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COMMON FIXED POINT FOR GENERALIZED CONTRACTION IN B-MULTIPLICATIVE METRIC SPACES WITH APPLICATIONS

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Abstract. The desired outcome of this paper is to extend the result of Al-Mazrooei et al. (Journal of Mathematical Analysis, 8(3):157-166, 2017) by applying new contractive condition only on a closed set instead of a whole set and by using b-multiplicative metric spaces instead of multiplicative metric spaces. We apply our result to obtain unique common solution of Fredholm multiplicative integral equations. An example and a result on F-contraction are also presented. Our results generate many new results in b-multiplicative metric spaces and b-metric spaces.

1. Introduction and Preliminaries

Bakhtin [7] was the first who had given the idea of b-metric. After that, Czerwik [9] gave an axiom and formally defined a b-metric space. For further results on b-metric space, see [17, 27]. Ozaksar and Cevical [16] investigated multiplicative metric space and proved its topological properties. Mongkolkeha et al. [15] described the concept of multiplicative proximal contraction mapping and proved best proximity point theorems for such mappings. Recently, Abbas et al. [1] proved some common fixed points results of quasi weak commutative mappings on a closed ball in the setting of multiplicative metric spaces. For further results on multiplicative metric space, see [2, 4, 10, 11, 14]. In 2017, Ali et al. [5] introduced the notion of b-multiplicative and proved some fixed point result. As an application, they established an existence theorem for the solution of a system of Fredholm multiplicative integral equations. Shoaib et al. [27] discussed some results for mappings satisfying contraction condition only on a closed ball in b-metric spaces. For further results on closed ball, see [18, 19, 21, 22, 23, 24, 25, 26, 28, 29]. In this paper, we proved a result in [4] by applying contractive condition only on a closed set instead of a whole space and for b-multiplicative metric space instead of multiplicative metric space. Moreover, we obtained corresponding new results on closed ball in b-metric spaces. Example is given which shows the effectiveness of the new results. We also showed that a specific type of generalization of F-contraction is not real. An

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application on integral equations is also given. The following definitions and results are used to understand the paper.

Definition 1.1 [5] Let W be a non-empty set and let $s \geq 1$ be a given real number. A mapping $m_b : W \times W \to [1, \infty)$ is called a b-multiplicative metric with coefficient s, if the following conditions hold:

(i) $m_b(w, y) > 1$ for all $w, y \in W$ with $w \neq y$ and $m_b(w, y) = 1$ if and only if $w = y$.

(ii) $m_b(w, y) = m_b(y, w)$ for all $w, y \in W$.

(iii) $m_b(w, z) \leq [m_b(w, y).m_b(y, z)]^s$ for all $w, y, z \in W$.

The triplet (W, m_b) is called b-multiplicative metric space. If $r > 1$, $u \in W$, then $\overline{B_{m_b}(u, r)} = \{v : m_b(u, v) < r\}$ is called a closed ball in (W, m_b).

Example 1.2 [5] Let $W = [0, \infty)$. Define a mapping $m_a : W \times W \to [1, \infty)$

$$m_a(w, y) = a^{(w-y)^2},$$

where $a > 1$ is any fixed real number. Then for each a, m_a is b-multiplicative metric on W with $s = 2$. Note that m_a is a not multiplicative metric on W.

Definition 1.3 [5] Let (W, m_b) be a b-multiplicative metric space.

(i) A sequence $\{w_n\}$ is convergent iff there exist $w \in W$ such that

$$m_b(w_n, w) \to 1, \text{ as } n \to +\infty.$$

(ii) A sequence $\{w_n\}$ is called $b-$multiplicative Cauchy iff

$$m_b(w_m, w_n) \to 1, \text{ as } m, n \to +\infty.$$

(iii) A b-multiplicative metric space (W, m_b) is said to be complete if every multiplicative Cauchy sequence in Y is convergent to some $y \in W$.

Definition 1.4 [17] Let W be a non-empty set and $s \geq 1$ be a real number. A mapping $b\colon W \times W \to \mathbb{R}^+ \cup \{0\}$ is said to be b-metric with coefficient s, if for all $w, y, z \in W$, the following conditions hold:

(i) $b(w, y) = 0$ if and only if $w = y$;

(ii) $b(w, y) = b(y, w)$;

(iii) $b(w, z) \leq s\,[b(w, y) + b(y, z)]$.

The pair (W, b) is called b-metric space. If $r > 0$, $u \in W$, then $\overline{B_b(u, r)} = \{v : b(u, v) < r\}$ is called a closed ball in (W, b).

Remark 1.5 [5] Every b-metric space (W, b) generates a $b-$multiplicative metric space (W, m_b) defined as

$$m_b\,(x, y) = e^{b(x,y)}.$$

Remark 1.6 Let (W, m_b) be a $b-$multiplicative metric space generated by b-metric space (W, b), $r > 0$ and $x_0 \in W$. If $\overline{B_b(x_0, r)}$ and $\overline{B_{m_b}(x_0, e^r)}$ are closed balls in (W, b) and (W, m_b) respectively, then $\overline{B_b(x_0, r)} = \overline{B_{m_b}(x_0, e^r)}$.

Definition 1.7 Let $S, T : X \to X$, $A \subseteq X$ and $M_A(S, T)$ be the family of all functions $a : X \times X \to [0, 1)$ with following assertions

$$a(TSx, y) \leq a(x, y) \text{ and } a(x, STy) \leq a(x, y), \text{ for all } x, y \in A.$$

If we take $A = X$, then $M_A(S, T)$ become $M(S, T)$, which is defined in [3]. Now, for a single mapping $S : X \to X$, we define the family $M_A(S)$ of all functions $a : X \times X \to [0, 1)$ with following assertions

$$a(S^2x, y) \leq a(x, y) \text{ and } a(x, S^2y) \leq a(x, y), \text{ for all } x, y \in A.$$

Proposition 1.8 Let $S, T : X \to X$ be self mappings, $A \subseteq X$ and $x_0 \in X$, we define the sequence $\{x_n\}$ by $x_{2n+1} = Sx_{2n}$, $x_{2n+2} = Tx_{2n+1}$ for all integers $n \geq 0$. If $\{x_n\}$ is a sequence in A and $a \in M_A(S,T)$, then $a(x_{2n,}y) \leq a(x_0, y)$ and $a(x, x_{2n+1}) \leq a(x, x_1)$ for all $x, y \in A$ and integers $n \geq 0$. Also, same is valid if $a \in M_A(S)$.

2. Main Result

Theorem 2.1 Let (X, m_b) be a complete b- multiplicative metric space and $S, T : X \to X$ be self-mappings. If there exist mappings $\alpha, \beta, \nu, \xi \in M_A(S,T)$, $A = \overline{B_{m_b}(x_0, r)}$, $x_0 \in X$ and $r > 1$ such that:

$$m_b(x_0, Sx_0) \leq r^{\frac{(1-\ sh)}{s}},$$

where $sh < 1$, $h = \max\{h_1, h_2\}$ and

$$h_1 = \frac{\alpha(x_0,\ x_1) + \beta(x_0,\ x_1) + s\xi(x_0, x_1)}{1 - \nu(x_0,\ x_1) - s\xi(x_0,\ x_1)},\ h_2 = \frac{\alpha(x_0,\ x_1) + \nu(x_0,\ x_1) + s\xi(x_0, x_1)}{1 - \beta(x_0,\ x_1) - s\xi(x_0,\ x_1)}.$$

Also, if $\overline{B_{m_b}(x_0, r)}$ is closed and x, y belongs to $\overline{B_{m_b}(x_0, r)}$, then this implies

$$m_b(Sx, Ty) \leq (m_b(x, y))^{\alpha(x,y)}.(m_b(x, Sx))^{\beta(x,y)}.(m_b(y, Ty))^{\nu(x,y)}.$$

$$(m_b(y, Sx).m_b(x, Ty))^{\xi(x,y)}. \tag{2.1}$$

Then S and T have a unique common fixed point in $\overline{B_{m_b}(x_0, r)}$.

Proof. Let x_0 be a given point in X. Let we construct sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Sx_{2n},\ x_{2n+2} = Tx_{2n+1},$$

for $n = 0, 1, 2, ...$ Now we show that $\{x_n\}$ is a sequence in $\overline{B_{m_b}(x_0, r)}$. Note that

$$m_b(x_0, x_1) = m_b(x_0, Sx_0) \leq r^{\frac{(1\ -\ sh)}{s}} \leq r. \tag{2.2}$$

Hence $x_1 \in \overline{B_{m_b}(x_0, r)}$. Assume $x_2, x_3, ...x_j \in \overline{B_{m_b}(x_0, r)}$ for some $j \in \mathbb{N}$. Then, if $j = 2k + 1$

$$\begin{aligned}
m_b(x_{2k+1}, x_{2k+2}) &= m_b(Sx_{2k}, Tx_{2k+1}) \\
&\leq (m_b(x_{2k}, x_{2k+1}))^{\alpha(x_{2k},\ x_{2k+1})}.(m_b(x_{2k,}Sx_{2k}))^{\beta(x_{2k},\ x_{2k+1})} \\
&\quad .(m_b(x_{2k+1}, Tx_{2k+1}))^{\nu(x_{2k},\ x_{2k+1})} \\
&\quad .(m_b(x_{2k+1,}Sx_{2k}).m_b(x_{2k,}Tx_{2k+1}))^{\xi(x_{2k},x_{2k+1})} \\
&\leq (m_b(x_{2k}, x_{2k+1}))^{\alpha(x_{2k},\ x_{2k+1})}.(m_b(x_{2k,}x_{2k+1}))^{\beta(x_{2k},x_{2k+1})} \\
&\quad .(m_b(x_{2k+1}, x_{2k+2}))^{\nu(x_{2k},\ x_{2k+1})}.(m_b(x_{2k}, x_{2k+2}))^{\xi(x_{2k},\ x_{2k+1})}.
\end{aligned}$$

From the Proposition 1.8 and by triangle inequality, we have

$$\begin{aligned}
& m_b(x_{2k+1}, x_{2k+2}) \\
\leq\ & (m_b(x_{2k}, x_{2k+1}))^{\alpha(x_0,\ x_{2k+1})}.(m_b(x_{2k}, x_{2k+1}))^{\beta(x_0,\ x_{2k+1})} \\
& .(m_b(x_{2k+1}, x_{2k+2}))^{\nu(x_0,\ x_{2k+1})}.(m_b(x_{2k}, x_{2k+1})^s.m_b(x_{2k+1}, x_{2k+2})^s)^{\xi(x_0,\ x_{2k+1})}.
\end{aligned}$$

Again from the Proposition 1.8, we have

$$\begin{aligned}
& m_b(x_{2k+1}, x_{2k+2}) \\
\leq\ & (m_b(x_{2k}, x_{2k+1}))^{\alpha(x_0,\ x_1)}.(m_b(x_{2k}, x_{2k+1}))^{\beta(x_0,\ x_1)} \\
& .(m_b(x_{2k+1}, x_{2k+2}))^{\nu(x_0,\ x_1)}.(m_b(x_{2k}, x_{2k+1})^s.m_b(x_{2k+1}, x_{2k+2})^s)^{\xi(x_0,\ x_1)} \\
\leq\ & (m_b(x_{2k}, x_{2k+1}))^{\alpha(x_0,\ x_1)+\beta(x_0,\ x_1)+s\xi(x_0,\ x_1)}.(m_b(x_{2k+1}, x_{2k+2}))^{\nu(x_0,\ x_1)+s\xi(x_0,\ x_1)} \\
\leq\ & (m_b(x_{2k}, x_{2k+1}))^{\frac{\alpha(x_0,\ x_1)+\beta(x_0,\ x_1)+s\xi(x_0,\ x_1)}{1-\nu(x_0,\ x_1)-s\xi(x_0,\ x_1)}} = (m_b(x_{2k}, x_{2k+1}))^{h_1}
\end{aligned}$$

$$m_b(x_{2k+1}, x_{2k+2}) \leq (m_b(x_{2k}, x_{2k+1})^h. \tag{2.3}$$

Similarly If $j = 2k$, we have

$$\begin{aligned}
m_b(x_{2k}, x_{2k+1}) &= m_b(Tx_{2k-1}, Sx_{2k}) = m_b(Sx_{2k}, Tx_{2k-1}) \\
&\leq (m_b(x_{2k-1}, x_{2k}))^{\alpha(x_{2k},x_{2k-1})}.(m_b(x_{2k}, x_{2k+1}))^{\beta(x_{2k},x_{2k-1})} \\
&\quad .(m_b(x_{2k-1}, x_{2k}))^{\nu(x_{2k},x_{2k-1})}.m_b(x_{2k-1}, x_{2k+1}))^{\xi(x_{2k},x_{2k-1})}.
\end{aligned}$$

Again from the Proposition 1.8, we have

$$\begin{aligned}
& m_b(x_{2k,}x_{2k+1}) \\
\leq\ & (m_b(x_{2k-1}, x_{2k}))^{\alpha(x_0,x_1)}.(m_b(x_{2k}, x_{2k+1}))^{\beta(x_0,x_1)} \\
& .(m_b(x_{2k-1}, x_{2k}))^{\nu(x_0,x_1)}.(m_b(x_{2k-1}, x_{2k}).m_b(x_{2k}, x_{2k+1}))^{s\xi(x_0,x_1)} \\
\leq\ & (m_b(x_{2k-1}, x_{2k}))^{\alpha(x_0,x_1)+\nu(x_0,x_1)+s\xi(x_0,x_1)} \\
& .(m_b(x_{2k}, x_{2k+1}))^{\beta(x_0,x_1)+s\xi(x_0,x_1)} \\
\leq\ & (m_b(x_{2k-1}, x_{2k})^{\frac{\alpha(x_0,x_1)+\nu(x_0,x_1)+s\xi(x_0,x_1)}{1-[\beta(x_0,x_1)+s\xi(x_0,x_1)]}} = (m_b(x_{2k-1}, x_{2k})^{h_2}.
\end{aligned}$$

$$m_b(x_{2k}, x_{2k+1}) \leq (m_b(x_{2k-1}, x_{2k})^h. \tag{2.4}$$

Thus from (2.3) and (2.4), we conclude that for all $k \in \mathbb{N}$

$$m_b(x_k, x_{k+1}) \leq m_b(x_{k-1}, x_k)^h \leq .... \leq m_b(x_0, x_1)^{h^k}. \tag{2.5}$$

Now,

$$\begin{aligned}
m_b(x_0, x_{j+1}) &\leq m_b(x_0, x_1)^s.m_b(x_1, x_2)^{s^2}....m_b(x_j, x_{j+1})^{s^{\ j+1}} \\
&\leq m_b(x_0, x_1)^{sh^0}.m_b(x_0, x_1)^{s^2h^1}....m_b(x_0, x_1)^{s^{\ j+1}h^{\ j}} \\
&\leq m_b(x_0, x_1)^{s(s^0h^0+s^1h^1+s^2h^2+....+s^{\ j}h^{\ j})} \\
m_b(x_0,x_{j+1}) &\leq m_b(x_0,x_1)^{s(\frac{1-(sh)^j}{1-sh})}.
\end{aligned}$$

Since $x_1 \in \overline{B_{m_b}(x_0, r)}$, we have

$$\begin{aligned}
m_b(x_0, x_{j+1}) &\leq \left(r^{\frac{1-sh}{s}}\right)^{s(\frac{1-(sh)^j}{1-sh})} \\
&= (r)^{1-(sh)^j} \leq r,
\end{aligned}$$

This implies $x_{j+1} \in \overline{B_{m_b}(x_0, r)}$. By induction on n, we conclude that $\{x_n\} \in \overline{B_{m_b}(x_0, r)}$ for all $n \in \mathbb{N}$. Therefore

$$m_b(x_n, x_{n+1}) \leq m_b(x_0, x_1)^{h^n} \text{ for all } n \in \mathbb{N}. \tag{2.6}$$

We claim that the sequence $\{x_n\}$ satisfies the multiplicative Cauchy criterion for convergence in $(\overline{B_{m_b}(x_0,r)},m_b)$. Let $m,n>0$ with $m>n$ as $m=n+p$; $p\in\mathbb{N}$.

$$\begin{aligned}
& m_b(x_n,x_m)\\
\leq\; & m_b(x_n,x_{n+1})^s.m_b(x_{n+1},x_{n+2})^{s^2}....m_b(x_{n+p-1},x_{n+p})^{s^p}\\
\leq\; & (m_b(x_0,x_1)^{sh^n}.(m_b(x_0,x_1)^{s^2h^{n+1}}....(m_b(x_0,x_1))^{s^ph^{n+p-1}}\\
\leq\; & (m_b(x_0,x_1))^{sh^n+s^2h^{n+1}+.....+s^ph^{n+p-1}}\\
<\; & (m_b(x_0,x_1))^{sh^n+s^2h^{n+1}......}=(m_b(x_0,x_1))^{\frac{sh^n}{1-sh}}\\
\leq\; & (m_b(x_0,x_1))^{\frac{sh^n}{1-sh}}.
\end{aligned}$$

Taking limit as $m,n\to\infty$, we get $m_b(x_n,x_m)\to 1$. Hence the sequence $\{x_n\}$ is a multiplicative Cauchy sequence. As the closed set $(\overline{B_{m_b}(x_0,r)},m_b)$ is complete. So, the completeness of $(\overline{B_{m_b}(x_0,r)},m_b)$ follows that $x_n\to x^*\in\overline{B_{m_b}(x_0,r)}$. So

$$m_b(x_n,x^*)\to 1,\ \text{ as } n\to+\infty. \tag{2.7}$$

Now, we have to show that x^* is a fixed point of mapping T.

$$\begin{aligned}
& m_b(x_{2n+1},Tx^*)\\
\leq\; & (m_b(x_{2n},x^*))^{\alpha(x_{2n},x^*)}.(m_b(x_{2n},Sx_{2n}))^{\beta(x_{2n},x^*)}\\
& .(m_b(x^*,Tx^*))^{\nu(x_{2n},x^*)}.(m_b(x^*,Sx_{2n}).m_b(x_{2n},Tx^*))^{\xi(x_{2n},x^*)}\\
\leq\; & (m_b(x_{2n},x^*))^{\alpha(x_{2n},x^*)}.(m_b(x_{2n},x_{2n+1}))^{\beta(x_{2n},x^*)}\\
& .(m_b(x^*,Tx^*))^{\nu(x_{2n},x^*)}.(m_b(x^*,x_{2n+1}).m_b(x_{2n},Tx^*))^{\xi(x_{2n},x^*)}.
\end{aligned}$$

From the Proposition 1.8, we have

$$\begin{aligned}
& m_b(x_{2n+1},Tx^*)\\
\leq\; & (m_b(x_{2n},x^*))^{\alpha(x_0,x^*)}.(m_b(x_{2n},x_{2n+1}))^{\beta(x_0,x^*)}\\
& .(m_b(x^*,Tx^*))^{\nu(x_0,x^*)}.(m_b(x^*,x_{2n+1}).m_b(x_{2n},Tx^*))^{\xi(x_0,x^*)}\\
\leq\; & (m_b(x_{2n},x^*))^{\alpha(x_0,x^*)}.(m_b(x_{2n},x_{2n+1}))^{\beta(x_0,x^*)}\\
& .(m_b(x^*,Tx^*))^{\nu(x_0,x^*)}.(m_b(x^*,x_{2n+1})\\
& .m_b(x_{2n},x^*)^s.m_b(x^*,Tx^*)^s)^{\xi(x_0,x^*)}.
\end{aligned}$$

Taking limit as $n\to\infty$ and by inequality (2.7), we have

$$\lim_{n\to\infty} m_b(x_{2n+1},Tx^*)\leq(m_b(x^*,Tx^*))^{\nu(x_0,x^*)+s\xi(x_0,x^*)}.$$

Now,

$$m_b(x^*,Tx^*)\leq(m_b(x^*,x_{2n+1}).m_b(x_{2n+1},Tx^*))^s.$$

Taking limit as $n\to\infty$ and by inequality (2.7), we have

$$m_b(x^*,Tx^*)\leq(m_b(x^*,Tx^*))^{s\nu(x_0,x^*)+s^2\xi(x_0,x^*)},$$

which implies that

$$(m_b(x^*,Tx^*))^{1-[s\nu(x_0,x^*)+s^2\xi(x_0,x^*)]}\leq 1,$$

which further implies that

$$(m_b(x^*,Tx^*))\leq 1^{\frac{1}{1-[s\nu(x_0,x^*)+s^2\xi(x_0,x^*)]}}\leq 1.$$

Thus x^* is a fixed point of mapping T. Now,

$$\begin{aligned}
& m_b(Sx^*, x_{2n+2}) \\
\leq\ & (m_b(x^*, x_{2n+1}))^{\alpha(x^*, x_{2n+1})}.(m_b(x^*, Sx^*))^{\beta(x^*, x_{2n+1})} \\
& .(m_b(x_{2n+1}, Tx_{2n+1}))^{\nu(x^*, x_{2n+1})}.(m_b(x_{2n+1}, Sx^*).m_b(x^*, Tx_{2n+1}))^{\xi(x^*, x_{2n+1})} \\
\leq\ & (m_b(x^*, x_{2n+1}))^{\alpha(x^*, x_{2n+1})}.(m_b(x^*, Sx^*))^{\beta(x^*, x_{2n+1})} \\
& .(m_b(x_{2n+1}, x_{2n+2}))^{\nu(x^*, x_{2n+1})}.(m_b(x_{2n+1}, Sx^*).m_b(x^*, x_{2n+2}))^{\xi(x^*, x_{2n+1})}.
\end{aligned}$$

From Proposition 1.8, We have

$$\begin{aligned}
& m_b(Sx^*, x_{2n+2}) \\
\leq\ & (m_b(x^*, x_{2n+1}))^{\alpha(x^*, x_1)}.(m_b(x^*, Sx^*))^{\beta(x^*, x_1)} \\
& .(m_b(x_{2n+1}, x_{2n+2}))^{\nu(x^*,\ x_1)}.(m_b(x_{2n+1}, Sx^*).m_b(x^*, x_{2n+2}))^{\xi(x^*,\ x_1)} \\
\leq\ & (m_b(x^*, x_{2n+1}))^{\alpha(x^*, x_1)}.(m_b(x^*, Sx^*))^{\beta(x^*, x_1)}.(m_b(x_{2n+1}, x_{2n+2}))^{\nu(x^*, x_1)} \\
& .(m_b(x_{2n+1}, x^*)^s.m_b(x^*, Sx^*)^s.m_b(x^*, x_{2n+2}))^{\xi(x^*, x_1)}.
\end{aligned}$$

Taking the limit as $n \longrightarrow \infty$, we get

$$\lim_{n\to\infty} m_b(Sx^*, x_{2n+2}) \leq (m_b(x^*, Sx^*))^{\beta(x^*, x_1)+s\xi(x^*, x_1)}.$$

By using above inequality and the triangle inequality, we have

$$(m_b(x^*, Sx^*))^{1-[s\beta(x_\circ, x^*)+s^2\xi(x_\circ, x^*)]} \leq 1,$$

which further implies that

$$(m_b(x^*, Sx^*)) \ \leq (1)^{\frac{1}{1-[\beta(x_\circ, x^*)+\ s\delta(x_\circ, x^*)]}} \leq 1.$$

Thus x^* is a fixed point of mapping S. Hence x^* is a common fixed point of mapping S and T. Let u be another common fixed point of the mappings S and T other than x^*. Now consider

$$\begin{aligned}
m_b(x^*, u) &= m_b(Sx^*, Tu) \\
&\leq (m_b(x^*, u))^{\alpha(x^*, u)}.(m_b(x^*, Sx^*))^{\beta(x^*, u)} \\
&\quad .(m_b(u, Tu))^{\nu(x^*, u)}.(m_b(u, Sx^*).m_b(x^*, Tu))^{\xi(x^*, u)} \\
&\leq (m_b(x^*, u))^{\alpha(x^*, u)}.(m_b(x^*, x^*))^{\beta(x^*, u)}.(m_b(u, u))^{\nu(x^*, u)}. \\
&\quad .(m_b(u, x^*).m_b(x^*, u))^{\xi(x^*, u)} \\
&\leq (m_b(x^*, u))^{\alpha(x^*, u)+2\xi(x^*, u)}.
\end{aligned}$$

This implies that

$$(m_b(x^*, u))^{1-[\alpha(x^*, u)+2\xi(x^*, u)]} \leq 1,$$

which further implies that

$$m_b(x^*, u) \leq (1)^{\frac{1}{1-[\alpha(x^*, u)+2\xi(x^*, u)]}} \leq 1.$$

which is a contradiction to the fact that $x^* \neq u$. Thus x^* is a unique common fixed point of the mapping S and T in $\overline{B_{m_b}(x_0, r)}$. □

Example 2.2 Let $X = [0, \infty)$ be endowed with a b-multiplicative metric with $s = 2$.

$$m_b(x, y) = \left\{ \begin{array}{ll} 2^{(x+y)^2} & \text{if } x \neq y \\ 1 & \text{if } x = y \end{array} \right\}.$$

Define

$$S\colon X \to X,\ Sx = \begin{cases} \frac{3x}{10} & \text{if } 0 \le x \le 3 \\ x^5 + \sqrt{x} + 6 & \text{otherwise.} \end{cases}$$

$$T\colon X \to X,\ Tx = \begin{cases} \frac{x}{9} & \text{if } 0 \le x \le 3 \\ 4x^6 + \sqrt{7x} + 9 & \text{otherwise.} \end{cases}$$

Define $\alpha(x,y) = \frac{3}{10}$, $\beta(x,y) = xy^4$, $\xi(x,y) = \frac{x+y}{70}$, $\nu(x,y) = \frac{(x-2y)^3}{40}$. Consider $x_0 = 1$, $r = 2^{16}$, then $\overline{B_{m_b}(x_0,r)} = [0,3]$. Clearly $\alpha,\beta,\xi,\nu \in M_A(S,T)$, where $A = \overline{B_{m_b}(x_0,r)}$. Now $x_1 = S1 = \frac{3}{10}$, $\alpha(x_0,\ x_1) = \frac{3}{10}$, $\beta(x_0,\ x_1) = \frac{81}{10000}$, $\xi(x_0,\ x_1) = \frac{13}{700}$, $\nu(x_0,\ x_1) = \frac{1}{625}$. Now, $h = \max\{h_1,h_2\} \approx \max\{0.355, 0.359\} = 0.359$. So, $sh < 1$. We know that

$$\begin{aligned} \left(1+\frac{3}{10}\right)^2 &< \frac{16(1-2(0.359))}{2} \\ \text{or } 2^{\left(1+\frac{3}{10}\right)^2} &< 2^{\frac{16(1-2(0.359))}{2}} \\ \text{or } m_b(x_0,Sx_0) &\le r^{\frac{(1-\ sh)}{s}}. \end{aligned}$$

For each $x,y \in \overline{B_{m_b}(x_0,r)}$, we have

$$\begin{aligned} 2^{\left(\frac{3x}{10}+\frac{y}{9}\right)^2} &\le (2^{(x+y)^2})^{\frac{3}{10}}.(2^{(x+\frac{3x}{10})^2})^{xy^4}.(2^{(y+\frac{y}{9})^2})^{\frac{(x-2y)^3}{40}}.(2^{(y+\frac{3x}{10})^2}.2^{(x+\frac{y}{9})^2})^{\frac{x+y}{70}} \\ \text{or } m_b(Sx,Ty) &\le (m_b(x,y))^{\alpha(x,y)}.(m_b(x,Sx))^{\beta(x,y)}.(m_b(y,Ty))^{\nu(x,y)} \\ &\quad .(m_b(y,Sx).m_b(x,Ty))^{\xi(x,y)}. \end{aligned}$$

Thus, all conditions of Theorem 2.1 hold. Therefore, S and T have a unique common fixed point in $\overline{B_{m_b}(x_0,r)}$. Note that $\alpha,\beta,\xi,\nu \notin M(S,T)$, so the result in [4] can not be applied to ensure the existence of a unique common fixed point.

If we take $\beta(x,y) = 0$ in Theorem 2.1, then we obtain the following result.

Theorem 2.3 Let (X,m_b) be a complete b- multiplicative metric space and $S,T : X \to X$ be self-mappings. If there exist mappings $\alpha,\nu,\xi \in M_A(S,T)$, $A = \overline{B_{m_b}(x_0,r)}$, $x_0 \in X$ and $r > 1$ such that:

$$m_b(x_0,Sx_0) \le r^{\frac{(1-\ sh)}{s}},$$

where $sh < 1$, $h = \max\{h_1,h_2\}$ and

$$h_1 = \frac{\alpha(x_0,\ x_1) + s\xi(x_0,x_1)}{1-\nu(x_0,\ x_1) - s\xi(x_0,\ x_1)},\ h_2 = \frac{\alpha(x_0,\ x_1)+\nu(x_0,\ x_1)+s\xi(x_0,x_1)}{1-s\xi(x_0,\ x_1)}.$$

Also, if $\overline{B_{m_b}(x_0,r)}$ is closed and x,y belongs to $\overline{B_{m_b}(x_0,r)}$, then this implies

$$\begin{aligned} m_b(Sx,Ty) &\le (m_b(x,y))^{\alpha(x,y)}.(m_b(y,Ty))^{\nu(x,y)}. \\ &\quad (m_b(y,Sx).m_b(x,Ty))^{\xi(x,y)}. \end{aligned}$$

Then S and T have a unique common fixed point in $\overline{B_{m_b}(x_0,r)}$.

If we take $\beta(x,y) = \nu(x,y) = 0$ in Theorem 2.1, then we obtain the following result.

Theorem 2.4 Let (X,m_b) be a complete b- multiplicative metric space and $S,T : X \to X$ be self-mappings. If there exist mappings $\alpha,\xi \in M_A(S,T)$, $A = \overline{B_{m_b}(x_0,r)}$, $x_0 \in X$ and $r > 1$ such that:

$$m_b(x_0,Sx_0) \le r^{\frac{(1-\ sh)}{s}},$$

where $sh < 1$, $h = \max\{h_1, h_2\}$ and

$$h_1 = \frac{\alpha(x_0,\ x_1) + s\xi(x_0, x_1)}{1 - s\xi(x_0,\ x_1)},\ h_2 = \frac{\alpha(x_0,\ x_1) + s\xi(x_0, x_1)}{1 - s\xi(x_0,\ x_1)}.$$

Also, if $\overline{B_{m_b}(x_0, r)}$ is closed and x, y belongs to $\overline{B_{m_b}(x_0, r)}$, then this implies

$$m_b(Sx, Ty) \le (m_b(x, y))^{\alpha(x,y)}.(m_b(y, Sx).m_b(x, Ty))^{\xi(x,y)}.$$

Then S and T have a unique common fixed point in $\overline{B_{m_b}(x_0, r)}$.

If we take $\beta(x, y) = \xi(x, y) = 0$ in Theorem 2.1, then we obtain the following result.

Theorem 2.5 Let (X, m_b) be a complete b- multiplicative metric space and $S, T : X \to X$ be self-mappings. If there exist mappings $\alpha, \nu \in M_A(S, T)$, $A = \overline{B_{m_b}(x_0, r)}$, $x_0 \in X$ and $r > 1$ such that:

$$m_b(x_0, Sx_0) \le r^{\frac{(1-\ sh)}{s}},$$

where $sh < 1$, $h = \max\{h_1, h_2\}$ and

$$h_1 = \frac{\alpha(x_0,\ x_1)}{1 - \nu(x_0,\ x_1)},\ h_2 = \alpha(x_0,\ x_1) + \nu(x_0,\ x_1).$$

Also, if $\overline{B_{m_b}(x_0, r)}$ is closed and x, y belongs to $\overline{B_{m_b}(x_0, r)}$, then this implies

$$m_b(Sx, Ty) \le (m_b(x, y))^{\alpha(x,y)}.(m_b(y, Ty))^{\nu(x,y)}.$$

Then S and T have a unique common fixed point in $\overline{B_{m_b}(x_0, r)}$.

If we take $\beta(x, y) = \nu(x, y) = \xi(x, y) = 0$ in Theorem 2.1, then we obtain the following result.

Theorem 2.6 Let (X, m_b) be a complete b- multiplicative metric space and $S, T : X \to X$ be self-mappings. If there exist mappings $\alpha \in M_A(S, T)$, $A = \overline{B_{m_b}(x_0, r)}$, $x_0 \in X$ and $r > 1$ such that:

$$m_b(x_0, Sx_0) \le r^{\frac{(1-\ s\alpha(x_0,x_1))}{s}},$$

where $s\alpha(x_0,\ x_1) < 1$. Also, if $\overline{B_{m_b}(x_0, r)}$ is closed and x, y belongs to $\overline{B_{m_b}(x_0, r)}$, then this implies

$$m_b(Sx, Ty) \le (m_b(x, y))^{\alpha(x,y)}.$$

Then S and T have a unique common fixed point in $\overline{B_{m_b}(x_0, r)}$.

If we take $S = T$ in Theorem 2.1, then we obtain the following result.

Theorem 2.7 Let (X, m_b) be a complete b- multiplicative metric space and $S : X \to X$ be self-mappings. If there exist mappings $\alpha, \beta, \nu, \xi \in M_A(S)$, $A = \overline{B_{m_b}(x_0, r)}$, $x_0 \in X$ and $r > 1$ such that:

$$m_b(x_0, Sx_0) \le r^{\frac{(1-\ sh)}{s}},$$

where $sh < 1$, $h = \max\{h_1, h_2\}$ and

$$h_1 = \frac{\alpha(x_0,\ x_1) + \beta(x_0,\ x_1) + s\xi(x_0, x_1)}{1 - \nu(x_0,\ x_1) - s\xi(x_0,\ x_1)},\ h_2 = \frac{\alpha(x_0,\ x_1) + \nu(x_0,\ x_1) + s\xi(x_0, x_1)}{1 - \beta(x_0,\ x_1) - s\xi(x_0,\ x_1)}.$$

Also, if $\overline{B_{m_b}(x_0, r)}$ is closed and x, y belongs to $\overline{B_{m_b}(x_0, r)}$, then this implies

$$\begin{aligned} m_b(Sx, Sy) \quad \le \quad & (m_b(x, y))^{\alpha(x,y)}.(m_b(x, Sx))^{\beta(x,y)}.(m_b(y, Sy))^{\nu(x,y)}. \\ & (m_b(y, Sx).m_b(x, Sy))^{\xi(x,y)}. \end{aligned}$$

Then S has a unique fixed point in $\overline{B_{m_b}(x_0, r)}$.

If we take whole space instead of closed ball in Theorem 2.1, then we obtain the following result.

Theorem 2.8 Let (X, m_b) be a complete b- multiplicative metric space and $S, T : X \to X$ be self-mappings. If there exist mappings $\alpha, \beta, \nu, \xi \in M_A(S, T)$, $x_0 \in X$ and $r > 1$ such that:

$$m_b(x_0, Sx_0) \leq r^{\frac{(1-\ sh)}{s}},$$

where $sh < 1$, $h = \max\{h_1, h_2\}$ and

$$h_1 = \frac{\alpha(x_0,\ x_1) + \beta(x_0,\ x_1) + s\xi(x_0, x_1)}{1 - \nu(x_0,\ x_1) - s\xi(x_0,\ x_1)},\ h_2 = \frac{\alpha(x_0,\ x_1) + \nu(x_0,\ x_1) + s\xi(x_0, x_1)}{1 - \beta(x_0,\ x_1) - s\xi(x_0,\ x_1)}.$$

Also, if $\overline{B_{m_b}(x_0, r)}$ is closed and x, y belongs to $\overline{B_{m_b}(x_0, r)}$, then this implies

$$\begin{aligned} m_b(Sx, Ty) \quad \leq \quad & (m_b(x, y))^{\alpha(x,y)}.(m_b(x, Sx))^{\beta(x,y)}.(m_b(y, Ty))^{\nu(x,y)}. \\ & (m_b(y, Sx).m_b(x, Ty))^{\xi(x,y)}. \end{aligned}$$

Then S and T have a unique common fixed point in $\overline{B_{m_b}(x_0, r)}$.

If we take multiplicative metric space instead of b- multiplicative metric space in Theorem 2.1, then we obtain the following result.

Theorem 2.9 Let (X, m) be a complete multiplicative metric space and $S, T : X \to X$ be self-mappings. If there exist mappings $\alpha, \beta, \nu, \xi \in M_A(S, T)$, $A = \overline{B_m(x_0, r)}$, $x_0 \in X$ and $r > 1$ such that:

$$m_b(x_0, Sx_0) \leq r^{(1-\ h)},$$

where $h < 1$, $h = \max\{h_1, h_2\}$ and

$$h_1 = \frac{\alpha(x_0,\ x_1) + \beta(x_0,\ x_1) + \xi(x_0,\ x_1)}{1 - \nu(x_0,\ x_1) - \xi(x_0,\ x_1)},\ h_2 = \frac{\alpha(x_0,\ x_1) + \nu(x_0,\ x_1) + \xi(x_0,\ x_1)}{1 - \beta(x_0,\ x_1) - \xi(x_0,\ x_1)}.$$

Also, if $\overline{B_m(x_0, r)}$ is closed and x, y belongs to $\overline{B_m(x_0, r)}$, then this implies

$$\begin{aligned} m(Sx, Ty) \quad \leq \quad & (m(x, y))^{\alpha(x,y)}.(m(x, Sx))^{\beta(x,y)}.(m(y, Ty))^{\nu(x,y)}. \\ & (m(y, Sx).m(x, Ty))^{\xi(x,y)}. \end{aligned}$$

Then S and T have a unique common fixed point in $\overline{B_m(x_0, r)}$.

If we take whole space instead of closed ball and multiplicative metric space instead of b- multiplicative metric space in Theorem 2.1, then we obtain the following result. In this result, we have omitted the condition $m_b(x_0, Sx_0) \leq r^{\frac{(1-\ sh)}{s}}$, because it was applied to restrict the sequence in a closed ball.

Theorem 2.10 Let (X, m) be a complete multiplicative metric space and $S, T : X \to X$ be self-mappings. If there exist mappings $\alpha, \beta, \nu, \xi \in M(S, T)$, $x_0 \in X$ and $\alpha(x_0, x_1) + \beta(x_0, x_1) + \nu(x_0, x_1) + 2\xi(x_0, x_1) < 1$ such that:

$$\begin{aligned} m(Sx, Ty) \quad \leq \quad & (m(x, y))^{\alpha(x,y)}.(m(x, Sx))^{\beta(x,y)}.(m(y, Ty))^{\nu(x,y)}. \\ & (m(y, Sx).m(x, Ty))^{\xi(x,y)},\ \text{for all } x, y \in X. \end{aligned}$$

Then S and T have a unique common fixed point in X.

Proof. (X, m) is a complete b-multiplicative metric space with $s = 1$. Now,

$$\alpha(x_0, x_1) + \beta(x_0, x_1) + \nu(x_0, x_1) + 2\xi(x_0, x_1) < 1$$

implies

$$\begin{aligned} h_1 &= \frac{\alpha(x_0,\ x_1)+\beta(x_0,\ x_1)+s\xi(x_0,x_1)}{1-\nu(x_0,\ x_1)-s\xi(x_0,\ x_1)}<1,\\ h_2 &= \frac{\alpha(x_0,\ x_1)+\nu(x_0,\ x_1)+s\xi(x_0,x_1)}{1-\beta(x_0,\ x_1)-s\xi(x_0,\ x_1)}<1. \end{aligned}$$

Hence $sh<1$, $h=\max\{h_1,h_2\}$. As the condition holds for all $x,y\in X$ then it obviously holds for it's closed subsets. Now, by Theorem 2.1, S and T have a unique common fixed point in X. □

Now, we present the b-metric version of Theorem 2.1.

Theorem 2.11 Let (X,b) be a complete b-metric space and $S,T:X\to X$ be self-mappings. If there exist mappings $\alpha,\beta,\nu,\xi\in M_A(S,T)$, $A=\overline{B_b(x_0,r)}$, $x_0\in X$ and $r>0$ such that:

$$b(x_0,Sx_0)\le\frac{r(1-sh)}{s}, \tag{2.8}$$

where $sh<1$, $h=\max\{h_1,h_2\}$ and

$$h_1=\frac{\alpha(x_0,\ x_1)+\beta(x_0,\ x_1)}{1-\nu(x_0,\ x_1)-s\xi(x_0,\ x_1)},\ h_2=\frac{\alpha(x_0,\ x_1)+\nu(x_0,\ x_1)}{1-\beta(x_0,\ x_1)-s\xi(x_0,\ x_1)}.$$

Also, if $\overline{B_b(x_0,r)}$ is closed and x,y belongs to $\overline{B_b(x_0,r)}$, then this implies

$$\begin{aligned} b(Sx,Ty)\le\alpha(x,y)b(x,y)+\beta(x,y)b(x,Sx)+\nu(x,y)b(y,Ty)+\\ \xi(x,y)(b(y,Sx)+b(x,Ty)). \end{aligned} \tag{2.9}$$

Then S and T have a unique common fixed point in $\overline{B_b(x_0,r)}$.

Proof. Define $m_b(x,y)=e^{b(x,y)}$. Then by Remark 1.5 (W,m_b) is a $b-$multiplicative metric space. By taking exponential on both sides of inequality (2.7), we have

$$\begin{aligned} e^{b(x_0,Sx_0)} &\le e^{\frac{r(1-sh)}{s}},\\ \text{or } m_b(x_0,Sx_0) &\le \varepsilon^{\frac{(1-\ sh)}{s}} \end{aligned}$$

where $\varepsilon=e^r>1$. Now, by taking exponential on both sides of inequality (2.8) and by using Remark 1.6, we have

$$\begin{aligned} e^{b(Sx,Ty)} \le\ & e^{\alpha(x,y)b(x,y)}.e^{\beta(x,y)b(x,Sx)}.e^{\nu(x,y)b(y,Ty)}.\\ & e^{\xi(x,y)(b(y,Sx)+b(x,Ty))}, \end{aligned}$$

for all x,y belong to the closed set $\overline{B_b(x_0,r)}$. Now by using Remark 1.5 and Remark 1.6, we have

$$\begin{aligned} m_b(Sx,Ty) \le\ & (m_b(x,y))^{\alpha(x,y)}.(m_b(x,Sx))^{\beta(x,y)}.(m_b(y,Ty))^{\nu(x,y)}.\\ & (m_b(y,Sx).m_b(x,Ty))^{\xi(x,y)}. \end{aligned}$$

for all x,y belong to the closed set $\overline{B_{m_b}(x_0,\varepsilon)}$. Now by Theorem 2.1, S and T have a unique common fixed point in $\overline{B_{m_b}(x_0,\varepsilon)}$ or $\overline{B_b(x_0,r)}$.

□

Now, we present a corresponding result for a strictly increasing mapping F. We give a short and simple proof. Other recent results in literature see [3, 8, 19] can be proved and improved in a similar way by using strictly increasing mapping F instead of mapping F introduced by Wardowski [32]. This also shows that this type of generalization of the result of Wardowski is not a real generalization.

Theorem 2.12 Let (X,b) be a complete b-metric space, $S,T : X \to X$ be self-mappings and F be a strictly increasing mapping. If there exist mappings $\alpha,\beta,\nu,\xi \in M_A(S,T)$, $A=\overline{B_b(x_0,r)}$, $x_0 \in X$ and $r>0$ such that:

$$b(x_0,Sx_0) \leq \frac{r(1-sh)}{s},$$

where $sh<1$, $h=\max\{h_1,h_2\}$ and

$$h_1=\frac{\alpha(x_0,\ x_1)+\beta(x_0,\ x_1)}{1-\nu(x_0,\ x_1)-s\xi(x_0,\ x_1)},\ h_2=\frac{\alpha(x_0,\ x_1)+\nu(x_0,\ x_1)}{1-\beta(x_0,\ x_1)-s\xi(x_0,\ x_1)}.$$

Also, if $\overline{B_b(x_0,r)}$ is closed, x,y belongs to $\overline{B_b(x_0,r)}$ and $\tau>0$, then this implies

$$\tau+F\left(b(Sx,Ty)\right) \leq F\left(\begin{array}{c}\alpha(x,y)b(x,y)+\beta(x,y)b(x,Sx)+\nu(x,y)b(y,Ty)\\ +\xi(x,y)(b(y,Sx)+b(x,Ty))\end{array}\right). \tag{2.10}$$

Then S and T have a unique common fixed point in $\overline{B_b(x_0,r)}$.

Proof. Since $\tau>0$, then inequality (2.10) implies

$$F\left(b(Sx,Ty)\right) < F\left(\begin{array}{c}\alpha(x,y)b(x,y)+\beta(x,y)b(x,Sx)+\nu(x,y)b(y,Ty)\\ +\xi(x,y)(b(y,Sx)+b(x,Ty))\end{array}\right).$$

As F is a strictly increasing mapping, so

$$\begin{aligned} b(Sx,Ty) &< \alpha(x,y)b(x,y)+\beta(x,y)b(x,Sx)+\nu(x,y)b(y,Ty) \\ &\quad +\xi(x,y)(b(y,Sx)+b(x,Ty)). \end{aligned}$$

So, all hypotheses of Theorem 2.11 are satisfied and hence S and T have a unique common fixed point in $\overline{B_b(x_0,r)}$. □

Example 2.13 Let $X=\mathbb{R}$ endowed with the b-metric $b(x,y)=|x-y|$ for all $x,y\in X$ and $f:X\to X$ be defined by

$$fx=\left\{\begin{array}{c} -\frac{1}{2}x \text{ if } \quad x\in[-9,11] \\ 2x \text{ if } x\in\mathbb{R}\backslash[-9,11] \end{array}\right\}$$

Let $r=10$ and $x_0=1$, then $\overline{B_b(x_0,r)}=[-9,11]$ is closed. Take $\alpha(x,y)=\frac{1}{2}$, $\beta(x,y)=\nu(x,y)=\frac{1}{9}$, $\xi(x,y)=\frac{1}{18}$, then

$$sh=h_1=h_2=\frac{\frac{1}{2}+\frac{1}{9}+\frac{1}{18}}{1-\frac{1}{9}-\frac{1}{18}}<1.$$

If x,y belong to $\overline{B_b(x_0,r)}$, then

$$\begin{aligned} b(fx,fy) &\leq \alpha(x,y)b(x,y)+\beta(x,y)b(x,fx)+\nu(x,y)b(y,fy)+ \\ &\quad \xi(x,y)(b(y,fx)+b(x,fy)). \end{aligned}$$

So, inequality (2.8) holds. Also,

$$b(x_0,fx_0) \leq \frac{r(1-sh)}{s}.$$

So, all hypotheses of Theorem 2.11 are satisfied and therefore, f has a unique fixed point.

3. Application

Let $X = C([a,b],\mathbb{R}_+)$, $a > 0$ and $\mathbb{R}_+ = (0,\infty)$, be the space of all positive, continuous real valued functions, endowed with the b-multiplicative metric

$$m_b(x,y) = \sup_{t\in[a,b]}\left\{\max\left\{\left|\frac{x(t)}{y(t)}\right|^2, \left|\frac{y(t)}{x(t)}\right|^2\right\}\right\}$$

Define $\overline{B(x_0(t),r)} = \{y(t) : \sup_{t\in[a,b]}\left\{\max\left\{\left|\frac{x_0(t)}{y(t)}\right|^2, \left|\frac{y(t)}{x_0(t)}\right|^2\right\}\right\} \leq r\}$.

As an application, we give an existence theorem for the Fredholm multiplicative integral equations of the following type.

$$x(t) = \int_a^b Q_1(t,s,x(s))^{ds},\ t,s\in[a,b] \tag{3.1}$$

$$x(t) = \int_a^b Q_2(t,s,x(s))^{ds},\ t,s\in[a,b] \tag{3.2}$$

where $Q_1, Q_2 \colon [a,b]\times[a,b]\times\mathbb{R}_+ \to \mathbb{R}_+$ are integrable functions.

Theorem 3.1 Let $X = C([a,b],\mathbb{R}_+)$, $a > 0$ and let the mappings $S,T\colon X\to X$,

$$\begin{aligned} Sx(t) &= \int_a^b Q_1(t,s,x(s))^{ds} \\ Tx(t) &= \int_a^b Q_2(t,s,x(s))^{ds} \end{aligned}$$

where $Q_1, Q_2 \colon [a,b]\times[a,b]\times\mathbb{R}_+ \to \mathbb{R}_+$ are integrable functions. Assume that the following conditions hold:

(i) for each $t,s\in[a,b]$ and $x,y\in\overline{B(x_0(t),r)}$, $x_0(t)\in X$, $r > 1$, there exists a function $\beta\in M_A(S,T)$, $A = \overline{B(x_0(t),r)}$, such that

$$\left|\frac{Q_1(t,s,x(s))}{Q_2(t,s,y(s))}\right| \leq \left(\left|\frac{x(s)}{y(s)}\right|\right)^{\beta(x,y)};$$

(ii) the function $\beta(x,y)$ is such that $2\beta(x_0,x_1) < \frac{1}{b-a}$;

(iii)

$$\sup_{t\in[a,b]}\left\{\max\left\{\left|\frac{x_0(t)}{x_1(t)}\right|^2, \left|\frac{x_1(t)}{x_0(t)}\right|^2\right\}\right\} \leq r^{\frac{1-\ 2\beta(x_0,x_1)(b-a)}{2}};$$

Then the integral equations (3.1) and (3.2) have a unique common solution.

Proof. Let $x, y \in \overline{B(x_0(t), r)}$. Now, we have

$$\begin{aligned}
\left|\frac{Sx(t)}{Ty(t)}\right|^2 &\leq \left(\int_a^b \left|\frac{Q_1(t,s,x(s))}{Q_2(t,s,y(s))}\right|^{ds}\right)^2 \\
&\leq \left(\int_a^b \left(\left|\frac{x(s)}{y(s)}\right|^{\beta(x,y)}\right)^{ds}\right)^2 \\
&\leq \left(\int_a^b \left(m_b(x,y)^{\frac{\beta(x,y)}{2}}\right)^{ds}\right)^2 \\
&= \left(\left(m_b(x,y)^{b-a}\right)^{\frac{\beta(x,y)}{2}}\right)^2 \\
&= m_b(x,y)^{\beta(x,y)(b-a)} \text{ for each } t \in [a,b].
\end{aligned}$$

Thus, we get $m_b(Sx, Ty) \leq m_b(x,y)^{\alpha(x,y)}$, $\alpha(x,y) = \beta(x,y)(b-a)$. As $2\beta(x_0, x_1) < \frac{1}{b-a}$, so $s\alpha(x_0,\, x_1) < 1$. Also, hypothesis (iii) implies

$$m_b(x_0, Sx_0) \leq r^{\left(\frac{1-s\alpha(x_0,\, x_1)}{s}\right)}.$$

Therefore by Theorem 2.6, there exists a unique common fixed point of the operators S and T. Hence, the integral equations (3.1) and (3.2) have a unique common solution. □

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