\varepsilon^{(L)}\,(a_{23}(t,\theta)) = 1 \tag{4.25}$$

and

$$\begin{aligned} 0 = \;& b\cdot s\,(a_{21}(t,\theta))+s(A(t,\theta)[\varepsilon^{(L)}\,(a_{21}(t,\theta))+s\,(a_{21}(t,\theta))] \\ & +\varepsilon^{(L)}\left(\tfrac{A_1(t,\theta)}{B_1(t,\theta)}\right)b\cdot s\,(a_{23}(t,\theta)) \\ & +s\left(\tfrac{A_1(t,\theta)}{B_1(t,\theta)}\right)[b\cdot\{\varepsilon^{(L)}\,(a_{23}(t,\theta))+s\,(a_{23}(t,\theta))\} \\ & +s(A(t,\theta)\{s\,(a_{23}(t,\theta))+\varepsilon^{(L)}\,(a_{23}(t,\theta))\}]. \end{aligned} \tag{4.26}$$

□

Theorem 4.4. *Let M_1, M_2 be supersmooth supercurves which are given by coordinate neighbourhoods (V, c) and (V, c^*) in $\left(B_L^{m+n}\right)_1$, respectively. Then (M_1, M_2) is Bertrand supercurve couple if and only if*

$$\lambda\varepsilon^{(L)}\left(a_{65}(t,\theta)\right) + \mu\varepsilon^{(L)}\left(a_{67}(t,\theta)\right) = 1$$

where λ, μ are superconstants and $a_{65}(t,\theta)$, $a_{67}(t,\theta)$ are supercurvatures in M_1.

Proof. If (M_1, M_2) is Bertrand supercurve couple from *Theorem* 4.2, we have

$$\begin{aligned} e_5^*(t,\theta)\tfrac{dt^*}{dt} = \quad & (1-(-1)^{|t||b+s(A(t,\theta))|}(b+s(A(t,\theta))))a_{65}(t,\theta))e_5(t,\theta) \\ & +(-1)^{|t||b+s(A(t,\theta))|}(b+s(A(t,\theta))))a_{67}(t,\theta)e_7(t,\theta) \end{aligned} \tag{4.27}$$

where $s(A(t,\theta))$ is a odd part of supervariable $A(t,\theta)$ and where b is a superconstant. Differentiating both sides of the expression (4.27) with respect to t and from the equation (3.13), an equation

$$\begin{aligned} a_{67}^*(t,\theta)e_6^*(t,\theta)\tfrac{dt^*}{dt} = \quad & G_1A_1(t,\theta)e_5(t,\theta) + G_1B_1(t,\theta)e_3(t,\theta) \\ & +[(-1)^{|t||A_1(t,\theta)|}a_{56}(t,\theta) \\ & -(-1)^{|t||B_1(t,\theta)|}B_1(t,\theta)a_{67}(t,\theta)]e_6(t,\theta) \end{aligned} \tag{4.28}$$

is obtained that $A_1(t,\theta) = (1-(-1)^{|t||b+s(A(t,\theta))|}(b+s(A(t,\theta)))a_{65}(t,\theta))$ and $B_1(t,\theta) = (-1)^{|t||b+s(A(t,\theta))|}(b+s(A(t,\theta)))a_{67}(t,\theta)$. Since (M_1, M_2) is Bertrand supercurve couple, we have

$$e_5^*(t,\theta) = A_1(t,\theta)e_5(t,\theta) + B_1(t,\theta)e_7(t,\theta) \tag{4.29}$$

where $A_1(t,\theta)$ and $B_1(t,\theta)$ are supervariables. Differentiating the equation (4.29) with respect to t and using the equation (3.9) gives

$$\begin{aligned} a_{65}^*(t,\theta)e_6^*(t,\theta)\tfrac{dt^*}{dt} = \quad & G_1A_1(t,\theta)e_5(t,\theta) + G_1B_1(t,\theta)e_7(t,\theta) \\ & +\left[A_1(t,\theta)a_{56}(t,\theta) - B_1(t,\theta)a_{67}(t,\theta)\right]e_7(t,\theta). \end{aligned} \tag{4.30}$$

From the equation (4.28) and $\{e_6(t,\theta), e_6^*(t,\theta)\}$ is a linearly dependent set, we get

$$\varepsilon'^{(L)}\left(G_1A_1(t,\theta)\right) = 0 \text{ and } \varepsilon'^{(L)}\left(G_1B_1(t,\theta)\right) = 0. \tag{4.31}$$

Then, using $\varepsilon'^{(L)}\left(A_1(t,\theta)\right)$ =superconstant and $\varepsilon'^{(L)}\left(B_1(t,\theta)\right)$ =superconstant

$$\varepsilon'^{(L)}\left(\frac{A_1(t,\theta)}{B_1(t,\theta)}\right) = a \tag{4.32}$$

is written where a is a supernumber. From the equation (4.28) and (4.29), we have

$$\begin{aligned} B_1(t,\theta) = \quad & (-1)^{|t||A(t,\theta)|}(b+s(A(t,\theta))))a_{65}(t,\theta)B_1(t,\theta) \\ & +(-1)^{|t||A(t,\theta)|}A_1(t,\theta)\left(b+s(A(t,\theta))\right))a_{67}(t,\theta). \end{aligned} \tag{4.33}$$

If we divide the equation (4.33) with $B_1(t,\theta)$ and seperate into the even and odd parts, then we get

$$\begin{aligned} (-1)^{|t||A(t,\theta)|} = \quad & (b+s(A(t,\theta)))\left[\varepsilon'^{(L)}\left(a_{65}(t,\theta)\right) + s\left(a_{65}(t,\theta)\right)\right] \\ & +\left[\varepsilon'^{(L)}\left(\tfrac{A_1(t,\theta)}{B_1(t,\theta)}\right) + s\left(\tfrac{A_1(t,\theta)}{B_1(t,\theta)}\right)\right] \\ & \cdot(b+s(A(t,\theta))) \\ & \cdot\left[\varepsilon'^{(L)}\left(a_{67}(t,\theta)\right) + s\left(a_{67}(t,\theta)\right)\right]. \end{aligned} \tag{4.34}$$

From even and odd parts of the equation (4.34), we get

$$\lambda\varepsilon'^{(L)}\left(a_{65}(t,\theta)\right) + \mu\varepsilon'^{(L)}\left(a_{67}(t,\theta)\right) = 1 \tag{4.35}$$

and

$$\begin{aligned} 0 = \quad & b\cdot s\left(a_{65}(t,\theta)\right) + s(A(t,\theta)[\varepsilon'^{(L)}\left(a_{65}(t,\theta)\right) + s\left(a_{65}(t,\theta)\right)] \\ & +\varepsilon'^{(L)}\left(\tfrac{A_1(t,\theta)}{B_1(t,\theta)}\right) b\cdot s\left(a_{67}(t,\theta)\right) \\ & +s\left(\tfrac{A_1(t,\theta)}{B_1(t,\theta)}\right)[b\cdot\{\varepsilon'^{(L)}\left(a_{67}(t,\theta)\right) + s\left(a_{67}(t,\theta)\right)\} \\ & +s(A(t,\theta)\{s\left(a_{67}(t,\theta)\right) + \varepsilon'^{(L)}\left(a_{67}(t,\theta)\right)\}]. \end{aligned} \tag{4.36}$$

□

Example Let B_L^{2+2} be a $(2,2)$ dimensional total super-Euclidean space, $V \subset B_L^{1,1}$ be an open subset,

$$\begin{array}{llll} c: & V\subset B_L^{1,1} & \to & B_L^{2,2} \\ & (t,\theta) & \mapsto & c(t,\theta) = \left(t^2+2, \theta\beta^2, \theta+2\beta^1 t-3, \theta t^2\right) \end{array} \tag{4.37}$$

be a supercurve. Supercurve $c(t,\theta)$ is supersmooth because the functions

$$c^1(t,\theta) = t^2+2,\ c^2(t,\theta) = \theta\beta^2,\ c^3(t,\theta) = \theta+2\beta^1 t-3,\ c^4(t,\theta) = \theta t^2 \tag{4.38}$$

are supersmooth. If we compute $G_1c(t,\theta), G_2c(t,\theta)$ and $G_1G_2c(t,\theta)$, then we have

$$\begin{aligned} G_1c(t,\theta) &= \left(2t, 0, 2\beta^1, 2\theta\cdot t\right) \\ G_2c(t,\theta) &= \left(0, \beta^2, 1, t^2\right) \\ G_1G_2c(t,\theta) &= (0,0,0,2t) \end{aligned} \tag{4.39}$$

Because of satisfying the equation (3.5), we get

$$\begin{aligned} \varepsilon^{(L)}\left(\langle G_2c(t,\theta), G_1G_2c(t,\theta)\rangle\right) &= 2t > 0 \\ \varepsilon^{(L)}\left(\langle G_2c(t,\theta), G_2c(t,\theta)\rangle\right) &= 0 \\ \varepsilon^{(L)}\left(\langle G_1G_2c(t,\theta), G_1G_2c(t,\theta)\rangle\right) &= 0 \end{aligned}$$

$e_1(t,\theta)$, $e_3(t,\theta)$ and $e_4(t,\theta)$ are obtained by

$$\begin{aligned} e_1(t,\theta) = \quad & \left(1, 0, 2\beta^1\cdot(2t)^{-1}, \theta\right) \\ e_3(t,\theta) = \quad & \left(0, \beta^2\cdot(2t)^{-1}, (2t)^{-1}, 2^{-1}t\right) \\ e_4(t,\theta) = \quad & (0,0,0,2t). \end{aligned} \tag{4.40}$$

Computing $G_1c(t,\theta), G_2c(t,\theta)$ and $G_1G_2c(t,\theta)$, the supervectors

$$\{G_1c(t,\theta), G_2c(t,\theta), G_1G_2c(t,\theta)\}$$

are linearly independent and then, system of the supervectors

$$\{e_1(t,\theta), e_2(t,\theta), e_3(t,\theta), e_4(t,\theta)\}$$

is Frenet frame of supercurve c. Let the matrix $M(t,\theta)$ be

$$M(t,\theta) = \begin{bmatrix} 1 & 0 & \theta & -2\beta^1(2t)^{-1} \\ 0 & \beta^2(2t)^{-1} & 2^{-1}t & -(2t)^{-1} \\ 0 & 0 & 2t & 0 \end{bmatrix} \tag{4.41}$$

and $e_2(t,\theta) = \left(-2\beta^2\beta^1(2t)^{-1}, -1, 0, -\beta^2\right)$ is computed. $a_{12}(t,\theta)$ and $a_{33}(t,\theta)$,

$$a_{12}(t,\theta) = \beta^1\beta^2 t^{-2} \tag{4.42}$$

and

$$a_{33}(t,\theta) = -t^{-1} \tag{4.43}$$

are obtained. Finally, the matrix

$$A=\begin{pmatrix} 0 & \beta^1\beta^2t^{-2} & 0 & 0 \\ -\beta^1\beta^2t^{-2} & 0 & 0 & 0 \\ 0 & 0 & -t^{-1} & 0 \\ 0 & 0 & 0 & -t^{-1} \end{pmatrix} \tag{4.44}$$

can be obtained.

Example Let B_L^{2+2} be a $(2,2)$ dimensional total super-Euclidean space, $V\subset B_L^{1,1}$ be an open subset and

$$\begin{array}{rcl} c^*:V\subset B_L^{1,1} & \to & B_L^{2,2} \\ (t,\theta) & \mapsto & c^*(t,\theta)=(t^2+1,\theta\beta^2-\dfrac{t}{\beta^2\beta^1},2\beta^1t+\theta-3,\theta t^2-\dfrac{t}{\beta^1}) \end{array} \tag{4.45}$$

be a supercurve. $c^*(t,\theta)$ supercurve is supersmooth because the functions

$$\begin{array}{llll} c^{*1}(t,\theta) & =t^2+1, & c^{*2}(t,\theta) & =\theta\beta^2-\left(\beta^2\beta^1\right)^{-1}t, \\ c^{*3}(t,\theta) & =2\beta^1t+\theta-3, & c^{*4}(t,\theta) & =\theta.t^2-\left(\beta^1\right)^{-1}t \end{array}$$

are supersmooth. We compute $G_1c^*(t,\theta),G_2c^*(t,\theta)$ and $G_1G_2c^*(t,\theta)$ as

$$\begin{array}{ll} G_1c^*(t,\theta) & =\left(2t,-\left(\beta^2\beta^1\right)^{-1},2\beta^1,2\theta t-\left(\beta^1\right)^{-1}\right) \\ G_2c^*(t,\theta) & =\left(0,\beta^2,1,t^2\right) \\ G_1G_2c^*(t,\theta) & =(0,0,0,2t)\,. \end{array}$$

Because of satisfying the equation (3.5),

$$\begin{array}{ll} \varepsilon^{(L)}\left(\langle G_2c^*(t,\theta),G_1G_2c^*(t,\theta)\rangle\right) & =2t>0 \\ \varepsilon^{(L)}\left(\langle G_2c^*(t,\theta),G_2c^*(t,\theta)\rangle\right) & =0 \\ \varepsilon^{(L)}\left(\langle G_1G_2c^*(t,\theta),G_1G_2c^*(t,\theta)\rangle\right) & =0 \end{array}$$

$e_1^*(t,\theta)$, $e_3^*(t,\theta)$ and $e_4^*(t,\theta)$ are obtained as

$$\begin{array}{ll} e_1^*(t,\theta)= & \left(1,-\left(\beta^2\beta^1\right)^{-1}(2t)^{-1},2\beta^1(2t)^{-1},\theta-\left(\beta^1\right)^{-1}(2t)^{-1}\right) \\ e_3^*(t,\theta)= & \left(0,\beta^2(2t)^{-1},(2t)^{-1},2^{-1}t\right) \\ e_4^*(t,\theta)= & (0,0,0,2t)\,. \end{array}$$

Computing $G_1c^*(t,\theta),G_2c^*(t,\theta)$ and $G_1G_2c^*(t,\theta)$, the supervectors $\{G_1c^*(t,\theta),G_2c^*(t,\theta),G_1G_2c^*(t,\theta)\}$ are linearly independent and then system of the supervectors $\{e_1^*(t,\theta),e_2^*(t,\theta),e_3^*(t,\theta),e_4^*(t,\theta)\}$ is Frenet frame of supercurve c. Let the matrix $M(t,\theta)$ be

$$M(t,\theta)=\begin{bmatrix} 1 & -\left(\beta^2\beta^1\right)^{-1}(2t)^{-1} & \theta-\left(\beta^1\right)^{-1}(2t)^{-1} & -2\beta^1(2t)^{-1} \\ 0 & \beta^2(2t)^{-1} & 2^{-1}t & -(2t)^{-1} \\ 0 & 0 & 2t & 0 \end{bmatrix} \tag{4.46}$$

and $e_2^*(t,\theta)=\left(-\left(\left(\beta^2\beta^1\right)^{-1}+2\beta^2\beta^1\right)(2t)^{-1},-1,0,-\beta^2\right)$ is computed. $a_{12}^*(t,\theta)$ and $a_{33}^*t,\theta)$,

$$a_{12}^*(t,\theta)=\frac{2\left(\beta^2\beta^1\right)^2-1}{2\beta^2\beta^1t^2} \tag{4.47}$$

and

$$a_{33}^*(t,\theta)=-t^{-1} \tag{4.48}$$

are obtained. Finally, the matrix

$$A=\begin{pmatrix} 0 & \dfrac{2\left(\beta^2\beta^1\right)^2-1}{2\beta^2\beta^1t^2} & 0 & 0 \\ \dfrac{1-2\left(\beta^2\beta^1\right)^2}{2\beta^2\beta^1t^2} & 0 & 0 & 0 \\ 0 & 0 & -t^{-1} & 0 \\ 0 & 0 & 0 & -t^{-1} \end{pmatrix} \tag{4.49}$$

can be obtained. Finally, since $\varepsilon^{(L)}\left\{e_2(t,\theta), e_2^*(t,\theta)\right\}$ is linearly dependent and

$$\varepsilon^{(L)}\left\langle e_1^*(t,\theta), e_2(t,\theta)\right\rangle = 0,$$

we can say that $(c(t,\theta), c^*(t,\theta))$ is Bertrand supercurve couple. The distance of Bertrand supercurve couple, as in equation (4.23),

$$d(c(t,\theta), c^*(t,\theta)) = 1 \tag{4.50}$$

is easily found.

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article. This study was supported by Research Projects Funding Unit of Eskisehir Osmangazi University (Project no: ESOGU-BAP: 2013-291) in Eskisehir, Turkey.

References

[1] Y.A. Aminov, *Differential Geometry and Topology of Curves*, Gordon and Breach Science Publishers (Singapore), (2000).
[2] H. Balgetir, M. Bektas, M. Ergut, *Bertrand curves for nonnull curves in 3-dimensional Lorentzian space*, Hadronic J., **27** (2004) 229-236.
[3] H. Balgetir, M. Bektas, J.-I. Inoguchi, *Null Bertrand curves in Minkowski 3-space and their characterizations*, Note Mat., **23** (2004) 7 - 13.
[4] C. Bartocci, U. Bruzzo and D.H. Ruiperez, *The Geometry of Supermanifolds (Mathematics and Its Applications)*, Springer (Amsterdam), (1991).
[5] M. Batchelor, *Two approaches to supermanifolds*, Trans. Amer. Math. Soc., **258** (1980) 257-270.
[6] F. A Berezin and Leites D. A., *Supermanifolds*, Soviet Mathematics. Doklady, **16** (1975) 1218-1222.
[7] F. A Berezin, *Introduction to superanalysis*, D. Reidel Publishing Campany (Holland), (1987).
[8] J. M. Bertrand, *Memoire sur la theorie des courbes a double courbure*, J. Math. Pures Appl., **15** (1850) 332 - 350.
[9] Y.M. Cheng, C.C. Lin, *On the generalized Bertrand curves in Euclidean n-spaces*, Note Mat., **29 2** (2009) 33-39.
[10] J.H. Choi, T.H. Kang and Y. H., Kim, *Bertrand curves in 3-dimensional space forms*, Appl. Math. Comput., **219** (2012) 1040-1046.
[11] V. G. Cristea, *Existence and uniqueness theorem for Frenet frame supercurves*, Note Mat., **24 1** (2005) 143-167.
[12] M. Dede, C. Ekici and I. A. Güven, *Directional Bertrand curves,* GU. J. Sci., **31 1** (2018) 202-211.
[13] B. DeWitt, *Supermanifolds*, Cambridge University press (Cambridge), (1992).
[14] N. Ekmekçi and K. Ilarslan, *On Bertrand curves and their characterization*, Differ. Geom. Dyn. Sys, **3 2**(2001) 17-24.
[15] S. Ersoy, M. Tosun, *Timelike Bertrand curves in semi-Euclidean space*, Int. J. Math. Stat., **14 2** (2013) 78-89.
[16] A. Görgülü and E. Özdamar, *A generalizations of the Bertrand curves as general inclined curves in E^n*, Communications Faculty Of Scıences De La Faculte Des Scıences Unıversıty Of Ankara Series A1, **35** (1998) 53-60.

[17] A. Inoque and Y. Maeda, *Foundations of calculus on super Euclidean space based on a Fréchet-Grassmann algebra*, Kodai Math. J., **14** (1991) 72-112.
[18] A. Jadczyk and K. Pilch, *Superspaces and supersymmetries*, Comm. Math. Phys., **78** (1981) 373-390.
[19] B., Konstant, Graded manifolds, Graded lie theory and prequantization, *Differential geometric methods in mathematical physics, in: K. Bleuler and A. Reetz, (Ed.),* Lect. Notes Math., **570** (1977) 177-306.
[20] D. A. Leites, *Introduction to the theory of supermanifolds*, Russian Math. Surveys, **35 1** (1980) 1-64.
[21] H. Matsuda, and S. Yorozu, *Notes on Bertrand curves*, Yokohama Math. J., **50** (2003) 41-58.
[22] L.R. Pears, *Bertrand curves in Riemannian space*, J. Lond. Math. Soc. (1), **10 3** (1935) 180-183.
[23] A. Rogers, *A global theory of supermanifolds,* J. Math. Phys., **21** (1980) 1352-1365.
[24] A. Rogers, *Graded Manifolds, supermanifolds and infinite-dimensional Grassmann algebras*, Comm. Math. Phys., **105** (1986) 375-384.
[25] A. Rogers, *Supermanifolds Theory and Applications*, World Scientific Publishing Company (Singapore), (2007).
[26] J.C. Saint-Venant, *Mémoire sur les lignes courbes non planes*, Journal d'Ecole Polytechnique (Paris) (1845).
[27] Y. Tunçer and S. Ünal, *New representations of Bertrand pairs in Euclidean 3-space*, Appl. Math. Comput., **219** (2012) 1833-1842.
[28] Y.C. Wong and H.-F. Lai, *A critical examination of the theory of curves in three dimensional differential geometry*, Tohoku Math. J., **19** (1967) 1-31.

Hatice Tozak
Department of Mathematics-Computer Eskişehir Osmangazi University, Postal 26480, Eskişehir, Turkey
E-mail address: hatice.tozak@gmail.com

Cumali Ekici
Department of Mathematics-Computer Eskişehir Osmangazi University, Postal 26480, Eskişehir, Turkey
E-mail address: cumali.ekici@gmail.com

Cansel Yormaz
Department of Mathematics Pamukkale University, Postal 20160, Denizli, Turkey
E-mail address: c_aycan@pau.edu.tr