Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 13 Issue 1(2021), Pages 41-56.

IDENTIFYING THE UNKNOWN SOURCE OF TIME FRACTIONAL DIFFUSION EQUATION ON A COLUMNAR SYMMETRIC DOMAIN

LE DINH LONG

Abstract. In this paper, I deal with the inverse problem of identifying the unknown source of time-fractional diffusion equation on a columnar symmetric domain. This problem is ill-posed. Firstly, we establish the conditional stability for this inverse problem. Then the regularization solution is obtained by using the Tikhonov regularization method and the error estimates are derived under the a priori and a posteriori choice rules of the regularization parameter.

1. Introduction

Determine the source of the problem inverse problem most common in heat conduction. These problems have been studied for decades by significant in many applications such as science and engineering in groundwater migration, identify and control sources of pollution, environmental protection [1]. The inverse heat source problems have extensive application background and important theoretic significance, so this have a long development history [2]–[6]. These problems are classical ill-posed problems, and some theories and extremely effective algorithms have been obtained. For instance, uniqueness and conditional stability results can be seen in [7]. This problem is usually uncorrected in the Hadamard sense, that is, the solution existing is not constantly dependent on the measured data. For that reason, it is very difficult to model the numbers for computation. Therefore, calibration methods and stability estimates are provided to correct the problem. A lot of regularization have been studied to deal with the inverse problem for time-fractional diffusion equation on a columnar axis-symmetric domain such as these methods include the fundamental solution method [8], [9], boundary element method [10] and [11] with iterative algorithm method, a mollification method [13],

2000 *Mathematics Subject Classification.* 35K05, 35K99, 47J06, 47H10x .
Key words and phrases. Source problem; Fractional pseudo-parabolic problem; Ill-posed problem; Convergence estimates; Regularization.

Submitted December 11, 2020. Published January 20, 2021.
Communicated by E. Karapinar.

[14]. In this work, we consider the following problem:

$$\begin{cases} D_t^\gamma u(r,t) - \dfrac{1}{r}u_r(r,t) - u_{rr}(r,t) = p(t)f(r), \ (r,t) \in (0,r_0)\times(0,T), \\ u(r,0) = a(r), & 0 \le r \le r_0, \\ u(r_0,t) = 0, & 0 \le t \le T, \\ \lim\limits_{r\to 0} u(r,t) \text{ bounded }, & 0 \le t \le T, \\ c_1 u(r,T) + c_2 \displaystyle\int_0^T u(r,t)dt = b(r), & 0 \le r \le r_0, \end{cases} \tag{1.1}$$

In case $c_1 = 1$ and $c_2 = 0$ then the final condition becomes $u(r,T) = b(r)$, there have been published a lot of research results (see [19]-[20], [21]). For instance, in [19], using the spectral method, Chu Li Fu and his colleagues surveyed the mathematical model by the following radial heat equation. In their study, estimation errors between the exact solution and is established strictly as the logarithmic-type is given under a suitable choice of regularization parameter. Next, in [20], using the spectral method, a Holder type estimate of the error between the approximate solution and the exact solution is obtained by C-L. Fu and his group, see [21], based on a modified Tikhonov regularization method, authors studied proposed for solving this inverse problem (1.1) with $p(t)f(r) = 0$. They showed a quite sharp estimate of the error between the approximate solution and the exact solution is obtained with a suitable choice of regularization parameter. Problem (1.1) in case $0 < \gamma < 1$, and $p(t) = 1$, seeing [15], the researchers have solved the (1.1) by using the Tikhonov regularization method, they show the error estimates are derived under the a priori and a posteriori choice rules of the regularization parameter, additional three numerical examples are presented to illustrate the validity and effectiveness of their method. However, in our study, we investigate case $F(r,t) = q(t)f(r)$, and $q(t) > 0$, by assuming that the time-fractional source term $q \in L^\infty(0,T)$ is known, the space-dependent source term $f(r)$ is unknown. Regarding the problem (1.1), Fan Yang and his group identify value for a time-fractional diffusion equation on a columnar axis-symmetric domain such as the inverse source problems [16], with the Tikhonov regularization method, and the initial value problem [18] with the Fractional Tikhonov regularization method. Authors identify the initial value for a time-fractional diffusion equation on a columnar axis-symmetric domain and Two different kinds of fractional Tikhonov methods are used to solve this problem. We use the data $c_1 u(r,T) + c_2 \int_0^T u(r,t)dt = b(r)$ to determine $f(r)$, instead of $u(r,T) = b(r)$ with r_0 is the radius, $c_1, c_2 \ge 0$, and $f(r)$ is the unknown heat source. To the best of ourknowledgement, there are several papers for the time-fractional diffusion equation on a columnar symmetric domain. therefore, we can say our results is one of the first results. Next, the Caputo fractional derivative D_t^γ is defined as follows:

$$D_t^\gamma u(r,t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{u_s(r,s)}{(t-s)^\gamma} ds, \ 0 < \gamma < 1, \tag{1.2}$$

where Γ is the gamma function, and $u(r,0) = a(r)$. In practice, the exact data (a,b,p) is noised by measured data $(a^\varepsilon, b^\varepsilon, p^\varepsilon)$ satisfy:

$$\|b^\varepsilon - b\|_{L^2(0,r_0;r)} \le \varepsilon,\ \|a^\varepsilon - a\|_{L^2(0,r_0;r)} \le \varepsilon,\ \|p^\varepsilon - p\|_{L^\infty(0,T)} \le \varepsilon. \tag{1.3}$$

where $\varepsilon > 0$ is the measurable error noise level, $L^2(0, r_0; r)$ denotes the Hilbert space of squares Lesbegue measurable functions with weight r defined on $(0, r_0)$. Our target in this paper is to apply the Landweber method to solve the inverse source problem in a general bounded domain. The Landweber iterative method is a very popular algorithm and regularization method in inverse problem research. Here, we show the convergent rate between the exact solution and its approximations under a *a-priori* parameter choice rule and *a-posteriori* parameter choice rule. The case $L^2 - norm$ used to evaluate the error estimation.

The outline of the paper is given as follows: In Section 2, we give some preliminary theoretical results. Ill-posed analysis and conditional stability are obtained in Section 3. In Section 4, we propose the iterated Landweber regularization method and give a convergence estimate under an a-priori regularization parameter choice rule and an a-posteriori regularization parameter choice rule for the deterministic case, respectively. The concluding remarks are shown in Section 5.

2. Statement of the problem

Throughout this paper, we denote by $L^2(0, r_0; r)$ the Hilbert space of Lebesgue measurable function f with weight r on $[0, r_0]$. $\langle\rangle$ and $\|\cdot\|$ denote inner product and norm on $L^2(0, r_0; r)$, respectively. Specifically, the norm and the inner product in $L^2(0, r_0; r)$ are defined as follows:

$$\|f\| := \|f\|_{L^2(0,r_0;r)} = \left(\int_0^{r_0} r\big|f(r)\big|^2 dr\right)^{\frac{1}{2}},\ \langle f, g\rangle = \int_0^{r_0} rf(r)g(r)dr, \tag{2.1}$$

for $f, g \in L^2(0, r_0; r)$. For $s > 0$, defining

$$\mathbb{H}^s(0, r_0; r) = \Big\{\nu \in L^2(0, r_0; r) : \sum_{j=1}^{\infty} \big(\frac{\lambda_j}{r_0}\big)^{4s} \big|\langle \nu, \xi_j\rangle\big|^2 < +\infty\Big\}, \tag{2.2}$$

where $\langle . \rangle$ is the inner product in $L^2(0, r_0; r)$, then $\mathbb{H}^s(0, r_0; r)$ is a Hilbert space equipped with the norm

$$\|\nu\|_{\mathbb{H}^s(0,r_0;r)} = \left(\sum_{j=1}^{\infty} \big(\frac{\lambda_j}{r_0}\big)^{4s} \big|\langle \nu, \xi_j\rangle\big|^2\right)^{\frac{1}{2}}. \tag{2.3}$$

Definition 2.1. *(See* [22]*) For any constant γ and $\kappa \in \mathbb{R}$, the Mittag-Leffler function is defined as:*

$$E_{\gamma,\alpha}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\gamma j + \alpha)},\ z \in \mathbb{C}, \tag{2.4}$$

where $\gamma > 0$ and $\alpha \in \mathbb{R}$ are arbitrary constant.

Lemma 2.1 ([24])**.** *For $0 < \gamma < 1$, $y > 0$, we get $0 \le E_{\gamma,1}(-y) < 1$. Therefore, $E_{\gamma,1}(-y)$ is completely monotonic, that is*

$$(-1)^c \frac{d^c}{dy^c} E_{\gamma,1}(-y) \ge 0,\ y \ge 0. \tag{2.5}$$

Lemma 2.2. [15] *For $\gamma > 0$ anh $\beta \in \mathbb{R}$, then we get*

$$E_{\gamma,\beta}(z) = zE_{\gamma,\gamma+\beta}(z) + \frac{1}{\Gamma(\beta)}, z \in \mathbb{C}. \tag{2.6}$$

Lemma 2.3. [16] *Assuming that $0 < \gamma_0 < \gamma_1 < 1$, then there exist constants $\mathcal{A}_1$ and $\mathcal{A}_2$ depending only on γ, γ_1 such that*

$$\frac{\mathcal{A}_1}{\Gamma(1-\gamma)} \frac{1}{1-z} \leq E_{\gamma,1}(z) \leq \frac{\mathcal{A}_2}{\Gamma(1-z)} \frac{1}{1-z}, \ z \geq 0. \tag{2.7}$$

Lemma 2.4. [18] *For $\lambda_j \geq \lambda_1 > 0$, then there exists constant $\mathcal{A}_1$ and $\mathcal{A}_2$ depending only on $\gamma, T, \lambda_1, r_0$ such that*

$$\frac{r_0^2 \ \mathcal{A}_1}{\lambda_j^2} \leq E_{\gamma,1}\big(- (\frac{\lambda_j}{r_0})^2 T^\gamma \big) \leq \frac{r_0^2 \ \mathcal{A}_2}{\lambda_j^2} \cdot \tag{2.8}$$

Proof. This proof can be found in [25]. □

Lemma 2.5. *Let $\mathcal{A}_3, \mathcal{A}_4 \geq 0$ satisfy $\mathcal{A}_3 \leq |p(t)| \leq \mathcal{A}_4$, $\forall t \in [0,T]$, let choose $\varepsilon \in \big(0, \frac{\mathcal{A}_3}{2}\big)$, by denoting $\mathcal{B}(\mathcal{A}_3, \mathcal{A}_4) = \mathcal{A}_4 + \frac{\mathcal{A}_3}{2}$, we get*

$$\frac{\mathcal{A}_3}{2} \leq |p^\varepsilon(t)| \leq \mathcal{B}(\mathcal{A}_3, \mathcal{A}_4). \tag{2.9}$$

Proof. This proof can be found at [23]. □

Lemma 2.6. [25] *For $\xi_j > 0$, $\gamma > 0$, and positive integer $j \in \mathbb{N}$, we have:*

$$\frac{d}{dt}\big(tE_{\gamma,2}(-\xi_j t^\gamma)\big) = E_{\gamma,1}(-\xi_j t^\gamma), \ \frac{d}{dt}\big(E_{\gamma,1}(-\xi_j t^\gamma)\big) = -\xi_j t^{\gamma-1} E_{\gamma,\gamma}(-\xi_j t^\gamma). \tag{2.10}$$

Lemma 2.7. *For any $\zeta, m, \beta, p > 0$, $\mathcal{N}$ is a positive constant, we get*

$$F_1(\zeta) = \Big(1 - m\frac{\mathcal{N}^2}{\zeta^4}\Big)^{2\beta} \zeta^{-4p} \leq \Big(\frac{p}{m\mathcal{N}^2}\Big)^p \big(p+2\beta\big)^{-p}. \tag{2.11}$$

Proof. Taking the derivative of $F_1'(\zeta) = 4\Big(1 - m\frac{\mathcal{N}}{\zeta^4}\Big)^{2\beta} \zeta^{-4p-1} \Big(m\mathcal{N}^2(2\beta+p) - p\zeta^4\Big)$. To solve $F_1'(\zeta_0) = 0$, we get $\zeta_0 = \Big[\frac{m\mathcal{N}^2(p+2\beta)}{p}\Big]^{\frac{1}{4}}$, so $F_1(\zeta_0) = \Big(\frac{2\beta}{p+2\beta}\Big)^{2\beta} \Big[\frac{p}{m\mathcal{N}^2(p+2\beta)}\Big]^p \leq \Big(\frac{p}{m\mathcal{N}^2}\Big)^p \big(p+2\beta\big)^{-p}$, this implies that $F_1(\zeta) \leq \Big(\frac{p}{m\mathcal{N}^2}\Big)^p \big(p+2\beta\big)^{-p}$. □

Lemma 2.8. *For any $\zeta, m, \beta, p > 0$, Q is a positive constant, we get*

$$F_2(\zeta) = \Big(1 - m\frac{Q^2}{\zeta^4}\Big)^{2\beta-2} \zeta^{-4p-4} \leq \Big(\frac{p+1}{2\beta m Q^2}\Big)^{p+1}. \tag{2.12}$$

Proof. Taking the derivative of $F_2'(\zeta) = \dfrac{4\Big(1 - m\frac{Q^2}{\zeta^4}\Big)^{2\beta-2} \zeta^{-4p-5}}{\zeta^4 - mQ^2} \Big[2(\beta-1)mQ^2 - (\zeta^4 - mQ^2)(p+1)\Big]$.

To solve $F_2'(\zeta_0) = 0$, we get $\zeta_0 = \Big[\frac{(p+2\beta-1)mQ^2}{p+1}\Big]^{\frac{1}{4}}$, so

$$F_2(\zeta_0) = \Big(\frac{2\beta-2}{p+2\beta-1}\Big)^{2\beta-2} \Big(\frac{p+1}{mQ^2(p+2\beta-1)}\Big)^{p+1} \leq \Big(\frac{p+1}{2\beta m Q^2}\Big)^{p+1},$$

this implies that $F_2(\zeta) \leq \Big(\frac{p+1}{2\beta m Q^2}\Big)^{p+1}$. □

Lemma 2.9. *For any $0<\gamma<1$, and the fact that $E_{\gamma,1}(-t^\gamma)$ is completely monotonic, with*
$\mathcal{C}_j(\gamma,t,r_0,\varsigma)=(t-\varsigma)^{\gamma-1}E_{\gamma,\gamma}\big(-(\frac{\lambda_j}{r_0})^2(t-\varsigma)^\gamma\big)$, *we get*

$$a)\ \Big(\frac{r_0}{\lambda_j}\Big)^2\Big(1-E_{\gamma,1}\big(-(\frac{\lambda_1}{r_0})^2T^\gamma\big)\Big)\le\int_0^T\mathcal{C}_j(\gamma,T,r_0,\varsigma)d\varsigma\le\Big(\frac{r_0}{\lambda_j}\Big)^2, \tag{2.13}$$

$$b)\ \Big(\frac{r_0}{\lambda_j}\Big)^2T\Big(1-E_{\gamma,2}\big(-(\frac{\lambda_1}{r_0})^2T^\gamma\big)\Big)\le\int_0^T\Big(\int_0^t\mathcal{C}_j(\gamma,t,r_0,\varsigma)d\varsigma\Big)dt\le\Big(\frac{r_0}{\lambda_j}\Big)^2\mathcal{A}_5. \tag{2.14}$$

Proof. The proof of part a) in this lemma can be found in the reference [27]. Next, using the Lemma 2.6, we prove the inequality 2.17 .

$$\begin{aligned}
\int_0^T\Big(\int_0^t\mathcal{C}_j(\gamma,t,r_0,\varsigma)d\varsigma\Big)dt&=\Big(\frac{r_0}{\lambda_j}\Big)^2\int_0^T\Big(1-E_{\gamma,1}(-(\frac{\lambda_j}{r_0})^2t^\gamma)\Big)dt\\
&=\Big(\frac{r_0}{\lambda_j}\Big)^2\int_0^Tdt-\Big(\frac{r_0}{\lambda_j}\Big)^2\int_0^TE_{\gamma,1}(-(\frac{\lambda_j}{r_0})^2t^\gamma)dt=\Big(\frac{r_0}{\lambda_j}\Big)^2T-\Big(\frac{r_0}{\lambda_j}\Big)^2\int_0^T\frac{d}{dt}tE_{\gamma,2}(-(\frac{\lambda_j}{r_0})^2t^\gamma)dt\\
&=\Big(\frac{r_0}{\lambda_j}\Big)^2T-\Big(\frac{r_0}{\lambda_j}\Big)^2TE_{\gamma,2}\big(-(\frac{\lambda_j}{r_0})^2t^\gamma\big)=\Big(\frac{r_0}{\lambda_j}\Big)^2T\Big(1-E_{\gamma,2}\big(-(\frac{\lambda_j}{r_0})^2T^\gamma\big)\Big)\\
&\ge\Big(\frac{r_0}{\lambda_j}\Big)^2T\Big(1-E_{\gamma,2}\big(-(\frac{\lambda_1}{r_0})^2T^\gamma\big)\Big).
\end{aligned}\tag{2.15}$$

$$\begin{aligned}
\int_0^T\Big(\int_0^t\mathcal{C}_j(\gamma,t,r_0,\varsigma)d\varsigma\Big)dt&=\Big(\frac{r_0}{\lambda_j}\Big)^2\int_0^T\Big(1-E_{\gamma,1}(-(\frac{\lambda_j}{r_0})^2t^\gamma)\Big)dt\\
&\le\Big(\frac{r_0}{\lambda_j}\Big)^2\int_0^Tdt+\Big(\frac{r_0}{\lambda_j}\Big)^2\int_0^TE_{\gamma,1}\big(-(\frac{\lambda_j}{r_0})^2t^\gamma\big)dt\\
&\le\Big(\frac{r_0}{\lambda_j}\Big)^2T+\Big(\frac{r_0}{\lambda_j}\Big)^2\int_0^T\frac{\mathcal{A}_2}{\big(\frac{\lambda_j}{r_0}\big)^2t^\gamma}dt=\Big(\frac{r_0}{\lambda_j}\Big)^2T+\Big(\frac{r_0}{\lambda_j}\Big)^4\frac{\mathcal{A}_2\ T^{1-\gamma}}{1-\gamma}\\
&\le\Big(\frac{r_0}{\lambda_j}\Big)^2\underbrace{\Big(T+\Big(\frac{r_0}{\lambda_1}\Big)^2\frac{\mathcal{A}_2T^{1-\gamma}}{1-\gamma}\Big)}_{:=\mathcal{A}_5}.
\end{aligned}\tag{2.16}$$

□

3. Ill-posed analysis and conditional stability

Using the separation of variables and Laplace transform of Mittag Leffler function, we obtain

Theorem 3.1. *Assume that* $p(t) \in L^{\infty}(0,T)$, $f(r), a(r), b(r) \in L^2(0, r_0; r)$, *the solution of problem* (1.1) *is:*

$$u(x,t) = \sum_{j=1}^{\infty} \Big(\int_0^t (t-\varsigma)^{\gamma-1} E_{\gamma,\gamma}(-(\frac{\lambda_j}{r_0})^2 (t-\varsigma)^{\gamma}) \ \langle f, \xi_j \rangle \ p(\varsigma) d\varsigma + E_{\gamma,1}\big(- (\frac{\lambda_j}{r_0})^2 t^{\gamma}\big) \langle a, \xi_j \rangle \Big) \xi_j(r). \tag{3.1}$$

whereby $\xi_j(r) = \dfrac{\sqrt{2}}{r_0 J_1(\lambda_j)} J_0\big(\dfrac{\lambda_j r}{r_0}\big)$, *inwhich* $j = 1, 2, 3, \ldots$, $\{\xi_j(r)\}$ *is an orthonormal basic an* $L^2[0, r_0; r]$, J_0 *and* J_1 *is a zero-order and first-order Bessel function.* $\{\xi_j\}_{j=1}^{\infty}$ *are the positive zeros of the zero-order Bessel function of the first kind* $J_0(\lambda)$ *and satisfy*

$$0 < \xi_1 < \xi_2 < \xi_3 < \ldots < \xi_j < \ldots, \lim_{j \to +\infty} \xi_j = +\infty. \tag{3.2}$$

Proof. From now on, for a shorter, by denoting $\mathcal{C}_j(\gamma, t, r_0, \varsigma) = (t-\varsigma)^{\gamma-1} E_{\gamma,\gamma}\big(- (\frac{\lambda_j}{r_0})^2 (t-\varsigma)^{\gamma}\big)$. Next, using the condition $c_1 u(r,T) + c_2 \int_0^T u(r,t) dt = b(r)$, we have

$$c_1 u(r,T) = c_1 \sum_{j=1}^{\infty} E_{\gamma,1}\big(- (\frac{\lambda_j}{r_0})^2 T^{\gamma}\big) \langle a, \xi_j \rangle \xi_j(r) + c_1 \sum_{j=1}^{\infty} \langle f, \xi_j \rangle \Big(\int_0^T \mathcal{C}_j(\gamma, T, r_0, \varsigma) p(\varsigma) d\varsigma \Big) \xi_j(r), \tag{3.3}$$

$$c_2 \int_0^T u(r,t) dt = c_2 \sum_{j=1}^{\infty} \int_0^T E_{\gamma,1}\big(- (\frac{\lambda_j}{r_0})^2 t^{\gamma}\big) dt \ \langle a, \xi_j \rangle \xi_j(r) + c_2 \sum_{j=1}^{\infty} \langle f, \xi_j \rangle \int_0^T \Big(\int_0^t \mathcal{C}_j(\gamma, t, r_0, \varsigma) p(\varsigma) d\varsigma \Big) dt \ \xi_j(r). \tag{3.4}$$

From (3.3) and (3.4), then

$$b(r) = c_1 \sum_{j=1}^{\infty} E_{\gamma,1}\big(- (\frac{\lambda_j}{r_0})^2 T^{\gamma}\big) \langle a, \xi_j \rangle \xi_j(r) + c_2 \sum_{j=1}^{\infty} \int_0^T E_{\gamma,1}\big(- (\frac{\lambda_j}{r_0})^2 t^{\gamma}\big) dt \ \langle a, \xi_j \rangle \xi_j(r)$$
$$+ c_1 \sum_{j=1}^{\infty} \langle f, \xi_j \rangle \Big(\int_0^T \mathcal{C}_j(\gamma, T, r_0, \varsigma) p(\varsigma) d\varsigma \Big) \xi_j(r) + c_2 \sum_{j=1}^{\infty} \langle f, \xi_j \rangle \int_0^T \Big(\int_0^t \mathcal{C}_j(\gamma, t, r_0, \varsigma) p(\varsigma) d\varsigma \Big) dt \ \xi_j(r). \tag{3.5}$$

From (3.5), we can see that

$$\langle b, \xi_j \rangle = c_1 \sum_{j=1}^{\infty} E_{\gamma,1}\big(- (\frac{\lambda_j}{r_0})^2 T^{\gamma}\big) \langle a, \xi_j \rangle \xi_j(r) + c_2 \sum_{j=1}^{\infty} \int_0^T E_{\gamma,1}\big(- (\frac{\lambda_j}{r_0})^2 t^{\gamma}\big) dt \ \langle a, \xi_j \rangle \xi_j(r)$$
$$+ \sum_{j=1}^{\infty} \langle f, \xi_j \rangle \Big(c_1 \int_0^T \mathcal{C}_j(\gamma, T, r_0, \varsigma) p(\varsigma) d\varsigma + c_2 \int_0^T \Big(\int_0^t \mathcal{C}_j(\gamma, t, r_0, \varsigma) p(\varsigma) d\varsigma \Big) dt \Big) \cdot \tag{3.6}$$

Through some basic transformations, which implies that

$$\big\langle f,\xi_j\big\rangle = \frac{\big\langle b,\xi_j\big\rangle - \big\langle a,\xi_j\big\rangle\Big(c_1 E_{\gamma,1}\big(-(\frac{\lambda_j}{r_0})^2 T^\gamma\big) + c_2\int\limits_0^T E_{\gamma,1}\big(-(\frac{\lambda_j}{r_0})^2 t^\gamma\big)dt\Big)}{c_1\int\limits_0^T \mathcal{C}_j(\gamma,T,r_0,\varsigma)p(\varsigma)d\varsigma + c_2\int\limits_0^T\Big(\int\limits_0^t \mathcal{C}_j(\gamma,t,r_0,\varsigma)p(\varsigma)d\varsigma\Big)dt}. \tag{3.7}$$

From (3.7), we conclude that

$$f(r) = \sum_{j=1}^{\infty}\frac{\big\langle b,\xi_j\big\rangle - \big\langle a,\xi_j\big\rangle\Big(c_1 E_{\gamma,1}\big(-(\frac{\lambda_j}{r_0})^2 T^\gamma\big) + c_2\int\limits_0^T E_{\gamma,1}\big(-(\frac{\lambda_j}{r_0})^2 t^\gamma\big)dt\Big)}{c_1\int\limits_0^T \mathcal{C}_j(\gamma,T,r_0,\varsigma)p(\varsigma)d\varsigma + c_2\int\limits_0^T\Big(\int\limits_0^t \mathcal{C}_j(\gamma,t,r_0,\varsigma)p(\varsigma)d\varsigma\Big)dt}\xi_j(r). \tag{3.8}$$

□

3.1. **The ill-posedness and stability of problem** (1.1)**.**

Theorem 3.2. *The inverse source problem is non-stability.*

Proof. A linear operator $\mathcal{P}: L^2(0,r_0;r)\to L^2(0,r_0;r)$ as follows.

$$\mathcal{P}f(r) = \int\limits_0^{r_0}\ell(r,\omega)f(\omega)d\omega = \varphi(r), \tag{3.9}$$

with

$$\varphi(r) = \big\langle b,\xi_j\big\rangle - \big\langle a,\xi_j\big\rangle\Big(c_1 E_{\gamma,1}(-(\frac{\lambda_j}{r_0})^2 T^\gamma) + c_2\int\limits_0^T E_{\gamma,1}(-(\frac{\lambda_j}{r_0})^2 t^\gamma)dt\Big), \tag{3.10}$$

and

$$\ell(r,\omega) = \sum_{j=1}^{\infty}\Bigg[c_1\int\limits_0^T \mathcal{C}_j(\gamma,T,r_0,\varsigma)p(\varsigma)d\varsigma + c_2\int\limits_0^T\Big(\int\limits_0^t \mathcal{C}_j(\gamma,t,r_0,\varsigma)p(\varsigma)d\varsigma\Big)dt\Bigg]\xi_j(r)\xi_j(\omega).$$

Due to $\ell(r,\omega)=\ell(\omega,r)$ we know $\mathcal{P}$ is self-adjoint operator. Next, we are going to prove its compactness. Defining the finite rank operators $\mathcal{P}_{\mathcal{N}_{Tr}}$ as follows

$$\mathcal{P}_{\mathcal{N}_{Tr}}\ f(x) = \sum_{j=1}^{\mathcal{N}_{Tr}}\Bigg[c_1\int\limits_0^T \mathcal{C}_j(\gamma,T,r_0,\varsigma)p(\varsigma)d\varsigma + c_2\int\limits_0^T\Big(\int\limits_0^t \mathcal{C}_j(\gamma,t,r_0,\varsigma)p(\varsigma)d\varsigma\Big)dt\Bigg]\big\langle f,\xi_j\big\rangle\xi_j(r). \tag{3.11}$$

Then, from (3.9) and (3.11), using the results from the Lemma 2.9, we have:

$$
\begin{aligned}
&\|\mathcal{P}_{\mathcal{N}_{Tr}} f-\mathcal{P} f\|^2_{L^2(0,r_0;r)}\\
&=\sum_{j=\mathcal{N}_{Tr}+1}^{\infty}\left|c_1\int_0^T \mathcal{C}_j(\gamma,T,r_0,\varsigma)p(\varsigma)d\varsigma+c_2\int_0^T\Big(\int_0^t \mathcal{C}_j(\gamma,t,r_0,\varsigma)p(\varsigma)d\varsigma\Big)dt\right|^2\left|\langle f,\xi_j\rangle\right|^2\\
&\le \sum_{j=\mathcal{N}_{Tr}+1}^{\infty}\frac{1}{\lambda_j^4}\Big[c_1\mathcal{A}_4r_0^2+c_2\mathcal{A}_4\mathcal{A}_5r_0^2\Big]^2\left|\langle f,\xi_j\rangle\right|^2\\
&\le \frac{\big[c_1\mathcal{A}_4r_0^2+c_2\mathcal{A}_4\mathcal{A}_5r_0^2\big]^2}{\lambda^4_{\mathcal{N}_{Tr}}}\sum_{j=\mathcal{N}_{Tr}+1}^{\infty}\left|\langle f,\xi_j\rangle\right|^2. \qquad (3.12)
\end{aligned}
$$

Therefore, $\|\mathcal{P}_{\mathcal{N}_{Tr}} f-\mathcal{P} f\|_{L^2(0,r_0;r)}$ in the sense of operator norm in $L(L^2(0,r_0;r);L^2(0,r_0;r))$ as $M\to\infty$. Also, $\mathcal{P}$ is a compact operator. Next, the singular values for the linear self-adjoint compact operator $\mathcal{P}$ are

$$
\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)=\left(c_1\int_0^T \mathcal{C}_j(\gamma,T,r_0,\varsigma)p(\varsigma)d\varsigma+c_2\int_0^T\Big(\int_0^t \mathcal{C}_j(\gamma,t,r_0,\varsigma)p(\varsigma)d\varsigma\Big)dt\right). \qquad (3.13)
$$

and corresponding eigenvectors is ξ_j which is known as an orthonormal basis in $L^2(0,r_0;r)$. From (3.9), the inverse source problem we introduced above can be formulated as an operator equation.

$$
\mathcal{P}f(r)=\varphi(r). \qquad (3.14)
$$

□

3.2. **Conditional stability of source term f.** In this section, we introduce a conditional stability by the following theorem.

Theorem 3.3. *If $\|f\|_{\mathbb{H}^{2s}(0,r_0;r)}\le\mathcal{M}$ for $\mathcal{M}$ is the positive constant, then we get*

$\|f\|_{L^2(0,r_0;r)}$ is defined in (3.18) ,

Proof. From (3.8), whereby $\varphi(r)$ is defined by (3.10) and Hölder inequality, we can see that:

$$
\begin{aligned}
\|f\|^2_{L^2(0,r_0;r)}&=\sum_{j=1}^{\infty}\frac{\left|\langle\varphi,\xi_j\rangle\right|^{\frac{2}{s+1}}\left|\langle\varphi,\xi_j\rangle\right|^{\frac{2s}{s+1}}}{\left|c_1\int_0^T \mathcal{C}_j(\gamma,T,r_0,\varsigma)p(\varsigma)d\varsigma+c_2\int_0^T\Big(\int_0^t \mathcal{C}_j(\gamma,t,r_0,\varsigma)p(\varsigma)d\varsigma\Big)dt\right|^2}\\
&\le\left(\sum_{j=1}^{\infty}\frac{\left|\langle\varphi,\xi_j\rangle\right|^2}{\left|c_1\int_0^T \mathcal{C}_j(\gamma,T,r_0,\varsigma)p(\varsigma)d\varsigma+c_2\int_0^T\Big(\int_0^t \mathcal{C}_j(\gamma,t,r_0,\varsigma)p(\varsigma)d\varsigma\Big)dt\right|^{2s+2}}\right)^{\frac{1}{s+1}}\left(\sum_{j=1}^{\infty}\left|\langle\varphi,\xi_j\rangle\right|^2\right)^{\frac{s}{s+1}}\\
&\le\left(\sum_{j=1}^{\infty}\frac{\left|\langle f,\xi_j\rangle\right|^2}{\left|c_1\int_0^T \mathcal{C}_j(\gamma,T,r_0,\varsigma)p(\varsigma)d\varsigma+c_2\int_0^T\Big(\int_0^t \mathcal{C}_j(\gamma,t,r_0,\varsigma)p(\varsigma)d\varsigma\Big)dt\right|^{2s}}\right)^{\frac{1}{s+1}}\|\varphi\|^{\frac{2s}{s+1}}_{L^2(0,r_0;r)}
\end{aligned}
\qquad (3.15)
$$

Due to Lemma 2.9, we can see that

$$\Big|c_1\int_0^T \mathcal{C}_j(\gamma,T,r_0,\varsigma)p(\varsigma)d\varsigma + c_2\int_0^T\Big(\int_0^t \mathcal{C}_j(\gamma,T,r_0,\varsigma)p(\varsigma)d\varsigma\Big)dt\Big|^{2s}$$
$$\geq \Big(\frac{r_0}{\lambda_j}\Big)^{4s}\underbrace{\Big(c_1\mathcal{A}_3\Big(1-E_{\gamma,1}(-(\frac{\lambda_1}{r_0})^2T^\gamma)\Big)+c_2\mathcal{A}_3T\Big(1-E_{\gamma,2}(-(\frac{\lambda_1}{r_0})^2T^\gamma)\Big)\Big)^{2s}}_{\big|Z(c_1,c_2,\gamma,T,\mathcal{A}_3,\lambda_1,r_0)\big|^{2s}}, \quad (3.16)$$

and this inequality leads to

$$\|f\|^2_{L^2(0,r_0;r)} \leq \big|Z(c_1,c_2,\gamma,T,\mathcal{A}_3,\lambda_1,r_0)\big|^{-2s}\sum_{j=1}^\infty\Big(\frac{\lambda_j}{r_0}\Big)^{4s}\big|\langle f,\xi_j\rangle\big|^2$$
$$\leq \frac{\|f\|^2_{\mathbb{H}^s(0,r_0;r)}}{\big|Z(c_1,c_2,\gamma,T,\mathcal{A}_3,\lambda_1,r_0)\big|^{2s}}. \quad (3.17)$$

Combining (3.16) and (3.17), we get

$$\|f\|^2_{L^2(0,r_0;r)} \leq \frac{\|f\|^{\frac{2}{s+1}}_{\mathbb{H}^s(0,r_0;r)}}{\big|Z(c_1,c_2,\gamma,T,\mathcal{A}_3,\lambda_1,r_0)\big|^{\frac{2s}{s+1}}}\|\varphi\|^{\frac{2s}{s+1}}_{L^2(0,r_0;r)}. \quad (3.18)$$

□

4. Landweber iteration regularization method and convergence rates

Now, we use the Landweber iterative method to obtain the regularization solution for problem (1.1). In here, we can see that the equation $\mathcal{P}f=\varphi$ in the form $f^\alpha = \big(I-\varsigma\mathcal{P}*\mathcal{P}\big)f^{\alpha-1}(r)+\varsigma\mathcal{P}^*\varphi(r)$ for some $\varsigma>0$ and give the following iterative form :

$$f^0(r):=0, \qquad f^\alpha(r) = \big(I-\varsigma\mathcal{P}^*\mathcal{P}\big)f^{\alpha-1}(r)+\varsigma\mathcal{P}^*\varphi(r), \qquad \alpha=1,2,3,... \quad (4.1)$$

where α is the iterative step number, the coefficient α needs to meet the condition $0<\alpha<\|\mathcal{P}\|^{-2}$, this implies that

$$f^{\alpha,\varepsilon}(r) = \varsigma\sum_{m=1}^{\alpha-1}\big(I-\varsigma\mathcal{P}^2\big)^m\mathcal{P}\varphi^\varepsilon(r). \quad (4.2)$$

With (3.13), it gives

$$f^{\alpha,\varepsilon}(r) = \mathcal{R}_\varsigma b^\varepsilon(r) = \sum_{j=1}^\infty \frac{1-\big(1-\varsigma\big|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p^\varepsilon)\big|^2\big)^\alpha}{\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p^\varepsilon)}\langle\varphi^\varepsilon,\xi_j\rangle\xi_j(r). \quad (4.3)$$

Before we go into proving the main theorem in subsection, we need the following lemma:

Lemma 4.1. *Let φ be given by (3.10) depends on a and b functions. Similarly, in a similar way we can find the function definition with the couple $\big(b^\varepsilon,a^\varepsilon\big)$ are observed data by (b,a) as follows $\langle\varphi^\varepsilon,\xi_j\rangle = \langle b^\varepsilon,\xi_j\rangle - \langle a^\varepsilon,\xi_j\rangle\Big(c_1E_{\gamma,1}(-(\frac{\lambda_j}{r_0})^2T^\gamma) + c_2\int_0^T E_{\gamma,1}(-(\frac{\lambda_j}{r_0})^2t^\gamma)dt\Big)$, denoting $\mathcal{A}_6 = \Big(c_1\mathcal{A}_2+c_2\frac{r_0^2}{\lambda_1^2}\frac{\mathcal{A}_2T^{1-\gamma}}{1-\gamma}\Big)^2$ then*

Proof. Using the inequality $(a+b)^2 \le 2a^2 + 2b^2, \forall a, b \ge 0$, and the properties of Mittag Leffler, see the Lemma 2.3, it gives

$$
\begin{aligned}
&\|\varphi^\varepsilon - \varphi\|^2_{L^2(0,r_0;r)} \\
&= \sum_{j=1}^{\infty} \Big\| \big\langle b^\varepsilon - b, \xi_j \big\rangle - \big\langle a^\varepsilon - a, \xi_j \big\rangle \Big(c_1 E_{\gamma,1}(-(\frac{\lambda_j}{r_0})^2 T^\gamma) + c_2 \int_0^T E_{\gamma,1}(-(\frac{\lambda_j}{r_0})^2 t^\gamma) d\varsigma \Big) \Big\|^2_{L^2(0,r_0;r)} \\
&\le 2 \sum_{j=1}^{\infty} \big| \big\langle b^\varepsilon - b, \xi_j \big\rangle \big|^2 + 2 \sum_{j=1}^{\infty} \Big\| \big\langle a^\varepsilon - a, \xi_j \big\rangle \Big(c_1 \mathcal{A}_2 + c_2 \int_0^T E_{\gamma,1}\big(-(\frac{\lambda_j}{r_0})^2 t^\gamma \big) dt \Big) \Big\|^2_{L^2(0,r_0;r)} \\
&\le 2\|b^\varepsilon - b\|^2_{L^2(0,r_0;r)} + 2\|a^\varepsilon - a\|^2_{L^2(0,r_0;r)} \Big(c_1 \mathcal{A}_2 + c_2 \frac{r_0^2}{\lambda_1^2} \frac{\mathcal{A}_2 T^{1-\gamma}}{1-\gamma} \Big)^2 \le 2\varepsilon^2 \big(1 + \mathcal{A}_6\big).
\end{aligned}
\tag{4.4}
$$

□

4.1. **An a priori parameter choice rule.**

Theorem 4.2. *Suppose that f is given by (3.8). Let $f^{\alpha,\varepsilon}$ is the its approximation, in here, we assume that conditions $f \in \mathbb{H}^s(0, r_0; r)$ and (1.3) hold. By choosing $\alpha = \big(\frac{\mathcal{M}}{\varepsilon}\big)^{\frac{2}{s+1}}$, then the following estimates:*

$$
\|f^{\alpha,\varepsilon} - f\|_{L^2(0,r_0;r)} \ \textit{\textbf{is of order}}\ \varepsilon^{\frac{s}{s+1}}. \tag{4.5}
$$

Proof. Using the triangle inequality, we get

$$
\|f^{\alpha,\varepsilon} - f\|_{L^2(0,r_0;r)} \le \|f^{\alpha,\varepsilon} - f^\alpha\|_{L^2(0,r_0;r)} + \|f^\alpha - f\|_{L^2(0,r_0;r)} \tag{4.6}
$$

We divide the proof into two steps:
<u>**Step 1:**</u> We have estimate $\|f^{\alpha,\varepsilon} - f^\alpha\|_{L^2(0,r_0;r)}$,

$$
\begin{aligned}
\|f^{\alpha,\varepsilon} - f^\alpha\|^2_{L^2(0,r_0;r)} &\le \underbrace{2 \sum_{j=1}^{\infty} \Big\| \frac{1 - \big(1 - \varsigma |\mathbf{E}_j^{c_1,c_2}(\gamma, T, t, r_0, p^\varepsilon)|^2\big)^\alpha}{\mathbf{E}_j^{c_1,c_2}(\gamma, T, t, r_0, p^\varepsilon)} \big\langle \varphi^\varepsilon - \varphi, \xi_j \big\rangle \Big\|^2_{L^2(0,r_0;r)}}_{\mathcal{I}_1^2} \\
&+ \underbrace{2 \sum_{j=1}^{\infty} \Big\| \Big[\frac{1 - \big(1 - \varsigma |\mathbf{E}_j^{c_1,c_2}(\gamma, T, t, r_0, p^\varepsilon)|^2\big)^\alpha}{\mathbf{E}_j^{c_1,c_2}(\gamma, T, t, r_0, p^\varepsilon)} - \frac{1 - \big(1 - \varsigma |\mathbf{E}_j^{c_1,c_2}(\gamma, T, t, r_0, p)|^2\big)^\alpha}{\mathbf{E}_j^{c_1,c_2}(\gamma, T, t, r_0, p)} \Big] \big\langle \varphi, \xi_j \big\rangle \Big\|^2_{L^2(0,r_0;r)}}_{\mathcal{I}_2^2}
\end{aligned}
\tag{4.7}
$$

$\boxed{\textit{Claim 1:}}$ Estimate of $\mathcal{I}_1^2$, with $0 < y < 1$, we know that $y \le \sqrt{y}$ and $(1-y)^\alpha \ge 1 - y\alpha$, with $\alpha > 0$, we obtain $1 - \big(1 - \varsigma |\mathbf{E}_j^{c_1,c_2}(\gamma, T, t, r_0, p^\varepsilon)|^2\big)^\alpha \le \varsigma^{\frac{1}{2}} \alpha^{\frac{1}{2}} \mathbf{E}_j^{c_1,c_2}(\gamma, T, t, r_0, p^\varepsilon)$, this is due to the use of the Bernoulli's inequality, combining the estimation from the Lemma 4.1, with $\mathcal{A}_6 = \Big(c_1 \mathcal{A}_2 + c_2 \frac{r_0^2}{\lambda_1^2} \frac{\mathcal{A}_2 T^{1-\gamma}}{1-\gamma} \Big)^2$, we have

$$
\mathcal{I}_1^2 \le 4\varsigma\alpha\varepsilon^2 \big(1 + \mathcal{A}_6\big). \tag{4.8}
$$

Claim 2: Estimation for $\mathcal{I}_2$, it is easy to see that $\dfrac{\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p^\varepsilon-p)}{\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p^\varepsilon)} \le \dfrac{2\varepsilon}{\mathcal{A}_3}$, we get

$$\begin{aligned}\mathcal{I}_2^2 &\le 4\sum_{j=1}^{\infty}\left\|\frac{\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p^\varepsilon-p)}{\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p^\varepsilon)\big(\frac{\lambda_j}{r_0}\big)^{2s}}\frac{\big(\frac{\lambda_j}{r_0}\big)^{2s}\langle\varphi,\xi_j\rangle}{\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)}\right\|^2_{L^2(0,r_0;r)}\\&+4\sum_{j=1}^{\infty}\left\|(1-\varsigma|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|^2)^\alpha\big(\frac{\lambda_j}{r_0}\big)^{-2s}\frac{\big(\frac{\lambda_j}{r_0}\big)^{2s}\langle\varphi,\xi_j\rangle}{\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)}\right\|^2_{L^2(0,r_0;r)}\\&+4\sum_{j=1}^{\infty}\left\|\frac{\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)(1-\varsigma|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p^\varepsilon)|^2)^\alpha}{\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p^\varepsilon)}\big(\frac{\lambda_j}{r_0}\big)^{-2s}\frac{\big(\frac{\lambda_j}{r_0}\big)^{2s}\langle\varphi,\xi_j\rangle}{\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)}\right\|^2_{L^2(0,r_0;r)}.\end{aligned}\tag{4.9}$$

From (4.9), using the estimation in Lemma 2.7, we can know that

$$\begin{aligned}\mathcal{I}_2^2 &\le 16r_0^{4s}\frac{\varepsilon^2\mathcal{M}^2}{\mathcal{A}_3^2\lambda_1^{4s}}+4r_0^{4s}\Big(\frac{s}{\varsigma Q^2}\Big)^s\big(s+2\alpha\big)^{-s}\mathcal{M}^2+16r_0^{4s}\Big(\frac{\mathcal{A}_4}{\mathcal{A}_3}\Big)^2\Big(\frac{s}{\varsigma\mathcal{N}^2}\Big)^s\big(s+2\alpha\big)^{-s}\mathcal{M}^2\\&\le\varepsilon^2\mathcal{M}^2\frac{16r_0^{4s}}{\mathcal{A}_3^2\lambda_1^{4s}}+\alpha^{-s}\mathcal{M}^2\Big(16r_0^{4s}\frac{\mathcal{A}_4^2}{\mathcal{A}_3^2}\Big(\frac{s}{2\varsigma\mathcal{N}^2}\Big)^s+4r_0^{4s}\big(\frac{s}{2\varsigma Q^2}\big)^s\Big)\cdot\end{aligned}\tag{4.10}$$

<u>**Step 2:**</u> Estimation for $\|f^\alpha-f\|_{L^2(0,r_0;r)}$, it gives

$$f^\alpha(r)-f(r)=\sum_{j=1}^{\infty}\big(1-\varsigma|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|^2\big)^\alpha\frac{\langle\varphi,\xi_j\rangle}{\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)}\xi_j(r)\tag{4.11}$$

From 4.11, we get

$$\|f^\alpha-f\|^2_{L^2(0,r_0;r)}=\sum_{j=1}^{\infty}\left\|\big(1-\varsigma|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|^2\big)^\alpha\Big(\frac{\lambda_j}{r_0}\Big)^{-2s}\Big(\frac{\lambda_j}{r_0}\Big)^{2s}\langle f,\xi_j\rangle\right\|^2_{L^2(0,r_0;r)}.\tag{4.12}$$

Apply the same proof as in claim 1, we receive

$$\begin{aligned}\|f^\alpha-f\|^2_{L^2(0,r_0;r)}&\le\mathcal{N}\sum_{j=1}^{\infty}r_0^{4s}\big(1-\varsigma|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|^2\big)^{2\alpha}\lambda_j^{-4s}\|f\|^2_{\mathbb{H}^s(0,r_0;r)}\\&\le r_0^{4s}\Big(\frac{s}{\varsigma Q^2}\Big)^s\big(s+2\alpha\big)^{-s}\mathcal{M}^2.\end{aligned}\tag{4.13}$$

Combining (4.7), (4.8), (4.9) and (4.13), taking the square root on the both side, we have

$$\|f^{\alpha,\varepsilon}-f\|_{L^2(0,r_0;r)}\le2\varsigma^{\frac12}\alpha^{\frac12}\varepsilon\big(1+\mathcal{A}_6^2\big)^{\frac12}+\varepsilon\mathcal{M}\Big(\frac{4r_0^{2s}}{\mathcal{A}_3\lambda_1^{2s}}\Big)+\alpha^{-\frac{s}{2}}\mathcal{M}(\mathcal{A}_7)^{\frac12}\cdot\tag{4.14}$$

Combining (4.14), and (4.32), we receive

$$\|f^{\alpha,\varepsilon}-f\|_{L^2(0,r_0;r)}\le\varepsilon^{\frac{s}{s+1}}\mathcal{M}^{\frac{1}{s+1}}\mathcal{H}\big(s,r_0,\varsigma,\delta,|Q|,\mathcal{A}_6,\mathcal{A}_7\big)\cdot\tag{4.15}$$

whereby $\mathcal{A}_7=\Big(16r_0^{4s}\dfrac{\mathcal{A}_4^2}{\mathcal{A}_3^2}\Big(\dfrac{s}{2\varsigma\mathcal{N}^2}\Big)^s+5r_0^{4s}\big(\dfrac{s}{2\varsigma Q^2}\big)^s\Big)$. For iterative step α is a integer, by choosing
$\alpha=\big(\frac{\mathcal{M}}{\varepsilon}\big)^{\frac{2}{s+1}}$, we obtain

$$\|f^{\alpha,\varepsilon}-f\|^2_{L^2(0,r_0;r)}\le\varepsilon^{\frac{s}{s+1}}\mathcal{M}^{\frac{1}{s+1}}\mathcal{A}_8\cdot\tag{4.16}$$

whereby $\mathcal{A}_8=\Big(2\varsigma^{\frac12}\big(1+\mathcal{A}_6\big)^{\frac12}+\varepsilon^{\frac{1}{s+1}}\mathcal{M}^{\frac{s}{s+1}}\Big(\dfrac{4r_0^{2s}}{\mathcal{A}_3\lambda_1^{2s}}\Big)+(\mathcal{A}_7)^{\frac12}\Big)$, $\mathcal{N}=\big[\mathcal{B}(\mathcal{A}_3,\mathcal{A}_4)c_1r_0^2+\mathcal{B}(\mathcal{A}_3,\mathcal{A}_4)\mathcal{A}_5c_2r_0^2\big]$, and $Q=\mathcal{A}_4c_1r_0^2+\mathcal{A}_4c_2r_0^2\mathcal{A}_5$. The proof of theorem is completed. $\square$

4.2. **An a posteriori parameter choice rule.** In this subsection, we give the *p*osteriori regularization choice rule. This algorithm will be stopped at the first occurrence of $\alpha = \alpha(\delta)$ with

$$\|\mathcal{P}f^{\alpha} - \varphi^{\varepsilon}\|_{L^2(0,r_0;r)} \leq \delta\varepsilon \leq \|\mathcal{P}f^{\alpha-1} - \varphi^{\varepsilon}\|_{L^2(0,r_0;r)}. \tag{4.17}$$

where $\|b^{\varepsilon}\| \geq \delta\varepsilon$. We need Lemma 4.3 to obtain the existence and uniqueness of (4.17).

Lemma 4.3. *Let $\mathcal{V}(\alpha) = \|\mathcal{P}f^{\alpha} - \varphi^{\varepsilon}\|_{L^2(0,r_0;r)}$, then we have*

(a) *$\mathcal{V}(\alpha)$ is a continous function;*

(b) $\lim_{\alpha\to 0}\mathcal{V}(\alpha) = \|\varphi^{\varepsilon}\|_{L^2(0,r_0;r)}$;

(c) $\lim_{\alpha\to +\infty}\mathcal{V}(\alpha) = 0$;

(d) *$\mathcal{V}(\alpha)$ is a strictly decreasing function for any $\alpha \in (0, +\infty)$.*

Proof. This can be easily found in the references [17] in Lemma 3.3. We omit here. □

Lemma 4.4. *Assume that $\delta > 1$, let $\alpha = \alpha(\varepsilon, \varphi^{\varepsilon}) \in \mathbb{N}_0$, choose from (4.17), then we obtain the regularization parameter α satisfies:*

$$\alpha \leq \left(\frac{s+1}{2\varsigma}\right)\left(\frac{r_0^{2s}\mathcal{M}}{Q^s\big(\delta - \sqrt{2}(1+\mathcal{A}_6)^{\frac{1}{2}}\big)\varepsilon}\right)^{\frac{2}{s+1}}. \tag{4.18}$$

Proof. From operator expression (4.2) and (3.10), we obtain

$$\mathcal{R}_{\alpha}\varphi^{\varepsilon}(r) = \sum_{j=1}^{\infty}\frac{1-\big(1-\varsigma|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|^2\big)^{\alpha}}{\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)}\langle\varphi^{\varepsilon},\xi_j\rangle\xi_j(r). \tag{4.19}$$

and thus

$$\|\mathcal{P}\mathcal{R}_{\alpha}\varphi - \varphi\|^2_{L^2(0,r_0;r)} = \sum_{j=1}^{\infty}\Big(1-\varsigma|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|^2\Big)^{2\alpha}\big|\langle\varphi,\xi_j\rangle\big|^2. \tag{4.20}$$

So $\|\mathcal{P}\mathcal{R}_{\alpha-1} - I\|_{L^2(0,r_0;r)} \leq 1$. From (4.19), we know that

$$\begin{aligned}
\|\mathcal{P}\mathcal{R}_{\alpha-1}\varphi - \varphi\|_{L^2(0,r_0;r)} &\geq \|\mathcal{P}\mathcal{R}_{\alpha-1}\varphi^{\varepsilon} - \varphi^{\varepsilon}\|_{L^2(0,r_0;r)} - \|(\mathcal{P}\mathcal{R}_{\alpha-1} - I)(\varphi - \varphi^{\varepsilon})\|_{L^2(0,r_0;r)} \\
&\geq \delta\varepsilon - \|(\mathcal{P}\mathcal{R}_{\alpha-1} - I)\|_{L^2(0,r_0;r)} \;\geq \delta\epsilon - \sqrt{2}\varepsilon\big(1+\mathcal{A}_6\big)^{\frac{1}{2}} \\
&\geq \big(\delta - \sqrt{2}(1+\mathcal{A}_6)^{\frac{1}{2}}\big)\varepsilon.
\end{aligned} \tag{4.21}$$

On the other hand, due to $\|f\|_{\mathbb{H}^s(0,r_0;r)} \le \mathcal{M}$, we know that

$$
\begin{aligned}
&\|\mathcal{P}\mathcal{R}_{\alpha-1}\varphi - \varphi\|^2_{L^2(0,r_0;r)}\\
&= \sum_{j=1}^{\infty} \Big\| \big(1 - \big(1 - \varsigma|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|^2\big)^{\alpha-1}\big)\langle \varphi, \xi_j\rangle - \langle \varphi,\xi_j\rangle \Big\|^2_{L^2(0,r_0;r)}\\
&= \sum_{j=1}^{\infty} \Big\| \big(1 - \varsigma|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|^2\big)^{\alpha-1} |\langle \varphi,\xi_j\rangle| \Big\|^2_{L^2(0,r_0;r)}\\
&= \sum_{j=1}^{\infty} \Big\| \big(1 - \varsigma|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|^2\big)^{\alpha-1} |\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)| \Big(\frac{\lambda_j}{r_0}\Big)^{-2s} \frac{\big(\frac{\lambda_j}{r_0}\big)^{2s}|\langle \varphi,\xi_j\rangle|}{\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)} \Big\|^2_{L^2(0,r_0;r)}\\
&= \sum_{j=1}^{\infty} \Big\| \big(1 - \varsigma|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|^2\big)^{\alpha-1} |\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)| \Big(\frac{\lambda_j}{r_0}\Big)^{-2s} \Big(\frac{\lambda_j}{r_0}\Big)^{2s} |\langle f,\xi_j\rangle| \Big\|^2_{L^2(0,r_0;r)}\\
&\le \sum_{j=1}^{\infty} r_0^{4s} \big(1 - \varsigma|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|^2\big)^{2\alpha-2} |\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|^2 \lambda_j^{-4s}\mathcal{M}^2. \qquad (4.22)
\end{aligned}
$$

From (4.22), we denote

$$
\mathcal{S}(j) := r_0^{4s}\big(1 - \varsigma|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|^2\big)^{2\alpha-2}|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|^2\lambda_j^{-4s-4}, \qquad (4.23)
$$

From (4.23), through several evaluation steps, we receive

$$
S(j) \le r_0^{4s}\ |Q|^2\ \Big(1 - \varsigma\frac{Q^2}{\lambda_j^4}\Big)^{2\alpha-2}\ \lambda_j^{-4s-4}. \qquad (4.24)
$$

So

$$
\big(\delta - \sqrt{2}(1+\mathcal{A}_6)^{\frac{1}{2}}\big)\varepsilon \le [\mathcal{S}(j)]^{\frac{1}{2}}\mathcal{M}. \qquad (4.25)
$$

Using the Lemma 2.8 and (4.24), we get

$$
\mathcal{S}(j) = r_0^{4s}|Q|^2\Big(1 - \varsigma\frac{Q^2}{\lambda_j^4}\Big)^{2\alpha-2}\lambda_j^{-4s-4} \le r_0^{4s}|Q|^{-2s}\Big(\frac{s+1}{2\alpha\varsigma}\Big)^{s+1}. \qquad (4.26)
$$

Combining (4.21) and (4.26), we can asert that

$$
\big(\delta - \sqrt{2}(1+\mathcal{A}_6)^{\frac{1}{2}}\big)\varepsilon \le r_0^{2s}|Q|^{-s}\Big(\frac{s+1}{2\alpha\varsigma}\Big)^{\frac{s+1}{2}}\mathcal{M}. \qquad (4.27)
$$

This implies that

$$
\alpha \le \Big(\frac{s+1}{2\varsigma}\Big)\Big(\frac{r_0^{2s}\mathcal{M}}{Q^s\big(\delta - \sqrt{2}(1+\mathcal{A}_6)^{\frac{1}{2}}\big)\varepsilon}\Big)^{\frac{2}{s+1}}. \qquad (4.28)
$$

whereby $Q = c_1\mathcal{A}_4 r_0^2 + c_2\mathcal{A}_4\mathcal{A}_5 r_0^2$. □

Theorem 4.5. *For $s > 0$, let $f(r)$ given by (3.8), and $f^{\alpha,\varepsilon}(r)$ be the regularization solution, assume that condition $f \in \mathbb{H}^s(0,r_0;r)$ and (1.3) holds. Then we have the following error estimate:*

$$
\|f^{\alpha,\varepsilon} - f\|_{L^2(0,r_0;r)}\ \textit{\textbf{is of order}}\ \varepsilon^{\frac{s}{s+1}}. \qquad (4.29)
$$

Proof. Using the triangle inequality, one has

$$
\|f^{\alpha,\varepsilon} - f\|_{L^2(0,r_0;r)} \le \|f^{\alpha,\varepsilon} - f^{\alpha}\|_{L^2(0,r_0;r)} + \|f^{\alpha} - f\|_{L^2(0,r_0;r)}. \qquad (4.30)
$$

Using the estimate (4.7) and the Lemma 4.4, we get

$$\|f^{\alpha,\varepsilon}-f^{\alpha}\|_{L^2(0,r_0;r)} \leq \underbrace{2\varsigma^{\frac{1}{2}}\alpha^{\frac{1}{2}}\varepsilon\big(1+\mathcal{A}_6\big)^{\frac{1}{2}}}_{\mathcal{L}_1} + \underbrace{\varepsilon\mathcal{M}\frac{4r_0^{2s}}{\mathcal{A}_3\lambda_1^{2s}}}_{\mathcal{L}_2} + \underbrace{\alpha^{-\frac{s}{2}}\mathcal{M}\mathcal{A}_9^{\frac{1}{2}}}_{\mathcal{L}_3}. \tag{4.31}$$

inwhich $\mathcal{A}_9 = \Big(\frac{16\mathcal{A}_4^2}{\mathcal{A}_3^2}\Big(\dfrac{s}{2\varsigma\mathcal{N}^2}\Big)^s + 4\Big(\dfrac{s}{2\varsigma Q^2}\Big)^s\Big)$. Subtituting (4.18) into (4.31), we can know that

$$\begin{aligned}
\mathcal{L}_1 &\leq \varepsilon^{\frac{s}{s+1}}\mathcal{M}^{\frac{1}{s+1}}\Big(\sqrt{2}\big[(s+1)(1+\mathcal{A}_6)\big]^{\frac{1}{2}}\Big(\frac{r_0}{|Q|^s\big(\delta-\sqrt{2}(1+\mathcal{A}_6)^{\frac{1}{2}}\big)}\Big)^{\frac{1}{s+1}}\Big),\\
\mathcal{L}_2 &\leq \varepsilon^{\frac{s}{s+1}}\mathcal{M}^{\frac{1}{s+1}}\Big(\varepsilon^{\frac{1}{s+1}}\mathcal{M}^{\frac{s}{s+1}}\frac{4r_0^{2s}}{\mathcal{A}_3\lambda_1^{2s}}\Big),\\
\mathcal{L}_3 &\leq \varepsilon^{\frac{s}{s+1}}\mathcal{M}^{\frac{1}{s+1}}\Big(\Big(\frac{2\varsigma}{s+1}\Big)^{\frac{s}{2}}\Big(\frac{|Q|^s\big(\delta-\sqrt{2}(1+\mathcal{A}_6)^{\frac{1}{2}}\big)^{\frac{1}{2}}}{r_0^{2s}}\Big)^{\frac{s}{s+1}}(\mathcal{A}_7)^{\frac{1}{2}}\Big).
\end{aligned} \tag{4.32}$$

Combining (4.14), and (4.32), we receive

$$\|f^{\alpha,\varepsilon}-f^{\alpha}\|_{L^2(0,r_0;r)} \leq \varepsilon^{\frac{s}{s+1}}\mathcal{M}^{\frac{1}{s+1}}\ \mathcal{H}\big(s,r_0,\varsigma,\delta,|Q|,\mathcal{A}_6,\mathcal{A}_7\big). \tag{4.33}$$

whereby

$$\begin{aligned}
\mathcal{H}\big(s,r_0,\varsigma,\delta,|Q|,\mathcal{A}_6,\mathcal{A}_7\big) &= \Big(\sqrt{2}\big[(s+1)(1+\mathcal{A}_6)\big]^{\frac{1}{2}}\Big(\frac{r_0}{|Q|^s\big(\delta-\sqrt{2}(1+\mathcal{A}_6)^{\frac{1}{2}}\big)}\Big)^{\frac{1}{s+1}}\\
&\quad+\varepsilon^{\frac{1}{s+1}}\mathcal{M}^{\frac{s}{s+1}}\frac{4r_0^{2s}}{\mathcal{A}_3\lambda_1^{2s}}+\Big(\frac{2\varsigma}{s+1}\Big)^{\frac{s}{2}}\Big(\frac{|Q|^s\big(\delta-\sqrt{2}(1+\mathcal{A}_6)^{\frac{1}{2}}\big)^{\frac{1}{2}}}{r_0^{2s}}\Big)^{\frac{s}{s+1}}(\mathcal{A}_7)^{\frac{1}{2}}\Big).
\end{aligned} \tag{4.34}$$

For the second, we can know that

$$\begin{aligned}
\mathcal{P}\big(f^{\alpha}-f\big) &= \sum_{j=1}^{\infty}\frac{1-\big(1-\varsigma|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|^2\big)^{\alpha}}{|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|}\big\langle\varphi,\xi_j\big\rangle\xi_j(r)\\
&= \sum_{j=1}^{\infty}\frac{1-\big(1-\varsigma|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|^2\big)^{\alpha}}{|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|}\big\langle\varphi-\varphi^{\varepsilon},\xi_j\big\rangle\xi_j(r)\\
&\quad+\sum_{j=1}^{\infty}\frac{1-\big(1-\varsigma|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|^2\big)^{\alpha}}{|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|}\big\langle\varphi^{\varepsilon},\xi_j\big\rangle\xi_j(r).
\end{aligned} \tag{4.35}$$

From (4.35), we can see that

$$\|\mathcal{P}\big(f^{\alpha}-f\big)\|_{L^2(0,r_0;r)} \leq \varsigma^{\frac{1}{2}}\alpha^{\frac{1}{2}}\varepsilon+\delta\varepsilon = \varepsilon\big(\varsigma^{\frac{1}{2}}\alpha^{\frac{1}{2}}+\delta\big). \tag{4.36}$$

Due to

$$\begin{aligned}
\big\|f^{\alpha}-f\big\|_{\mathbb{H}^s(0,r_0;r)} &= \Big(\sum_{j=1}^{\infty}\Big(\frac{\lambda_j}{r_0}\Big)^{4s}\big(1-\varsigma|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|^2\big)^{2\alpha}\frac{\big|\big\langle\varphi,\xi_j\big\rangle\big|^2}{|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|^2}\Big)^{\frac{1}{2}}\\
&\leq \Big(\sum_{j=1}^{\infty}\big(1-\varsigma|\mathbf{E}_j^{c_1,c_2}(\gamma,T,t,r_0,p)|^2\big)^{2\alpha}\Big(\frac{\lambda_j}{r_0}\Big)^{4s}\big|\big\langle f,\xi_j\big\rangle\big|^2\Big)^{\frac{1}{2}} \leq \mathcal{M}.
\end{aligned} \tag{4.37}$$

Because of Theorem 3.3, it is easy to see that

$$\left\|f^{\alpha}-f\right\|_{L^{2}(0,r_{0};r)}\leq\varepsilon^{\frac{s}{s+1}}\mathcal{M}^{\frac{1}{s+1}}\ (\varsigma^{\frac{1}{2}}\alpha^{\frac{1}{2}}+\delta)^{\frac{s}{s+1}}\left|\mathcal{Z}(c_1,c_2,\gamma,T,\mathcal{A}_3,\lambda_1,r_0)\right|^{-\frac{s}{s+1}}. \tag{4.38}$$

Combining (4.33) and (4.38), we conclude that

$$\|f^{\varepsilon,\alpha}-f\|_{L^{2}(0,r_{0};r)}\leq\ \textbf{is of order}\ \varepsilon^{\frac{s}{s+1}}. \tag{4.39}$$

□

5. Conclusion

In this paper, by using the Landweber method, we solved the unknown problem to recover the source term of time-fractional diffusion equation on a columnar symmetric domain. In the theoretical results, we show the error estimates between sought solution and regularized solution of both prior and posterior parameter choice rule methods based on a priori condition belongs to $\mathbb{H}^s(0,r_0;r)$. We can see that the convergence rate of the level is similarly.

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

References

[1] Li, G, Tan, Y, Cheng, J, Wang, X:, *Determining magnitude of groundwater pollution sources by data compatibility analysis.* Inverse Probl. Sci. Eng. 14, 287-300 (2006).
[2] J.R. Cannon, Determination of an unknown heat source from overspecified boundary data, SIAM J. Numer. Anal. 5 (1968), pp. 275–286.
[3] J.R. Cannon and P. Duchateau, Structural identification of an unknown source term in a heat equation, Inverse Probl. 14 (1998), pp. 535–551.
[4] J.R. Cannon and S.P. Esteva, An inverse problem for the heat equation, Inverse Probl. 2 (1986), pp. 395–403.
[5] P. Duchateau and W. Rundell, Unicity in an inverse problem for an unknown reaction term in a reation–diffusion–equation, J. Differ. Eqns. 59 (1985), pp. 155–164.
[6] F. Hettlich and W. Rundell, Identification of a discontinuous source in the heat equation, Inverse Probl. 17 (2001), pp. 1465–1482.
[7] J.R. Cannon and S.P. Esteva, Uniqueness and stability of 3D heat source, Inverse Probl. 7 (1991), pp. 57–62
[8] Micrzwiczak, M, Kolodziej, JA: *Application of the method of fundamental solutions and radical basis functions for inverse tranyient heat source problem.* Commun. Comput. Phys. 181, 2035-2043 (2010).
[9] Ahmadabadi, MN, Arab, M, Ghaini, *FMM: The method of fundamental solutions for the inverse space-dependent heat source problem*, Eng. Anal. Bound. Elem. 33, 1231-1235 (2009).
[10] A. Farcas, D. Lesnic: *The boundary-element method for the determination of a heat source dependent on one variable*, J. Eng. Math. 54, 375-388 (2006)
[11] T. Johansson, D. Lesnic, : *Determination of a spacewise dependent heat source*, J. Comput. Appl. Math. 209, 66-80 (2007)
[12] E. Klann, P. Maass, R. Ramlau, *Two-step regularization methods for linear inverse problems*, J. Inverse Ill-Posed Probl. 14 (6) (2006) 583–609.
[13] Z. Yi and D.A. Murio, *Source term identification in 1-D IHCP*, Comput. Math. Appl. 47, (2004),
pp. 1921 – 1933.
[14] Z. Yi and D.A. Murio, *Source term identification in 2-D IHCP*, Comput. Math. Appl. 47, (2004),
pp. 1517 – 1533.

[15] F. Yang, P. Zhang, X.-X. Li, X.-Y. Wa, *Tikhonov regularization method for identifying the space-dependent source for time-fractional diffusion equation on a columnar symmetric domain*, Yang et al. Advances in Difference Equations (2020) 2020:128, https://doi.org/10.1186/s13662-020-2542-1

[16] F. Yang, Y. P. Ren, X. X. Li, *Landweber iteration regularization method for identifying unknown source on a columnar symmetric domain*, Inverse Problems in Science and Engineering, Volume 26, 2018 - Issue 8, 10.1080/17415977.2017.1384825.

[17] E. Karapinar, D. Kumar, R. Sakthivel, N. H. Luc and N.H. Can, *Identifying the space source term problem for time-space-fractional diffusion equation*, Karapinar et al. Advances in Difference Equations (2020) 2020:557, https://doi.org/10.1186/s13662-020-02998-y

[18] F. Yang, Q. Pu, X. X. Li, *The fractional Tikhonov regularization methods for identifying the initial value problem for a time-fractional diffusion equation* , Journal of Computational and Applied Mathematics, Volume 380, 15 December 2020, 112998, https://doi.org/10.1016/j.cam.2020.112998.

[19] W. Cheng, Y.-J. Ma, C. L. Fu, *Identifying an unknown source term in radial heat conduction*, Inverse Problems in Science and Engineering, Vol.20, No.3, April 2012, 335-349.

[20] W. Cheng, C. L. Fu, *Identifying an unknown source term in a spherically symmetric parabolic equation*, Applied Mathematics Letters 26 (2013) 387–391.

[21] W. Cheng, Y.-J. Ma, C. L. Fu, *A regularization method for solving the radially symmetric backward heat conduction problem*, Applied Mathematics Letters 30 (2014) 38–43.

[22] I. Podlubny, Fractional Differential Equations, Mathematics in Science and Engineering, vol 198, Academic Press Inc, San Diego, CA, 1990.

[23] D. H. Q. N., L. D. Long, D. ORegand, T. B. Ngoc and N. H. Tuan, *Identification of the right-hand side in a bi-parabolic equation with final data*, APPLICABLE ANALYSIS, https://doi.org/10.1080/00036811.2020.1775817

[24] J.G. Wang, T. Wei, *Quasi-reversibility method to identify a space-dependent source for the time-fractional diffusion equation*, Applied Mathematical Modelling 39 (2015) 6139–6149.

[25] F. Yang, Q. Pu, X.-X. Li, *The fractional Tikhonov regularization methods for identifying the initial value problem for a time-fractional diffusion equation*, Journal of Computational and Applied Mathematics, https://doi.org/10.1016/j.cam.2020.112998 .

[26] A. Kirsch *An introduction to the mathematical theory of inverse problem.* Berlin: Springer-Verlag; 1996.

[27] H. T. Nguyen, D. L. Le, V. T. Nguyen, *Regularized solution of an inverse source problem for a time fractional diffusion equation*, Applied Mathematics Modelling, Volume 40, Issues 19–20, October 2016, Pages 8244-8264.

Division of Applied Mathematics, Thu Dau Mot University, Binh Duong Province, Vietnam

E-mail address: ledinhlong@tdmu.edu.vn