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ON A NONLOCAL FRACTIONAL SOBOLEV EQUATION WITH RIEMANN-LIOUVILLE DERIVATIVE

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Abstract. Our paper considers a nonlocal terminal in time problem for fractional diffusion equation. The derivative is taken as Riemann-Liouville. By applying some properties of the Mittag-Leffler function, we set some of the results about the existence, uniqueness and regularity of the mild solutions of the proposed problem in some suitable space. We obtain the lower bound and upper bound for the mild solution respecr to the given data. Finally, we obtain the asymptotic behaviour of the solution when parameter tends to zero.

1. Introduction

Fractional calculus has been around for a long time and has made an important contribution to modeling real-life phenomena. Although the amount of research on this topic is numerous and enormous, there is still a large number of unresolved non-local phenomena and many other interesting problems that have not been resolved. The selection of a suitable fraction operator depends on the physical system being studied and considered. In many types of fractional derivatives, many mathematicians are interested to study fractional diffusion equations with the Caputo derivative or the Riemann-Liouville derivative. There are many research regarding to application of fractional calculus such as D. Baleanu [25, 26, 27] and [6, 22, 21, 18]. Regarding these application of fractional calculus, we have the following references: [46, 47, 48, 49, 50] and and their references. In this paper, we consider the fractional Sobolev equation

$$\begin{cases} D^{\alpha}_{0^+}u + \mathcal{A}u + \nu D^{\alpha}_{0^+}\mathcal{A}u = 0, \ (x,t) \in \Omega \times (0,T), \\ u = 0, \ (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{1.1}$$

with the following integral condition

$$\beta u(x,T) + \gamma \int_0^T \psi(t)u(x,t)dt = f(x), \ x \in \partial\Omega, \tag{1.2}$$

where Ω is the bounded domain with the sufficiently smooth boundary $\partial\Omega$. The symbol $D^{\alpha}_{0^+}$ is the Riemann-Liouville fractional derivative of order α, $0 < \alpha \leq 1$,

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which is defined by

$$\begin{cases} D^{\alpha}_{0^+}v(t) =: \dfrac{d}{dt}\big(\mathbb{I}^{1-\alpha}_t v(t)\big), \quad \mathbb{I}^{\alpha}_t v(t) := \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t (t-r)^{-\alpha}v(r)dr, \\ D^{\alpha}_{0^+}v(t) =: \dfrac{d}{dt}v(t), \ \alpha = 1. \end{cases} \tag{1.3}$$

When $\alpha = 1$, Problem (1.1) becomes the following problem

$$\begin{cases} D_t u + \mathcal{A}u + \nu D_t\mathcal{A}u = 0, \ (x,t) \in \Omega \times (0,T), \\ u = 0, \ (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{1.4}$$

which is called the classical Sobolev equation, or pseudo-parabolic equation with classical derivative. This category of equations has been studied in some nice papers[40, 41] and references therein. They also have many important applications in physics, for example, the permeability of a homogeneous liquid, aggregation of populations [44].

According to common sense, we can divide the fractional diffusion problem into three common forms

- *Initial value problem:* A similar form of the above model (1.1) with the initial condition has been studied in [43] and references therein .
- *Terminal value problem:* In the condition (1.2), if $\gamma = 0, \beta = 1$ then it becomes terminal condition. This problem is often called backward problem for diffusion equation, see [21, 14]. The first result on this last problem probably comes from the article [45].
- *Nonlocal value problem:* To the best of our knowledge, there are not any works about the topic of nonlocal condition for fractional diffusion equations (1.1).

Some of the main contributions in the paper are detailed as follows

- The first result is related to the existence of solutions to the problem (1.1)–(1.2) in the space $L^p(0,T;H^s(\Omega))$ under the input data $f \in H^s(\Omega)$ and the case $\beta = 0, \gamma = 1$. We also study the lower bound of $\|u(.,T)\|_{H^s(\Omega)}$. This is a novel result of this paper.
- The second result is about the existence of the mild solution in the problem (1.1)–(1.2) on the space $L^p(0,T;H^s(\Omega))$ under the input data $f \in H^s(\Omega)$. We also study the asymptotic form when one parameter γ approaches 0.

This article is organized as follows. Section 2 gives some preliminary and mild solutions. In Section 3, we present our main results including two main theorems. Finally, the proof of some theorems is completed in section 4.

2. Preliminaries

Definition 2.1. *(see [44]) The Mittag-Leffler function $E_{\alpha,\beta}(\cdot)$ is*

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \ \alpha, \beta \in \mathbb{R}, \ \textit{and} \ \Re(\alpha), \Re(\beta) > 0, \ z \in \mathbb{C}. \tag{2.1}$$

Lemma 2.1. *(see [2]) Let $0 < \beta < 1$. Then there exist positive constants M_1, M_2 such that for any $z > 0$*

$$\frac{M_1}{1+z} \le E_{\beta,1}(-z) \le \frac{M_2}{1+z}, \tag{2.2}$$

Lemma 2.2. *(see [2]) Let $0 < \alpha < 1$ and $\lambda > 0$. Then*

i) $\partial_t(E_\alpha(-\lambda t^\alpha)) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)$, *for $t > 0$;*
ii) $\partial_t(t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)) = t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha)$, *for $t > 0$.*

Lemma 2.3. *([37])Let $0 < \alpha < 1$. Then the function $z \mapsto E_{\alpha,\alpha}(z)$ has no negative root. Moreover, there exists a constant C^+ such that*

$$0 \le E_{\alpha,\alpha}(-z) \le \frac{C^+}{1+z}, \ z > 0. \tag{2.3}$$

Let us give some property of the eigenvalues of the operator $\mathcal{A}$, see [44]. The following identity hold

$$\mathcal{A}\varphi_j(x) = -\lambda_j \varphi_j(x), \ x \in \Omega; \ \varphi_j = 0, \ x \in \partial\Omega, \ n \in \mathbb{N}, \tag{2.4}$$

where $\{\lambda_j\}_{j=1}^\infty$ are the eigenvalues of the operator $\mathcal{A}$ and $0 < \lambda_1 \le \lambda_2 \le \cdots \lambda_j \le ...$, and $\lim_{j\to\infty} \lambda_j = \infty$. For positive number $r \ge 0$, we also define the Hilber scale space

$$H^r(\Omega) = \left\{ w \in L^2(\Omega) : \sum_{j=1}^\infty \lambda_j^{2r} \langle w, \varphi_j \rangle^2 < +\infty \right\}, \tag{2.5}$$

with the following norm $\|u\|_{H^r(\Omega)} = \left(\sum_{j=1}^\infty \lambda_j^{2r} |\langle u, \varphi_j \rangle|^2 \right)^{\frac{1}{2}}$.

3. Main results

Theorem 3.1. *Let $\beta = 0, \gamma = 1$. Let $\psi \in L^1(0,T)$ and $f \in H^s(\Omega)$. Then problem (1.1)-(1.2) has a unique solution $u \in L^p(0,T; H^s(\Omega))$. And we have that*

$$\|u\|_{L^p(0,T;H^s(\Omega))} \le M(\alpha, T, \nu, \lambda_1) \frac{T^{\frac{1}{p}-1+\alpha}}{1+(\alpha-1)p} \|f\|_{H^s(\Omega)}, \tag{3.1}$$

$$\|u(.,T)\|_{H^s(\Omega)} \ge P(\alpha, T, \nu, \lambda_1) \|f\|_{H^s(\Omega)}. \tag{3.2}$$

Remark. *The study of lower bound of the norm of $u(x,T)$ is a difficult problem. The above result is one of our novelties.*

Theorem 3.2. *Let $\beta > 0, \gamma > 0$. Let $\psi \in L^\infty(0,T)$ and $f \in H^s(\Omega)$. Then problem (1.1)-(1.2) has a unique solution $u^{\beta,\gamma} \in L^p(0,T; H^s(\Omega))$ for any $1 < p < \frac{1}{\alpha}$. And we have that*

$$\|u^{\beta,\gamma}\|_{L^p(0,T;H^s(\Omega))} \le \beta^{-1} |\overline{C}(T)| \frac{T^{\frac{1}{p}-1+\alpha}}{1+(\alpha-1)p} \|f\|_{H^s(\Omega)}, \tag{3.3}$$

where $\overline{C}(T) = |C^+||C^-|T^{1-\alpha}$. Furthermore, the mild solution of nonlocal problem (1.1)-(1.2) converges to the mild solution of problem (1.1) with the following Cauchy terminal condition

$$\beta u(x,T) = f(x), \ x \in \Omega. \tag{3.4}$$

4. Proofs of main results

4.1. **Proof of Theorem** (3.1)**.** Let us assume that Problem (1.1) has a unique solution u. Now we use the separation of variables to yield the solution of (1.1). Suppose that the exact u is defined by Fourier series $u(x,t) = \sum_{j=1}^{\infty} u_j(t)\varphi_j(x)$, where $u_j(t) = \big\langle u(t,\cdot), \varphi_j(\cdot)\big\rangle$. It becomes to the fractional ordinary differential equation

$$D_{0+}^{\alpha} u_j(t) + \lambda_j \left(1 + \nu\lambda_j\right)^{-1} u_j(t) = 0.$$

Let $h = t^{1-\alpha}u|_{t=0}$. Then we get the following identity

$$u_j(t) = \Gamma(\alpha) t^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_j \left(1 + \nu\lambda_j\right)^{-1} t^{\alpha}\right) \big\langle h, \varphi_j \big\rangle \varphi_j. \tag{4.1}$$

The integral condition $\int_0^T \psi(t)u(x,t)dt = f(x)$ gives that

$$\int_0^T \psi(t) \left(\sum_{j=1}^{\infty} u_j(t)\varphi_j(x)\right) dt = \sum_{j=1}^{\infty} \big\langle f, \varphi_j \big\rangle \varphi_j(x).$$

Hence

$$\Gamma(\alpha)\big\langle h, \varphi_j \big\rangle \int_0^T \psi(t) t^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_j \left(1 + \nu\lambda_j\right)^{-1} t^{\alpha}\right) dt = \big\langle f, \varphi_j \big\rangle. \tag{4.2}$$

Due to the uniqueness property of Fourier expansion, we find that

$$\big\langle h, \varphi_j \big\rangle = \frac{\big\langle f, \varphi_j \big\rangle}{\Gamma(\alpha) \int_0^T \psi(t) t^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_j \left(1 + \nu\lambda_j\right)^{-1} t^{\alpha}\right) dt}. \tag{4.3}$$

Combining (4.1) and (4.3), we arrive at

$$u_j(t) = \frac{t^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_j \left(1 + \nu\lambda_j\right)^{-1} t^{\alpha}\right)}{\int_0^T \psi(t) t^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_j \left(1 + \nu\lambda_j\right)^{-1} t^{\alpha}\right) dt} \big\langle f, \varphi_j \big\rangle, \tag{4.4}$$

which allows us to obtain the explicit fomula

$$u(x,t) = t^{\alpha-1} \sum_{j=1}^{\infty} \frac{E_{\alpha,\alpha}\left(-\lambda_j \left(1 + \nu\lambda_j\right)^{-1} t^{\alpha}\right)}{\int_0^T \psi(t) t^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_j \left(1 + \nu\lambda_j\right)^{-1} t^{\alpha}\right) dt} \big\langle f, \varphi_j \big\rangle \varphi_j(x). \tag{4.5}$$

Let the function $\mathcal{H}_\alpha(y) = y^{\alpha-1}E_{\alpha,\alpha}(-y^\alpha)$. The derivative of $\mathcal{H}_\alpha$ is

$$\partial_y \mathcal{H}_\alpha(y) = -\Big[(-1)^2 \partial_y^{(2)} E_{\alpha,1}(-y^\alpha)\Big] \le 0,$$

where we notice from [38] that the function $E_{\alpha,1}(-y^\alpha)$ satisfy $(-1)^n \partial_y^{(n)} E_{\alpha,1}(-y^\alpha) \ge 0$ for all $y > 0$ and $n \ge 1$. Hence, we can deduce that $\mathcal{H}_\alpha(y) = y^{\alpha-1}E_{\alpha,\alpha}(-y^\alpha)$ is decreasing function on $(0,+\infty)$. We confirm that

$$\begin{aligned}
&E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}t^{\alpha}\right)\\
&=\left(\lambda_j\left(1+\nu\lambda_j\right)^{-1}\right)^{\frac{1}{\alpha}-1}t^{1-\alpha}\left[\left(\left(\lambda_j\left(1+\nu\lambda_j\right)^{-1}\right)^{\frac{1}{\alpha}}t\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\left(\left(\lambda_j\left(1+\nu\lambda_j\right)^{-1}\right)^{\frac{1}{\alpha}}t\right)^{\alpha}\right)\right]\\
&=\left(\lambda_j\left(1+\nu\lambda_j\right)^{-1}\right)^{\frac{1}{\alpha}-1}t^{1-\alpha}\mathcal{H}_{\alpha}\left(\left(\lambda_j\left(1+\nu\alpha_j\right)^{-1}\right)^{\frac{1}{\alpha}}t\right). \qquad (4.6)
\end{aligned}$$

Notice that $\left(\lambda_j\left(1+\nu\lambda_j\right)^{-1}\right)^{\frac{1}{\alpha}}t\le\nu^{-\frac{1}{\alpha}}t$. Hence, since the fact that the function $\mathcal{H}_{\alpha}(y)$ is decreasing, we infer that

$$\mathcal{H}_{\alpha}\left(\left(\lambda_j\left(1+\nu\lambda_j\right)^{-1}\right)^{\frac{1}{\alpha}}t\right)\ge\mathcal{H}_{\alpha}\left(\nu^{-\frac{1}{\alpha}}t\right)=\nu^{\frac{1-\alpha}{\alpha}}t^{\alpha-1}E_{\alpha,\alpha}\left(-\nu^{-1}t^{\alpha}\right). \qquad (4.7)$$

Therefore, we obtain

$$E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}t^{\alpha}\right)\ge\left(\lambda_j\left(1+\nu\lambda_j\right)^{-1}\right)^{\frac{1}{\alpha}-1}\nu^{\frac{1-\alpha}{\alpha}}E_{\alpha,\alpha}\left(-\nu^{-1}t^{\alpha}\right) \qquad (4.8)$$

Since the fact that

$$\left(\lambda_j\left(1+\nu\lambda_j\right)^{-1}\right)^{\frac{1}{\alpha}}t\le\nu^{-\frac{1}{\alpha}}t\le\nu^{-\frac{1}{\alpha}}T.$$

we also obtain that

$$\mathcal{H}_{\alpha}\left(\left(\lambda_j\left(1+\nu\lambda_j\right)^{-1}\right)^{\frac{1}{\alpha}}T\right)\ge\mathcal{H}_{\alpha}\left(\nu^{-\frac{1}{\alpha}}T\right)=\nu^{\frac{1-\alpha}{\alpha}}T^{\alpha-1}E_{\alpha,\alpha}\left(-\nu^{-1}T^{\alpha}\right). \qquad (4.9)$$

which allow us to get that

$$E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}t^{\alpha}\right)\ge\left(\lambda_j\left(1+\nu\lambda_j\right)^{-1}\right)^{\frac{1}{\alpha}-1}t^{1-\alpha}\nu^{\frac{1-\alpha}{\alpha}}T^{\alpha-1}E_{\alpha,\alpha}\left(-\nu^{-1}T^{\alpha}\right) \qquad (4.10)$$

Next, we need to estimate for integral quantity $\int_0^T\psi(t)t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}t^{\alpha}\right)dt$. Using (4.10), we obtain the following estimate

$$\begin{aligned}
&\int_0^T\psi(t)t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}t^{\alpha}\right)dt\\
&\ge\left(\lambda_j\left(1+\nu\lambda_j\right)^{-1}\right)^{\frac{1}{\alpha}-1}\nu^{\frac{1-\alpha}{\alpha}}T^{\alpha-1}E_{\alpha,\alpha}\left(-\nu^{-1}T^{\alpha}\right)\int_0^T\psi(t)dt\\
&=\left(\lambda_j\left(1+\nu\lambda_j\right)^{-1}\right)^{\frac{1}{\alpha}-1}\nu^{\frac{1-\alpha}{\alpha}}T^{\alpha-1}E_{\alpha,\alpha}\left(-\nu^{-1}T^{\alpha}\right)\|\psi\|_{L^1(0,T)}. \qquad (4.11)
\end{aligned}$$

From Lemma (2.3) and (4.11) and using Parseval's equality, we find that

$$
\begin{aligned}
&\|u(.,t)\|^2_{H^s(\Omega)}\\
&= t^{2\alpha-2}\sum_{j=1}^{\infty}\lambda_j^{2s}\left(\frac{E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}t^\alpha\right)}{\int_0^T\psi(t)t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}t^\alpha\right)dt}\right)^2\left\langle f,\varphi_j\right\rangle^2\\
&\le |C^+|^2t^{2\alpha-2}\left(\nu^{\frac{1-\alpha}{\alpha}}T^{\alpha-1}E_{\alpha,\alpha}\left(-\nu^{-1}T^\alpha\right)\|\psi\|_{L^1(0,T)}\right)^2\sum_{j=1}^{\infty}\frac{\lambda_j^{2s}(1+\lambda_j)^{\frac{2}{\alpha}}}{\lambda_j^{\frac{2}{\alpha}}}\left\langle f,\varphi_j\right\rangle^2
\end{aligned}
\tag{4.12}
$$

Using the fact that $\frac{\lambda_j^{2s}(1+\lambda_j)^{\frac{2}{\alpha}}}{\lambda_j^{\frac{2}{\alpha}}}\le\lambda_j^{2s}\left(\frac{1}{\lambda_1}+1\right)^{\frac{2}{\alpha}}$, we deduce that

$$
\begin{aligned}
\|u(.,t)\|^2_{H^s(\Omega)} &\le |M(\alpha,T,\nu,\lambda_1)|^2t^{2\alpha-2}\sum_{j=1}^{\infty}\lambda_j^{2s}\left\langle f,\varphi_j\right\rangle^2\\
&= |M(\alpha,T,\nu,\lambda_1)|^2t^{2\alpha-2}\|f\|^2_{H^s(\Omega)}
\end{aligned}
\tag{4.13}
$$

where we denote

$$M(\alpha,T,\nu,\lambda_1)=C^+\nu^{\frac{1-\alpha}{\alpha}}T^{\alpha-1}E_{\alpha,\alpha}\left(-\nu^{-1}T^\alpha\right)\|\psi\|_{L^1(0,T)}\left(\frac{1}{\lambda_1}+1\right)^{\frac{1}{\alpha}}.$$

Hence, we arrive at

$$
\begin{aligned}
\|u\|_{L^p(0,T;H^s(\Omega))} &= \left(\int_0^T\|u(.,t)\|^p_{H^s(\Omega)}dt\right)^{1/p}\le|M(\alpha,T,\nu,\lambda_1)|\|f\|_{H^s(\Omega)}\left(\int_0^T t^{(\alpha-1)p}dt\right)^{1/p}\\
&= |M(\alpha,T,\nu,\lambda_1)|\frac{T^{\frac{1}{p}-1+\alpha}}{1+(\alpha-1)p}\|f\|_{H^s(\Omega)}
\end{aligned}
\tag{4.14}
$$

where we note that $1<p<\frac{1}{1-\alpha}$. Let $t=T$ into (4.15), we obtain that

$$u(x,T)=T^{\alpha-1}\sum_{j=1}^{\infty}\frac{E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}T^\alpha\right)}{\int_0^T\psi(t)t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}t^\alpha\right)dt}\left\langle f,\varphi_j\right\rangle\varphi_j(x). \tag{4.15}$$

We recall from (4.8) that

$$E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}T^\alpha\right)\ge\left(\lambda_j\left(1+\nu\lambda_j\right)^{-1}\right)^{\frac{1}{\alpha}-1}\nu^{\frac{1-\alpha}{\alpha}}E_{\alpha,\alpha}\left(-\nu^{-1}T^\alpha\right). \tag{4.16}$$

It is not difficult to see that

$$
\begin{aligned}
&\int_0^T\psi(t)t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}t^\alpha\right)dt\\
&\qquad\le\sup_{0\le t\le T}|\psi(t)|\int_0^T t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}t^\alpha\right)dt\\
&\qquad=\|\psi\|_{L^\infty(0,T)}\frac{1+\lambda_j}{\lambda_j}\int_0^T\frac{\partial}{\partial t}\Big(-E_{\alpha,1}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}t^\alpha\right)\Big)dt\\
&\qquad=\|\psi\|_{L^\infty(0,T)}\frac{1+\lambda_j}{\lambda_j}\left(1-E_{\alpha,1}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}T^\alpha\right)\right)\le\|\psi\|_{L^\infty(0,T)}\frac{1+\lambda_j}{\lambda_j}.
\end{aligned}
\tag{4.17}
$$

Combining (4.15), (4.16), (4.17) yields that

$$\|u(.,T)\|^2_{H^s(\Omega)} = T^{2\alpha-2}\sum_{j=1}^{\infty}\lambda_j^{2s}\left(\frac{E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}T^\alpha\right)}{\int_0^T \psi(t)t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}t^\alpha\right)dt}\right)^2\left\langle f,\varphi_j\right\rangle^2$$
$$\geq \|\psi\|^{-2}_{L^\infty(0,T)}\nu^{\frac{2-2\alpha}{\alpha}}\left|E_{\alpha,\alpha}\left(-\nu^{-1}T^\alpha\right)\right|^2\sum_{j=1}^{\infty}\lambda_j^{2s}\left(\lambda_j\left(1+\nu\lambda_j\right)^{-1}\right)^{\frac{2}{\alpha}}\left\langle f,\varphi_j\right\rangle^2. \tag{4.18}$$

It is obvious to see that $\left(\lambda_j\left(1+\nu\lambda_j\right)^{-1}\right)^{\frac{2}{\alpha}} \geq \left(\lambda_1\left(1+\nu\lambda_1\right)^{-1}\right)^{\frac{2}{\alpha}}$. Therefore, we follows from... that

$$\|u(.,T)\|^2_{H^s(\Omega)} \geq P(\alpha,T,\nu,\lambda_1)^2\sum_{j=1}^{\infty}\lambda_j^{2s} = \left\langle f,\varphi_j\right\rangle^2 = P(\alpha,T,\nu,\lambda_1)^2\|f\|^2_{H^s(\Omega)}. \tag{4.19}$$

Here, we denote

$$P(\alpha,T,\nu,\lambda_1) = \nu^{\frac{1-\alpha}{\alpha}}E_{\alpha,\alpha}\left(-\nu^{-1}T^\alpha\right)\left(\lambda_1\left(1+\nu\lambda_1\right)^{-1}\right)^{\frac{1}{\alpha}}\|\psi\|^{-1}_{L^\infty(0,T)}. \tag{4.20}$$

4.2. **Proof of Theorem** (3.2)**.** The integral condition $\beta u^{\beta,\gamma}(x,T)+\gamma\int_0^T\psi(t)u^{\beta,\gamma}(x,t)dt = f(x)$ gives that

$$\beta u_j^{\beta,\gamma}(T) + \gamma\int_0^T\psi(t)\left(\sum_{j=1}^{\infty}u_j^{\beta,\gamma}(t)\varphi_j(x)\right)dt = \sum_{j=1}^{\infty}\left\langle f,\varphi_j\right\rangle\varphi_j(x).$$

Hence

$$\begin{aligned}&\left\langle h,\varphi_j\right\rangle\Gamma(\alpha)T^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}T^\alpha\right)\\&+\Gamma(\alpha)\left\langle h,\varphi_j\right\rangle\int_0^T\psi(t)t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}t^\alpha\right)dt = \left\langle f,\varphi_j\right\rangle.\end{aligned} \tag{4.21}$$

This implies that

$$\left\langle h,\varphi_j\right\rangle = \frac{\left\langle f,\varphi_j\right\rangle}{\beta\Gamma(\alpha)T^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}T^\alpha\right)+\gamma\Gamma(\alpha)Q_j(\psi,\alpha,T)}, \tag{4.22}$$

where

$$Q_j(\psi,\alpha,T) = \int_0^T\psi(t)t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}t^\alpha\right)dt$$

So, we obtain that

$$u^{\beta,\gamma}(x,t) = \sum_{j=1}^{\infty}\frac{t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}t^\alpha\right)\left\langle f,\varphi_j\right\rangle}{\beta T^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}T^\alpha\right)+Q_j(\psi,\alpha,T)}\varphi_j(x). \tag{4.23}$$

It is obvious to see that

$$\begin{aligned}&\beta T^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}T^\alpha\right)+Q_j(\psi,\alpha,T)\\&\geq \beta T^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}T^\alpha\right) \geq \beta T^{\alpha-1}\frac{C^-}{1+\lambda_j\left(1+\nu\lambda_j\right)^{-1}}\end{aligned} \tag{4.24}$$

and

$$t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}t^{\alpha}\right) \le t^{\alpha-1}\frac{C^+}{1+\lambda_j\left(1+\nu\lambda_j\right)^{-1}}. \tag{4.25}$$

Combining (4.23), (4.24), (4.25), we infer that

$$\begin{aligned}
&\|u^{\beta,\gamma}(.,t)\|^2_{H^s(\Omega)}\\
&=\sum_{j=1}^{\infty}\lambda_j^{2s}\left(\frac{t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}t^{\alpha}\right)}{\beta T^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}T^{\alpha}\right)+Q_j(\psi,\alpha,T)}\right)^2\left\langle f,\varphi_j\right\rangle^2\\
&\le \beta^{-2}|C^+|^2|C^-|^2T^{2-2\alpha}t^{2\alpha-2}\sum_{j=1}^{\infty}\lambda_j^{2s}\left\langle f,\varphi_j\right\rangle^2=\beta^{-2}|\overline{C}(T)|^2t^{2\alpha-2}\|f\|^2_{H^s(\Omega)}
\end{aligned} \tag{4.26}$$

where we recall that $\overline{C}(T)=|C^+||C^-|T^{1-\alpha}$. Therefore, we can deduce that

$$\|u^{\beta,\gamma}(.,t)\|_{H^s(\Omega)}\le\beta^{-1}|\overline{C}(T)|t^{\alpha-1}\|f\|_{H^s(\Omega)}. \tag{4.27}$$

Hence, we arrive at

$$\begin{aligned}
\|u^{\beta,\gamma}\|_{L^p(0,T;H^s(\Omega))}&=\left(\int_0^T\|u^{\beta,\gamma}(.,t)\|^p_{H^s(\Omega)}dt\right)^{1/p}\\
&\le\beta^{-1}|\overline{C}(T)|\|f\|_{H^s(\Omega)}\left(\int_0^T t^{(\alpha-1)p}dt\right)^{1/p}\\
&=\beta^{-1}|\overline{C}(T)|\frac{T^{\frac{1}{p}-1+\alpha}}{1+(\alpha-1)p}\|f\|_{H^s(\Omega)}
\end{aligned} \tag{4.28}$$

where we note that $1<p<\frac{1}{1-\alpha}$.

Let us review that

$$u^{\beta,\gamma}(x,t)=\sum_{j=1}^{\infty}\frac{t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}t^{\alpha}\right)\left\langle f,\varphi_j\right\rangle}{\beta T^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}T^{\alpha}\right)+\gamma Q_j(\psi,\alpha,T)}\varphi_j(x). \tag{4.29}$$

and

$$u^{\beta,0}(x,t)=\sum_{j=1}^{\infty}\frac{t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}t^{\alpha}\right)\left\langle f,\varphi_j\right\rangle}{\beta T^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}T^{\alpha}\right)}\varphi_j(x). \tag{4.30}$$

From two recent estimates, we find that

$$u^{\beta,\gamma}(x,t)-u^{\beta,0}(x,t)=\gamma\sum_{j=1}^{\infty}S_j(t,\alpha)\langle f,\varphi_j\rangle\varphi_j(x), \tag{4.31}$$

where

$$\begin{aligned}
&S_j(t,\alpha)\\
&=\frac{t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}t^{\alpha}\right)Q_j(\psi,\alpha,T)}{\beta T^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}T^{\alpha}\right)\left(\beta T^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}T^{\alpha}\right)+\gamma Q_j(\psi,\alpha,T)\right)}
\end{aligned}$$

First, we see that

$$\beta T^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}T^{\alpha}\right)+\gamma Q_j(\psi,\alpha,T)\Bigg)\geq\beta T^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}T^{\alpha}\right). \tag{4.32}$$

So, using the lower bound of the function $E_{\alpha,\alpha}(-z),\ z>0$ we know that

$$\begin{aligned}&\beta T^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}T^{\alpha}\right)\left(\beta T^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}T^{\alpha}\right)+\gamma Q_j(\psi,\alpha,T)\right)\\&\quad\geq\beta^2T^{2\alpha-2}\Big|E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}T^{\alpha}\right)\Big|^2\\&\quad\geq\beta^2T^{2\alpha-2}\Big|\frac{C^+}{1+T^{\alpha}\lambda_j\left(1+\nu\lambda_j\right)^{-1}}\Big|^2,\end{aligned} \tag{4.33}$$

and from the result (4.17), we find that

$$\begin{aligned}&E_{\alpha,\alpha}\left(-\lambda_j\left(1+\nu\lambda_j\right)^{-1}t^{\alpha}\right)Q_j(\psi,\alpha,T)\\&\quad\leq\frac{C^+}{1+\lambda_jt^{\alpha}\left(1+\nu\lambda_j\right)^{-1}}\|\psi\|_{L^\infty(0,T)}\frac{1+\lambda_j}{\lambda_j}\leq C^+\|\psi\|_{L^\infty(0,T)}\frac{1+\lambda_j}{\lambda_j}\end{aligned} \tag{4.34}$$

This leads to

$$S_j(t,\alpha)\leq\frac{t^{\alpha-1}C^+\|\psi\|_{L^\infty(0,T)}\frac{1+\lambda_j}{\lambda_j}}{\beta^2T^{2\alpha-2}\Big|\frac{C^+}{1+T^{\alpha}\lambda_j(1+\nu\lambda_j)^{-1}}\Big|^2}\leq\frac{t^{\alpha-1}C^+\|\psi\|_{L^\infty(0,T)}\frac{1+\lambda_1}{\lambda_1}}{\beta^2T^{2\alpha-2}\Big|\frac{C^+T^{\alpha}}{\nu}\Big|^2}=G(\nu,T,\alpha)t^{\alpha-1} \tag{4.35}$$

where $G(\nu,T,\alpha)=\frac{C^+\|\psi\|_{L^\infty(0,T)}\frac{1+\lambda_1}{\lambda_1}}{\beta^2T^{2\alpha-2}\Big|\frac{C^+T^{\alpha}}{\nu}\Big|^2}$. Combining (4.31) and (4.35), we find that

$$\begin{aligned}\|u^{\beta,\gamma}(.,t)-u^{\beta,0}(.,t)\|^2_{H^s(\Omega)}&=\gamma^2\sum_{j=1}^{\infty}\lambda_j^{2s}|S_j(t,\alpha)|^2\left\langle f,\varphi_j\right\rangle^2\\&\leq\gamma^2t^{2\alpha-2}|G(\nu,T,\alpha)|^2\sum_{j=1}^{\infty}\lambda_j^{2s}\left\langle f,\varphi_j\right\rangle^2\end{aligned} \tag{4.36}$$

which allows us to get that

$$\begin{aligned}\|u^{\beta,\gamma}(.,t)-u^{\beta,0}(.,t)\|_{L^p(0,T;H^s(\Omega))}&=\left(\int_0^T\|u^{\beta,\gamma}(.,t)-u^{\beta,0}(.,t)\|^p_{H^s(\Omega)}dt\right)^{1/p}\\&\leq\gamma G(\nu,T,\alpha)\left(\int_0^Tt^{(\alpha-1)p}dt\right)^{1/p}\|f\|_{H^s(\Omega)}\\&=\gamma G(\nu,T,\alpha)\frac{T^{\frac{1}{p}-1+\alpha}}{1+(\alpha-1)p}\|f\|_{H^s(\Omega)}.\end{aligned} \tag{4.37}$$

4.3. **Conclusion.** In this paper, We set some of the results about the existence, uniqueness, and regularity of the mild solutions of the proposed problem in some suitable space. The lower bound and upper bound for the mild solution with respect to the given data. Finally, we obtain the asymptotic behavior of the solution when the parameter tends to zero.

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