Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://bmathaa.org
Volume 10 Issue 4(2018), Pages 45-57.

AFFINE DIFFERENTIAL INVARIANTS OF A CURVE IN THE PLANE

UĞUR GÖZÜTOK, YASEMIN SAĞIROĞLU

Abstract. This study focused on the concept of arc length and curvature, being affine differential invariants of a curve in the plane. The Lie transformation groups theory implementing affine differential geometry was associated with these concepts in order to obtain the invariants by means of group operators.

1. Introduction

Affine differential geometry assume a significant importance in the field of geometry. Affine differential geometry, as a branch developing out from the classical differential geometry, was introduced in the early 1920's and, most notably, W . Blaschke is known as one of the primary contributors extending this field as a separate branch of study following his seminal study [7] in 1923. This was to be shortly followed by the important contributions of G. Fubini and E. Čech to the newly advancing field [2]. In addition to these studies, Felix Klein's Erlangen Program further stirred the new research areas hinted at by affine differential geometry through which Klein postulated a method for characterizing geometries based on the group theory.

Research proposals focusing on affine differential geometry attracted a significant amount of attention from numerous mathematicians since the beginning of the latter half of the 20th century. P.A. Schirokow in 1962 and S. Buchin in 1983, for instance, published two very influential papers [5, 6] in the field. Their seminal studies established the curve theory within the affine group alongside with its subgroups. Additionally, in 1994, K. Nomizu and T. Sasaki introduced affine immersions to the growing field [3]. These advances were followed by the recent incorporation of the equivalence of parametric curves by means of differential variants, with respect to the 2012 study published by Y. Sağıroğlu [10].

In affine differential geometry, the bulk of the research focus is on the curves and their invariants such as the curvature and the arc length. Obtaining invariants like these can be done through numerous methods that were previously demonstrated elsewhere [1, 4, 8, 9]. In [5], most notably, Schirokow demonstrates a method to

2000 *Mathematics Subject Classification.* 53A15, 53A55.
Key words and phrases. Affine differential invariants, Affine arc length, Affine curvature.

Submitted October 30, 2018. Published December 12, 2018.
Communicated by K. L. Duggal.

calculate the curvature and the arc length of a planar curve with respect to the geometry of affine group and all associated subgroups.

In this study, we would like to demonstrate the steps in obtaining the curvature and the arc length of a curve in the plane through the geometry of affine group and associated affine subgroups with the aid of operators postulated in the Lie method for the solution of differential equations invariant under transformation groups, or in other words, the Lie transformation groups theory.

2. On Lie Transformation Groups

In this section we would like to introduce some basic definitions and calculations from [5] regarding Lie transformation groups.

To start with, in an $n-$dimensional space, we denote the coordinates of a point x as $x_1, x_2, ..., x_n$. In this space, the definition of a transformation which transforms x to x' is

$$x'_i = \varphi_i(x_1, x_2, ..., x_n; a_1, a_2, ..., a_r), \quad i = 1, 2, ..., n \tag{2.1}$$

where the functions φ_i are analytic functions depending on x_i and a_α, $\alpha = 1, 2, ..., r$. The variables a_α are called the parameters determining the transformation (2.1). If there is no relationship between the parameters, that is,if the number of the parameters cannot be decreased any further, then the parameters are called the independent parameters or the principal parameters. Therefore, the set of images described in (2.1) is defined as $r-$parameter family. If we denote the transformation (2.1) with T_a, then the equation becomes

$$x' = T_a x. \tag{2.2}$$

Definition 2.1. *The $r-$parameter family of the transformations* (2.2) *is called $r-$parameter Lie group, if the following requirements hold:*

i. *The product of any two transformations in the family belongs to family.*
ii. *For any transformation, there exists an inverse transformation within the family.*

In (2.1), if we replace the parameters a_α with the product $c_\alpha \delta t$ where c_α are arbitrary constants and δt is an infinitesimal multiplier, then we end up with a transformation which transforms x to a point x' in a neighbourhood of x. This transformation is called an infinitesimal transformation of the group and is defined by the as:

$$x'_i = x_i + \sum_{\alpha=1}^{r} \left(\frac{\partial \varphi_i}{\partial a_\alpha} \right)_0 c_\alpha \delta t \tag{2.3}$$

where $\left(\frac{\partial \varphi_i}{\partial a_\alpha} \right)_0$ are obtained via substituting the parameters of the identity transformation with the terms $\frac{\partial \varphi_i}{\partial a_\alpha}$. We also denote $\left(\frac{\partial \varphi_i}{\partial a_\alpha} \right)_0$ with $\xi^i_\alpha(x)$.

Definition 2.2. *The operators of which associated coefficients are obtained by infinitesimal transformation of the group are called the infinitesimal operators of the group and are expressed as:*

$$X_\alpha = \xi^s_\alpha(x) \frac{\partial}{\partial x_s}, \, 1 \leq \alpha \leq r. \tag{2.4}$$

By incorporating linear combinations of the infinitesimal operators, we can rewrite the infinitesimal transformation (2.3) as

$$x_i' = x_i + (c_\alpha X_\alpha x_i)\delta t. \tag{2.5}$$

Proposition 2.3. *Any $r-$parameter Lie group has r infinitesimal operators.*

Example 2.4. *Consider the equi affine group in the plane. Transformations of the equi affine group are as follows:*

$$\begin{aligned} x_1 &= a_1x + b_1y + c_1, \\ y_1 &= a_2x + b_2y + c_2, \end{aligned} \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 1.$$

Since the determinant equals to 1, one of the parameters can be eliminated. Hence, the equi affine group is a $5-$parameter Lie group. The 5 infinitesimal operators of this group are as follows:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = y\frac{\partial}{\partial x}, \quad X_4 = x\frac{\partial}{\partial y}, \quad X_5 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

Now, consider an arbitrary $r-$parameter Lie group in the plane:

$$\begin{aligned} x_1 &= f_1(x, y; a_1, ..., a_r), \\ y_1 &= f_2(x, y; a_1, ..., a_r). \end{aligned} \tag{2.6}$$

In this group, assume that the curve $y = y(x)$ has continuous derivatives up to the order $(r-1)$. If we take the terms $x_1, y_1, y_1', ..., y_1^{(r-2)}$ as transformed points, then we end up having the following equation system:

$$\begin{aligned} x_1 &= f_1(x, y; a_1, ..., a_r), \\ y_1 &= f_2(x, y; a_1, ..., a_r), \\ y_1' &= f_3(x, y, y'; a_1, ..., a_r), \\ &\vdots \\ y_1^{(r-2)} &= f_r(x, y, y', ..., y^{(r-2)}; a_1, ..., a_r). \end{aligned} \tag{2.7}$$

The equation system (2.7) is called generalized group equations of the group. The $n-$tuple that consists of the terms $x_1, y_1, y_1', ..., y_1^{(r-2)}$ is called the element of order $(r-2)$ and is denoted by e. Since the functions in the equation (2.6) are analytic functions, we obtain:

$$a_1 = a_1(e, e_1), ..., a_r = a_r(e, e_1). \tag{2.8}$$

By differentiating the coordinates, through, we also derive:

$$\begin{aligned} dx_1 &= \frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial y}dy, \\ dy_1 &= \frac{\partial f_2}{\partial x}dx + \frac{\partial f_2}{\partial y}dy. \end{aligned}$$

Incorporating equations (2.8) within the above equation system, we get:

$$\begin{aligned} dx_1 &= \beta_1(e, e_1)dx + \beta_2(e, e_1)dy, \\ dy_1 &= \gamma_1(e, e_1)dx + \gamma_2(e, e_1)dy. \end{aligned} \tag{2.9}$$

Fixing the terms dx_1, dy_1, e_1, does not alter the equation (2.9). If we use the equations $\beta_1(e,e_1) = \lambda_1(e), \beta_2(e,e_1) = \lambda_2(e), \gamma_1(e,e_1) = \mu_1(e), \gamma_2(e,e_1) = \mu_2(e)$, then we derive the following differential forms:

$$\omega_1 = \lambda_1(e)dx + \lambda_2(e)dy,$$
$$\omega_2 = \mu_1(e)dx + \mu_2(e)dy.$$

Furthermore, since $dy = y'dx$, we also derive the following equations:

$$\omega_1 = [\lambda_1(e) + \lambda_2(e)y']dx = \lambda(e)dx,$$
$$\omega_2 = [\mu_1(e) + \mu_2(e)y']dx = \mu(e)dx.$$

Quotient of the coefficients in the above equations is an invariant of the group that depends on only e. By means of this invariants, an invariant form $\omega = \omega(e)dx$ can be defined.

Definition 2.5. *The invariant form mentioned above is called the arc element of a curve in the geometry of $r-$parameter Lie group and is expressed as $ds = w(e)dx$.*

Now let us consider the differential of the term $y_1^{(r-2)}$ in the generalized group equations (2.7):

$$dy_1^{(r-2)} = \frac{\partial f_r}{\partial x}dx + \frac{\partial f_r}{\partial y}dy + \frac{\partial f_r}{\partial y'}dy' + ... + \frac{\partial f_r}{\partial y^{(r-2)}}dy^{(r-2)}.$$

If we incorporate equation (2.8) in the above equation, we get:

$$dy_1^{(r-2)} = \alpha_1(e,e_1)dx + \alpha_2(e,e_1)dy + ... + \alpha_r(e,e_1)dy^{(r-2)}.$$

Additionally, if we adjust e_1, we derive the following differential form:

$$\alpha_1(e)dx + \alpha_2(e)dy + ... + \alpha_r(e)dy^{(r-2)}.$$

Moreover, since $dy = y'dx, dy' = y''dx, ..., dy^{(r-2)} = y^{(r-1)}dx$, we obtain the following invariant form:

$$[\alpha_1(e) + \alpha_2(e)y' + ... + \alpha_r(e)y^{(r-1)}]dx.$$

The above invariant form can be written shortly as:

$$[\tilde{\alpha}(e) + \tilde{\beta}(e)y^{(r-1)}]dx.$$

Definition 2.6. *The differential form obtained by dividing the above mentioned invariant form by the arc element is called the curvature of a curve in the geometry of $r-$parameter Lie group and is denoted by $k = \alpha(e) + \beta(e)y^{(r-1)}$.*

Example 2.7. *Consider the equi-centro affine group in the plane. Transformations of equi-centro affine group are as follows:*

$$\begin{matrix} x_1 = a_1x + b_1y, \\ y_1 = a_2x + b_2y, \end{matrix} \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 1.$$

Equi-centro affine group is a $3-$parameter Lie group. Let us take a curve that has continuous derivatives up to the order 2. In this case, the generalized group equations are obtained as follows:

$$x_1 = a_1x + b_1y$$
$$y_1 = a_2x + b_2y$$
$$y_1' = \frac{a_2 + b_2y'}{a_1 + b_1y'}.$$

Following the elementary calculations, we yield the formula for equi-centro affine arc length of a curve in the form

$$ds = (xy' - y)dx.$$

Now, we need to focus on the differential of y_1'. *We have*

$$dy_1' = \frac{y''}{xy' - y}dx,$$

and dividing this invariant form by the arc element $(xy' - y)dx$, *we obtain equi-centro affine curvature of a curve as follows:*

$$k = \frac{y''}{(xy' - y)^3}.$$

3. Group Operators Method for Curvature and Arc Length

We know that the r infinitesimal operators of an $r-$parameter Lie group in the plane are as follows:

$$X_\rho = \xi_\rho \frac{\partial}{\partial x} + \eta_\rho \frac{\partial}{\partial y}, \quad \rho = 1, 2, ..., r. \tag{3.1}$$

Definition 3.1. *Generalized form of order* $(r-1)$ *of an infinitesimal operator is defined by*

$$X_\rho^{(r-1)} = \xi_\rho \frac{\partial}{\partial x} + \eta_\rho \frac{\partial}{\partial y} + \eta_\rho' \frac{\partial}{\partial y'} + ... + \eta_\rho^{(r-1)} \frac{\partial}{\partial y^{(r-1)}} \tag{3.2}$$

where

$$\begin{aligned} \eta_\rho' &= \frac{d\eta_\rho}{dx} - y' \frac{d\xi_\rho}{dx}, \\ \eta_\rho'' &= \frac{d\eta_\rho'}{dx} - y'' \frac{d\xi_\rho}{dx}, \\ &\vdots \\ \eta_\rho^{(r-1)} &= \frac{d\eta_\rho^{(r-2)}}{dx} - y^{(r-1)} \frac{d\xi_\rho}{dx}. \end{aligned}$$

We know from Definition (2.6) that the curvature of a curve in the geometry of an $r-$parameter Lie group assumes the form $k = \alpha + \beta y^{(r-1)}$. The image of this equation under the infinitesimal transformation is expressed as

$$\frac{\delta k}{\delta t} = \frac{\delta \alpha}{\delta t} + \frac{\delta \beta}{\delta t} y^{(r-1)} + \beta \eta_\rho^{(r-1)} = 0 \tag{3.3}$$

where $\eta_\rho^{(r-1)} = \dfrac{d\eta_\rho^{(r-2)}}{dx} - y^{(r-1)} \dfrac{d\xi_\rho}{dx}$. After $\eta_\rho^{(r-1)}$ is calculated, we simplify the equation into

$$\eta_\rho^{(r-1)} = \gamma_\rho y^{(r-1)} + \delta_\rho.$$

Placing this equation within the equation (3.3), we get

$$\frac{\delta \alpha}{\delta t} + \beta \delta_\rho + y^{(r-1)} (\frac{\delta \beta}{\delta t} + \beta \gamma_\rho) = 0.$$

Above equation gives us the following equation system:

$$\begin{aligned}\frac{\delta\beta}{\delta t}+\beta\gamma_\rho &=0\\ \frac{\delta\alpha}{\delta t}+\beta\delta_\rho &=0.\end{aligned} \tag{3.4}$$

We begin with the first equation of the system (3.4):

$$\frac{\delta\beta}{\delta t}+\beta\gamma_\rho=\frac{\delta ln\beta}{\delta t}+\gamma_\rho=\frac{\partial ln\beta}{\partial x}\xi_\rho+\frac{\partial ln\beta}{\partial y}\eta_\rho+\frac{\partial ln\beta}{\partial y'}\eta'_\rho+...+\frac{\partial ln\beta}{\partial y^{(r-2)}}\eta_\rho^{(r-2)}+\gamma_\rho=0$$

where $\rho=1,2,...,r$. Also, since

$$\frac{\partial ln\beta}{\partial x}dx+\frac{\partial ln\beta}{\partial y}dy+\frac{\partial ln\beta}{\partial y'}dy'+...+\frac{\partial ln\beta}{\partial y^{(r-2)}}dy^{(r-2)}-dln\beta=0,$$

we have $r+1$ equations. Depending on the consistency of this equation system, the following equation also holds:

$$\begin{vmatrix}\xi_1 & \eta_1 & \eta'_1 & \dots & \eta_1^{(r-2)} & \gamma_1\\ \xi_2 & \eta_2 & \eta'_2 & \dots & \eta_2^{(r-2)} & \gamma_2\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \xi_r & \eta_r & \eta'_r & \dots & \eta_r^{(r-2)} & \gamma_r\\ dx & dy & dy' & \dots & dy^{(r-2)} & -dln\beta\end{vmatrix}=0.$$

By using elementary determinant properties, we derive from the above equation:

$$dln\beta=\frac{1}{\Delta}\begin{vmatrix}\xi_1 & \eta_1 & \eta'_1 & \dots & \eta_1^{(r-2)} & \gamma_1\\ \xi_2 & \eta_2 & \eta'_2 & \dots & \eta_2^{(r-2)} & \gamma_2\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \xi_r & \eta_r & \eta'_r & \dots & \eta_r^{(r-2)} & \gamma_r\\ dx & dy & dy' & \dots & dy^{(r-2)} & 0\end{vmatrix} \tag{3.5}$$

where

$$\Delta=\begin{vmatrix}\xi_1 & \eta_1 & \eta'_1 & \dots & \eta_1^{(r-2)}\\ \xi_2 & \eta_2 & \eta'_2 & \dots & \eta_2^{(r-2)}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \xi_r & \eta_r & \eta'_r & \dots & \eta_r^{(r-2)}\end{vmatrix}. \tag{3.6}$$

Similarly, for the second equation of the system (3.4), we have:

$$d\alpha=\frac{\beta}{\Delta}\begin{vmatrix}\xi_1 & \eta_1 & \eta'_1 & \dots & \eta_1^{(r-2)} & \delta_1\\ \xi_2 & \eta_2 & \eta'_2 & \dots & \eta_2^{(r-2)} & \delta_2\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \xi_r & \eta_r & \eta'_r & \dots & \eta_r^{(r-2)} & \delta_r\\ dx & dy & dy' & \dots & dy^{(r-2)} & 0\end{vmatrix}. \tag{3.7}$$

Finally, from the equations (3.5) and (3.7), we can obtain the curvature formula of a curve in the geometry of an $r-$parameter Lie group via group operators.

Now we focus on the arc length formula. For this, first of all, we know from Definition (2.5) that the arc length formula of a curve in the geometry of an

$r-$parameter Lie group has the form:

$$s=\int\varphi(x,y,y',...,y^{(r-2)})dx.$$

By using the equations $\delta x=\xi_\alpha\delta t$ and $\delta y=\eta_\alpha\delta t$, and if keeping in mind the image of the above equation under the infinitesimal transformation, we obtain for the right-hand side

$$s+\int(\frac{\delta\varphi}{\delta t}+\varphi\frac{d\xi_\alpha}{dx})dx\delta t.$$

Since the arc length is a group invariant, $\frac{\delta\varphi}{\delta t}+\varphi\frac{d\xi_\alpha}{dx}=0$. Hence, we get

$$\frac{\delta\varphi}{\delta t}+\varphi\frac{d\xi_\alpha}{dx}=\frac{\delta\varphi}{\delta t}+\frac{d\xi_\alpha}{dx}=\frac{\partial ln\varphi}{\partial x}\xi_\alpha+\frac{\partial ln\varphi}{\partial y}\eta_\alpha+\frac{\partial ln\varphi}{\partial y'}\eta'_\alpha$$
$$+...+\frac{\partial ln\varphi}{\partial y^{(r-2)}}\eta_\alpha^{(r-2)}+\frac{d\xi_\alpha}{dx}=0.$$

Following this step, consider

$$\frac{\partial ln\varphi}{\partial x}dx+\frac{\partial ln\varphi}{\partial y}dy+\frac{\partial ln\varphi}{\partial y'}dy'+...+\frac{\partial ln\varphi}{\partial y^{(r-2)}}dy^{(r-2)}-dln\varphi=0.$$

Moreover, consistency of these $r+1$ equations necessitates that the following determinant equation should also hold:

$$\begin{vmatrix}\xi_1 & \eta_1 & \eta'_1 & \dots & \eta_1^{(r-2)} & \frac{d\xi_1}{dx}\\ \xi_2 & \eta_2 & \eta'_2 & \dots & \eta_2^{(r-2)} & \frac{d\xi_2}{dx}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \xi_r & \eta_r & \eta'_r & \dots & \eta_r^{(r-2)} & \frac{d\xi_r}{dx}\\ dx & dy & dy' & \dots & dy^{(r-2)} & -dln\varphi\end{vmatrix}=0.$$

By using the determinant expansion, then, we derive the following equation that lets us obtain φ:

$$dln\varphi=\frac{1}{\Delta}\begin{vmatrix}\xi_1 & \eta_1 & \eta'_1 & \dots & \eta_1^{(r-2)} & \frac{d\xi_1}{dx}\\ \xi_2 & \eta_2 & \eta'_2 & \dots & \eta_2^{(r-2)} & \frac{d\xi_2}{dx}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \xi_r & \eta_r & \eta'_r & \dots & \eta_r^{(r-2)} & \frac{d\xi_r}{dx}\\ dx & dy & dy' & \dots & dy^{(r-2)} & 0\end{vmatrix} \qquad (3.8)$$

where Δ is identical to the term expressed in the equation (3.6).

Finally, we now can calculate φ with the arc length formula given for a curve in the geometry of an $r-$parameter Lie group.

4. Application to Affine Group and It's Subgroups

4.1. "k" and "ds" in the Geometry of Equi-Centro Affine Group. Equi-centro affine group in the plane is determined by the following transformations:

$$\begin{matrix}x_1=ax+by,\\ y_1=cx+dy,\end{matrix}\quad\begin{vmatrix}a & b\\ c & d\end{vmatrix}=1.$$

This stands for a 3−parameter Lie group. Therefore, the associated 3 infinitesimal operators are

$$y\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

Taking any operator in the form $X_\rho = \xi_\rho \frac{\partial}{\partial x} + \eta_\rho \frac{\partial}{\partial y}$, we obtain the coefficients ξ_ρ and η_ρ:

$$\begin{aligned} &\xi_1 = y, && \eta_1 = 0 \\ &\xi_2 = 0, && \eta_2 = x \\ &\xi_3 = x, && \eta_3 = -y, \end{aligned}$$

Additionally, the generalized form of the second order is expressed as

$$X''_\rho = \xi_\rho \frac{\partial}{\partial x} + \eta_\rho \frac{\partial}{\partial y} + \eta'_\rho \frac{\partial}{\partial y'} + \eta''_\rho \frac{\partial}{\partial y''}$$

The coefficients of the generalized operator are also derived from the formulas in (3.1):

$$\begin{aligned} &\eta'_1 = -{y'}^2 && \eta''_1 = -3y'y'' \\ &\eta'_2 = 1 && \eta''_2 = 0 \\ &\eta'_3 = -2y' && \eta''_3 = -3y''. \end{aligned}$$

Therefore, from equation (3.6) we obtain:

$$\Delta = \begin{vmatrix} y & 0 & -{y'}^2 \\ 0 & x & 1 \\ x & -y & -2y' \end{vmatrix} = (xy' - y)^2$$

and relying on equation (3.8) we get:

$$dln\varphi = \frac{1}{(xy'-y)^2} \begin{vmatrix} y & 0 & -{y'}^2 & y' \\ 0 & x & 1 & 0 \\ x & -y & -2y' & 1 \\ dx & dy & dy' & 0 \end{vmatrix} = dln(xy' - y)$$

hence $\varphi = xy' - y$. The equation that we obtained gives us the arc length formula of a curve in the geometry of equi-centro affine group:

$$ds = (xy' - y)dx. \tag{4.1}$$

Let σ represent the position vector of a point on the curve,where for any parameter t, the derivative of s with respect to t would be $\frac{ds}{dt} = \dot{s}$. Taking this into account for parametric curves, equation (4.1) can be expressed as

$$\dot{s} = (\sigma\dot{\sigma}). \tag{4.2}$$

Here, $(\sigma\dot{\sigma})$ denotes the determinant of the position vectors σ and $\dot{\sigma}$.

Now we move on to the curvature formula. The first step is to calculate the coefficients of $\eta''_\rho = \gamma_\rho y'' + \delta_\rho$, to be followed by the equations (3.5) and (3.7). Since

$\eta_1'' = -3y'y'', \eta_2'' = 0, \eta_3'' = -3y''$, we get the coefficients γ_ρ and δ_ρ as follows:

$$\begin{aligned} \gamma_1 &= -3y', & \delta_1 &= 0 \\ \gamma_2 &= 0, & \delta_2 &= 0 \\ \gamma_3 &= -3, & \delta_3 &= 0. \end{aligned}$$

From (3.5),

$$dln\beta = \frac{1}{(xy'-y)^2} \begin{vmatrix} y & 0 & -y'^2 & -3y' \\ 0 & x & 1 & 0 \\ x & -y & -2y' & -3 \\ dx & dy & dy' & 0 \end{vmatrix} = dln\left(\frac{1}{(xy'-y)^3}\right)$$

following which we get $\beta = \dfrac{1}{(xy'-y)^3}$.

From (3.7),

$$d\alpha = \frac{1}{(xy'-y)^5} \begin{vmatrix} y & 0 & -y'^2 & 0 \\ 0 & x & 1 & 0 \\ x & -y & -2y' & 0 \\ dx & dy & dy' & 0 \end{vmatrix}$$

where we have $\alpha = 0$. Finally, we yield the curvature formula for a curve in the geometry of equi-centro afiine group as

$$k = \frac{y''}{(xy'-y)^3}. \tag{4.3}$$

Let σ represent the position vector of a point on the curve. For any parameter t, equation (4.3) can be expressed as

$$k = \frac{(\dot{\sigma}\ddot{\sigma})}{(\sigma\dot{\sigma})^3}. \tag{4.4}$$

4.2. **"k" and "ds" in the Geometry of Centro Affine Group.** Centro affine group in the plane is determined by the following transformations:

$$\begin{aligned} x_1 &= ax + by, \\ y_1 &= cx + dy, \end{aligned} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$$

This group is a $4-$parameter Lie group. Therefore its 4 infinitesimal operators are as follows:

$$x\frac{\partial}{\partial x}, \quad y\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial y}, \quad y\frac{\partial}{\partial y}.$$

The coefficients of the infinitesimal operators are as follows:

$$\begin{aligned} \xi_1 &= x, & \eta_1 &= 0 \\ \xi_2 &= y, & \eta_2 &= 0 \\ \xi_3 &= 0, & \eta_3 &= x \\ \xi_4 &= 0, & \eta_4 &= y. \end{aligned}$$

Following this step, the coefficients of the generalized operator are derived as

$$\begin{array}{lll} \eta_1' = -y' & \eta_1'' = -2y'' & \eta_1''' = -3y''' \\ \eta_2' = -y'^2 & \eta_2'' = -3y'y'' & \eta_2''' = -3y''^2 - 4y'y''' \\ \eta_3' = 1 & \eta_3'' = 0 & \eta_3''' = 0 \\ \eta_4' = y' & \eta_4'' = y'' & \eta_4''' = y'''. \end{array}$$

Therefore,

$$\Delta = \begin{vmatrix} x & 0 & -y' & -2y'' \\ y & 0 & -y'^2 & -3y'y'' \\ 0 & x & 1 & 0 \\ 0 & y & y' & y'' \end{vmatrix} = -2y''(xy'-y)^2.$$

At this point, the arc length formula is obtained:

$$dln\varphi = \frac{1}{-2y''(xy'-y)^2} \begin{vmatrix} x & 0 & -y' & -2y'' & 1 \\ y & 0 & -y'^2 & -3y'y'' & -4y' \\ 0 & x & 1 & 0 & 0 \\ 0 & y & y' & y'' & 0 \\ dx & dy & dy' & dy'' & 0 \end{vmatrix} = dln\left(\frac{y''}{xy'-y}\right)^{\frac{1}{2}}.$$

Following that, the arc length formula for a curve in the geometry of centro affine group can also be expressed as

$$ds = \frac{y''^{\frac{1}{2}}}{(xy'-y)^{\frac{1}{2}}} dx. \tag{4.5}$$

Let σ represent the position vector of a point on the curve. For any parameter t, equation (4.5) can be expressed as

$$\dot{s} = \frac{(\dot{\sigma}\ddot{\sigma})^{\frac{1}{2}}}{(\sigma\dot{\sigma})^{\frac{1}{2}}}. \tag{4.6}$$

For the curvature formula, the coefficients α and β are as follows:

$$dln\beta = \frac{1}{-2y''(xy'-y)^2} \begin{vmatrix} x & 0 & -y' & -2y'' & -3 \\ y & 0 & -y'^2 & -3y'y'' & -4y' \\ 0 & x & 1 & 0 & 0 \\ 0 & y & y' & y'' & 1 \\ dx & dy & dy' & dy'' & 0 \end{vmatrix} = dln\left(\frac{(xy'-y)^{\frac{1}{2}}}{2y''^{\frac{3}{2}}}\right)$$

where

$$\beta = \frac{1}{2}\frac{(xy'-y)^{\frac{1}{2}}}{2y''^{\frac{3}{2}}}.$$

$$d\alpha = \frac{\alpha}{\Delta} \begin{vmatrix} x & 0 & -y' & -2y'' & 0 \\ y & 0 & -y'^2 & -3y'y'' & -3y''^2 \\ 0 & x & 1 & 0 & 0 \\ 0 & y & y' & y'' & 0 \\ dx & dy & dy' & dy'' & 0 \end{vmatrix} = d\left(\frac{-3xy''^{\frac{1}{2}}}{2(xy'-y)^{\frac{1}{2}}}\right)$$

and, therefore,

$$\alpha = -\frac{3}{2}\frac{-3xy''^{\frac{1}{2}}}{(xy'-y)^{\frac{1}{2}}}.$$

Drawing on equations hitherto, we express the curvature formula of a curve in the geometry of centro affine group as

$$k = \frac{1}{2}\frac{(xy' - y)^{\frac{1}{2}}}{2y''^{\frac{3}{2}}}y''' - \frac{3}{2}\frac{-3xy''^{\frac{1}{2}}}{(xy' - y)^{\frac{1}{2}}}. \tag{4.7}$$

By parametrizing of the equation (4.7) with t, we yield the curvature formula

$$k = \frac{1}{2}\frac{(\sigma\dot{\sigma})^{\frac{1}{2}}}{(\dot{\sigma}\ddot{\sigma})^{\frac{3}{2}}}\left[(\dot{\sigma}\dddot{\sigma}) - 3\frac{(\dot{\sigma}\ddot{\sigma})(\sigma\ddot{\sigma})}{(\sigma\dot{\sigma})}\right]. \tag{4.8}$$

4.3. **"k" and "ds" in the Geometry of Equi Affine Group.** Equi affine group in the plane is determined by the following transformations:

$$\begin{aligned} x_1 &= ax + by + e, \\ y_1 &= cx + dy + f, \end{aligned} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1.$$

This group is a 5$-$parameter Lie group. Therefore, its 5 infinitesimal operators are:

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad y\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

The coefficients of the infinitesimal operators are as follows:

$$\begin{aligned} &\xi_1 = 1, && \eta_1 = 0, \\ &\xi_2 = 0, && \eta_2 = 1, \\ &\xi_3 = y, && \eta_3 = 0, \\ &\xi_4 = 0, && \eta_4 = x, \\ &\xi_5 = x, && \eta_5 = -y. \end{aligned}$$

Following this, we derive the coefficients of the generalized operator:

$$\begin{array}{llll} \eta_1' = 0, & \eta_1'' = 0, & \eta_1''' = 0, & \eta_1^{(4)} = 0, \\ \eta_2' = 0, & \eta_2'' = 0, & \eta_2''' = 0, & \eta_2^{(4)} = 0, \\ \eta_3' = -y'^2, & \eta_3'' = -3y'y'', & \eta_3''' = -3y''^2 - 4y'y''', & \eta_3^{(4)} = -10y''y''' - 5y'y^{(4)} \\ \eta_4' = 1, & \eta_4'' = 0, & \eta_4''' = 0, & \eta_4^{(4)} = 0, \\ \eta_5' = -2y', & \eta_5'' = -3y'', & \eta_5''' = -4y''', & \eta_5^{(4)} = -5y^{(4)}. \end{array}$$

By using above terms, we obtain the arc length formula of a curve in the geometry of equi affine group as

$$ds = y''^{\frac{1}{3}}dx. \tag{4.9}$$

Following parametrization, we rewrite the formula as

$$\dot{s} = (\dot{\sigma}\ddot{\sigma})^{\frac{1}{3}}. \tag{4.10}$$

Similarly, we obtain the curvature formula of a curve in the geometry of equi affine group:

$$k = \frac{1}{3}\frac{y^{(4)}}{y''^{\frac{5}{3}}} - \frac{5}{9}\frac{y'''^2}{y''^{\frac{8}{3}}}. \tag{4.11}$$

After the parametrization, we rewrite the formula:

$$k = \frac{3(\dot{\sigma}\ddot{\sigma})(\dot{\sigma}\ddddot{\sigma}) + 12(\dot{\sigma}\ddot{\sigma})(\ddot{\sigma}\dddot{\sigma}) - 5(\dot{\sigma}\dddot{\sigma})^2}{9(\dot{\sigma}\ddot{\sigma})^{\frac{8}{3}}}. \tag{4.12}$$

4.4. **"k" and "ds" in the Geometry of Affine Group.** Affine group in the plane is determined by the following transformations:

$$\begin{aligned} x_1 &= ax + by + e, \\ y_1 &= cx + dy + f, \end{aligned} \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

This group is a $6-$parameter Lie group. Therefore, its 6 infinitesimal operators are:

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad y\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial x}, \quad y\frac{\partial}{\partial y}.$$

We obtain the affine arc length formula of a curve in the plane by means of these operators:

$$ds = \left(\frac{3y''y^{(4)} - 5y'''^2}{3y''^2} \right)^{\frac{1}{2}} dx. \tag{4.13}$$

Commencing on to the parametrization step, we rewrite the formula as

$$\dot{s} = \left(\frac{3(\dot{\sigma}\ddot{\sigma})(\dot{\sigma}\ddddot{\sigma}) + 12(\dot{\sigma}\ddot{\sigma})(\ddot{\sigma}\dddot{\sigma}) - 5(\dot{\sigma}\dddot{\sigma})^2}{3(\dot{\sigma}\ddot{\sigma})^2} \right)^{\frac{1}{2}}. \tag{4.14}$$

Similarly, we express the affine curvature formula of a curve in the plane as

$$k = -\frac{3}{2\zeta^{\frac{1}{2}}} \left(ln \frac{\zeta}{y''^{\frac{3}{2}}} \right)' \tag{4.15}$$

where $\zeta = \dfrac{y^{(4)}}{y''} - \dfrac{5y'''^2}{3y''^2}$. Following parametrization we rewrite the formula as

$$k = -\frac{3}{2\Lambda^{\frac{1}{2}}} \frac{d}{dt} \left(ln \frac{\Lambda}{(\dot{\sigma}\ddot{\sigma})^{\frac{3}{2}}} \right), \quad \Lambda^{\frac{1}{2}} = \dot{s}. \tag{4.16}$$

Acknowledgments. The first author would like to thank TUBITAK(The Scientific and Technological Research Council of Turkey) for their financial supports during his doctorate studies.

References

[1] E. Salkowski, *Affine Differentialgeometrie*, de Gruyter, Berlin, 1934.
[2] G. Fubini and E. Čech, *Geometria Proiettiva Differenziale*, ASCR, Bologna, 1926.
[3] K. Nomizu and T. Sasaki, *Affine Differential Geometrt: Geometry of Affine Immersions*, Cambridge University Press, Cambridge, 1994.
[4] M. Nadjafikhah, *Affine Differential Invariants for Planar Curves*, BJGA **7 (1)** (2002) 69–78.
[5] P.A. Schirokow and A.P. Schirokow, *Affine Differentialgeometrie*, Teubner, Leipzig, 1962.
[6] S. Buchin, *Affine Differential Geometry*, Science Press, China, 1983.
[7] W. Blaschke, *Vorlesungen Über Differentialgeometrie und Geometrische Grundlagen von Einsteins Relativitätstheorie II*, Springer, Berlin, 1923.
[8] Y. Sağıroğlu, *The Equivalence of Curves in $SL(n, \mathbb{R})$ and Its Application to Ruled Surfaces*, Appl. Math. Comput. **2018** (2011) 1019–1024.
[9] Y. Sağıroğlu and Ö. Pekşen, *The Equivalence of Equi-Affine Curves*, Turk. J. Math. **34** (2010) 95–104.
[10] Y. Sağıroğlu, *Affine Differential Invariants of Curves*, LAP, Saarbrücken, 2012.

Uğur Gözütok
Department of Mathematics, Karadeniz Technical University, 61080 Trabzon, TURKEY
E-mail address: ugurgozutok@ktu.edu.tr

Yasemin Sağiroğlu
Department of Mathematics, Karadeniz Technical University, 61080 Trabzon, TURKEY
E-mail address: sagiroglu.yasemin@gmail.com