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CHARACTERIZATION OF SMARANDACHE TRAJECTORY CURVES OF CONSTANT MASS POINT PARTICLES AS THEY MOVE ALONG THE TRAJECTORY CURVE VIA PAF

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Abstract. In this paper, Smarandache trajectory curves of constant mass point particles are described and evaluated as they move along the trajectory curve in Euclidean 3-space E^3 using its position adapted frame (PAF). We also look at the Frenet apparatus of these unique trajectories. We anticipate a new way of analysing particle kinematics that could be useful in some application areas of differential geometry and particle physics. We then give a computational examples to illustrate these curves.

1. Introduction

The local theory of space curves is essential in differential geometry. Curve-adapted moving frames are helpful instruments for studying this idea. Many authors have created new moving frames that share a basis vector with the Serret-Frenet frame (for examples, see [3, 20, 22]).

In Euclidean and Minkowski spaces, the Smarandache curve is a regular curve whose position vector is made up of Frenet frame vectors on another regular curve [1, 6, 11]. Smarandache curves in Minkowski and Euclidean spaces have lately been explored by several authors [2, 4, 7, 14, 17, 18, 19, 21, 23, 24].

According to the moving frame we're working on, a moving point particle with constant mass has a position vector in Euclidean 3-space. This particle can represent any location on the trajectory in this method. As a result, the kinematics of a moving particle and the differential geometry of the trajectory, which is the oriented arc sketched out by this particle, have a very close connection and also in the fluid and magnetic field (see [8, 9, 10]. In robotics, kinematics measurements are used to determine motion and acquire a desired location. In this example, moving frames have shown to be highly helpful instruments for investigating kinematic notions such as location, velocity, acceleration, and jerking vectors in the kinemics of a moving particle. We were encouraged to prepare this study because of the relevance of the position vector. Obtaining an equation that incorporates all

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of the position, velocity, acceleration, and jerk, as well as the connection between them, has a number of advantages for exploring robotics topics like minimal jerk trajectory development.

We explore the peculiar trajectories Smarandache curve of constant mass point particles are described and evaluated as they move along the trajectory curve according to position adapted frame (see [13]) in E^3 and compute the Frenet apparatus of these trajectories in the current work. Finally, we present the visuals of these unique paths.

2. Preliminaries

Assume that in E^3, a constant mass point particle goes along a unit speed trajectory curve $\zeta = \zeta(\varsigma)$. If $\{T, N, B\}$ represents the ζ moving Frenet frame, then $\{T, N, B\}$ has the important attributes: [5, 12, 15, 16]:

$$\begin{aligned} \dot{T}(\varsigma) &= \kappa(\varsigma)\, N(\varsigma), \\ \dot{N}(\varsigma) &= \kappa(\varsigma)\, T(\varsigma) + \tau(\varsigma)\, B(\varsigma), \\ \dot{B}(\varsigma) &= \tau(\varsigma)\, N(\varsigma), \end{aligned} \tag{2.1}$$

where $\left(\cdot = \dfrac{d}{d\varsigma}\right)$, $\langle T, T\rangle = \langle N, N\rangle = \langle B, B\rangle = 1$, $\langle T, B\rangle = \langle N, B\rangle = \langle T, N\rangle = 0$ and $\kappa(\varsigma)$, and $\tau(\varsigma)$ are the trajectory curvature functions.

In particle kinematics, the angular momentum vector of the abovementioned moving particle about the origin plays a significant role. It is calculated using the vector product of the moving particle's position vector and linear momentum vector, and is given by

$$H^0 = m\langle \zeta(\varsigma), B(\varsigma)\rangle \left(\frac{d\varsigma}{dt}\right) N(\varsigma) - m\langle \zeta(\varsigma), N(\varsigma)\rangle \left(\frac{d\varsigma}{dt}\right) B(\varsigma),$$

where m and t denote mass and time, appropriately. Presume that this vector doesn't really equal zero at any point anywhere along trajectory $\zeta(\varsigma)$. Even during motion of the moving particle, this assumption assures that the functions $\langle \zeta(\varsigma), B(\varsigma)\rangle$ and $\langle \zeta(\varsigma), N(\varsigma)\rangle$ do not equal zero at the same moment. As a result, we can state that the tangent line of $\zeta(\varsigma)$ never crosses the origin. Then, there is a position adapted frame (abbreviated PAF) represented by $\{T(\varsigma), G(\varsigma), P(\varsigma)\}$ along $\zeta(\varsigma)$ that is supplied as (see [13] for additional details):

$$\begin{aligned} \dot{T}(\varsigma) &= k_1(\varsigma)\, G(\varsigma) + k_2(\varsigma)\, P(\varsigma), \\ \dot{G}(\varsigma) &= -k_1(\varsigma)\, T(\varsigma) + k_3(\varsigma)\, P(\varsigma), \\ \dot{P}(\varsigma) &= -k_2(\varsigma)\, T(\varsigma) - k_3(\varsigma)\, G(\varsigma), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} G(\varsigma) &= \frac{\langle \zeta(\varsigma), B(\varsigma)\rangle}{\sqrt{\langle \zeta(\varsigma), N(\varsigma)\rangle^2 + \langle \zeta(\varsigma), B(\varsigma)\rangle^2}}\, N(\varsigma) + \frac{\langle \zeta(\varsigma), N(\varsigma)\rangle}{\sqrt{\langle \zeta(\varsigma), N(\varsigma)\rangle^2 + \langle \zeta(\varsigma), B(\varsigma)\rangle^2}}\, B(\varsigma), \\ P(\varsigma) &= \frac{-\langle \zeta(\varsigma), N(\varsigma)\rangle}{\sqrt{\langle \zeta(\varsigma), N(\varsigma)\rangle^2 + \langle \zeta(\varsigma), B(\varsigma)\rangle^2}}\, N(\varsigma) + \frac{\langle \zeta(\varsigma), B(\varsigma)\rangle}{\sqrt{\langle \zeta(\varsigma), N(\varsigma)\rangle^2 + \langle \zeta(\varsigma), B(\varsigma)\rangle^2}}\, B(\varsigma), \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} k_1(\varsigma) &= \kappa(\varsigma)\cos\Theta(\varsigma), \\ k_2(\varsigma) &= \kappa(\varsigma)\sin\Theta(\varsigma), \\ k_3(\varsigma) &= \tau(\varsigma) - \dot{\Theta}(\varsigma). \end{aligned} \tag{2.4}$$

The Frenet frame and PAF have the following relationship:

$$\begin{aligned} T(\varsigma) &= T(\varsigma), \\ G(\varsigma) &= \cos\Theta(\varsigma)N(\varsigma) - \sin\Theta(\varsigma)B(\varsigma), \\ P(\varsigma) &= \sin\Theta(\varsigma)N(\varsigma) + \cos\Theta(\varsigma)B(\varsigma), \end{aligned} \tag{2.5}$$

where $\Theta(\varsigma)$ is the angle formed by the vectors $B(\varsigma)$ and $P(\varsigma)$ when they are orientated favorably from $B(\varsigma)$ to $P(\varsigma)$. The following formula is used to determine the specified angle $\Theta(\varsigma)$:

$$\Theta(\varsigma) = \begin{cases} \arctan\left(-\dfrac{\langle\zeta(\varsigma), N(\varsigma)\rangle}{\langle\zeta(\varsigma), B(\varsigma)\rangle}\right) \text{ if } \langle\zeta(\varsigma), B(\varsigma)\rangle > 0, \\ \arctan\left(-\dfrac{\langle\zeta(\varsigma), N(\varsigma)\rangle}{\langle\zeta(\varsigma), B(\varsigma)\rangle}\right) + \pi \text{ if } \langle\zeta(\varsigma), B(\varsigma)\rangle < 0, \\ -\dfrac{\pi}{2} \text{ if } \langle\zeta(\varsigma), B(\varsigma)\rangle = 0 \text{ and } \langle\zeta(\varsigma), N(\varsigma)\rangle > 0, \\ \dfrac{\pi}{2} \text{ if } \langle\zeta(\varsigma), B(\varsigma)\rangle = 0 \text{ and } \langle\zeta(\varsigma), N(\varsigma)\rangle < 0. \end{cases} \tag{2.6}$$

3. Main results

In this section, we explore any moving point particle that meets the previous assumption (concerning angular momentum) and show the unit speed parameterization of the trajectory with $\zeta(\varsigma)$. We provide a special Smarandache trajectory curve according to PAF of $\zeta(\varsigma)$ in Euclidean 3-space E^3 also, we derive the Frenet apparatus of these curves. Besides, when the angle $\Theta(\varsigma) = \frac{\pi}{2}$ or $-\frac{\pi}{2}$, we examine certain aspects on it.

Definition 3.1. Let $\zeta = \zeta(\varsigma)$ be a trajectory unit speed curve in E^3. The TG-Smarandache trajectory curve via to PAF (2.2) of $\zeta(\varsigma)$ defined by

$$\varphi = \varphi(\varsigma^*) = \frac{1}{\sqrt{2}}\Big(a\,T(\varsigma) + b\,G(\varsigma)\Big), \quad a^2 + b^2 = 2. \tag{3.1}$$

Theorem 3.1. Let $\zeta = \zeta(\varsigma)$ be a trajectory unit speed curve of moving point particle of constant mass m in space E^3 via to PAF (2.2). If $\varphi : I \subset R \to E^3$ is the TG-Smarandache trajectory curve of ζ with non-zero curvature function, then its Frenet frame $\{T_\varphi, N_\varphi, B_\varphi\}$ is given by

$$\begin{bmatrix} T_\varphi \\ N_\varphi \\ B_\varphi \end{bmatrix} = \begin{bmatrix} \frac{-bk_1}{\sqrt{2k_1^2+(ak_2+bk_3)^2}} & \frac{ak_1}{\sqrt{2k_1^2+(ak_2+bk_3)^2}} & \frac{ak_2+bk_3}{\sqrt{2k_1^2+(ak_2+bk_3)^2}} \\ \frac{\vartheta_1}{\sqrt{\vartheta_1^2+\vartheta_2^2+\vartheta_3^2}} & \frac{\vartheta_2}{\sqrt{\vartheta_1^2+\vartheta_2^2+\vartheta_3^2}} & \frac{\vartheta_3}{\sqrt{\vartheta_1^2+\vartheta_2^2+\vartheta_3^2}} \\ \frac{a(\vartheta_3k_1-\vartheta_2k_2)-b\vartheta_2k_3}{\Delta_1} & \frac{a\vartheta_1k_2+b(\vartheta_1k_3+\vartheta_3k_1)}{\Delta_1} & \frac{(a\vartheta_1+b\vartheta_2)k_1}{\Delta_1} \end{bmatrix} \begin{bmatrix} T \\ G \\ P \end{bmatrix}, \tag{3.2}$$

where

$$\begin{aligned}\vartheta_1 &= -\big[a\kappa^2 + b\dot{k}_1 + bk_2k_3\big]\big[2k_1^2 + (ak_2+bk_3)^2\big] - bk_1\big[2k_1\dot{k}_1 + (ak_2+bk_3)(a\dot{k}_2 \\ &\quad + b\dot{k}_3)\big], \\ \vartheta_2 &= \big[a\dot{k}_1 - bk_1^2 - k_3(ak_2+bk_3)\big]\big[2k_1^2 + (ak_2+bk_3)^2\big] + ak_1\big[2k_1\dot{k}_1 + (ak_2+bk_3) \\ &\quad \times (a\dot{k}_2 + b\dot{k}_3)\big], \\ \vartheta_3 &= \big[a(\dot{k}_2 + k_1k_3) + b(\dot{k}_3 - k_1k_2)\big]\big[2k_1^2 + (ak_2+bk_3)^2\big] + (ak_2+bk_3)\big[2k_1\dot{k}_1 \\ &\quad + (ak_2+bk_3)(a\dot{k}_2 + b\dot{k}_3)\big], \\ \Delta_1 &= \sqrt{2k_1^2 + (ak_2+bk_3)^2}\sqrt{\vartheta_1^2 + \vartheta_2^2 + \vartheta_3^2}.\end{aligned} \tag{3.3}$$

Proof. Using (2.2) and differentiate (3.1) with regard to ς , we get

$$\dot{\varphi}(\varsigma^*) = \frac{d\varphi}{d\varsigma^*}\frac{d\varsigma^*}{d\varsigma} = \frac{1}{\sqrt{2}}\Big(-bk_1\,T(\varsigma) + ak_1\,G(\varsigma) + (ak_2+bk_3)P(\varsigma)\Big), \tag{3.4}$$

hence

$$T_\varphi(\varsigma^*) = \frac{-bk_1\,T(\varsigma) + ak_1\,G(\varsigma) + (ak_2+bk_3)P(\varsigma)}{\sqrt{2k_1^2 + (ak_2+bk_3)^2}}, \tag{3.5}$$

such that

$$\frac{d\varsigma^*}{d\varsigma} = \frac{\sqrt{2k_1^2 + (ak_2+bk_3)^2}}{\sqrt{2}}. \tag{3.6}$$

Then, we have

$$\dot{T}_\varphi(\varsigma^*) = \frac{\sqrt{2}\Big(\vartheta_1 T(\sigma) + \vartheta_2 B_1(\varsigma) + \vartheta_3 B_2(\varsigma)\Big)}{\big[2k_1^2 + (ak_2+bk_3)^2\big]^2}.$$

where

$$\begin{aligned}\vartheta_1 &= -\big[a\kappa^2 + b\dot{k}_1 + bk_2k_3\big]\big[2k_1^2 + (ak_2+bk_3)^2\big] - bk_1\big[2k_1\dot{k}_1 + (ak_2+bk_3)(a\dot{k}_2 \\ &\quad + b\dot{k}_3)\big], \\ \vartheta_2 &= \big[a\dot{k}_1 - bk_1^2 - k_3(ak_2+bk_3)\big]\big[2k_1^2 + (ak_2+bk_3)^2\big] + ak_1\big[2k_1\dot{k}_1 + (ak_2+bk_3) \\ &\quad \times (a\dot{k}_2 + b\dot{k}_3)\big], \\ \vartheta_3 &= \big[a(\dot{k}_2 + k_1k_3) + b(\dot{k}_3 - k_1k_2)\big]\big[2k_1^2 + (ak_2+bk_3)^2\big] + (ak_2+bk_3)\big[2k_1\dot{k}_1 \\ &\quad + (ak_2+bk_3)(a\dot{k}_2 + b\dot{k}_3)\big].\end{aligned}$$

Rather, the trajectory curvature and, as a result, the principal normal vector field of φ are

$$\kappa_\varphi(\varsigma^*) = \Big\|\dot{T}_\varphi(\varsigma^*)\Big\| = \frac{\sqrt{2}\sqrt{\vartheta_1^2 + \vartheta_2^2 + \vartheta_3^2}}{\big[2k_1^2 + (ak_2+bk_3)^2\big]^2},$$

and

$$N_\varphi(\varsigma^*) = \frac{\vartheta_1\,T(\varsigma) + \vartheta_2\,G(\varsigma) + \vartheta_3\,P(\varsigma)}{\sqrt{\vartheta_1^2 + \vartheta_2^2 + \vartheta_3^2}}.$$

On the other side, we have the ability to express ourselves.

$$B_\varphi(\varsigma^*) = \frac{1}{\Delta_1}\Big\{\big[a(\vartheta_3 k_1 - \vartheta_2 k_2) - b\vartheta_2 k_3\big]T(\varsigma) + \big[a\vartheta_1 k_2 + b(\vartheta_1 k_3 + \vartheta_3 k_1)\big]G(\varsigma) - k_1(a\vartheta_1 + b\vartheta_2)P(\varsigma)\Big\},$$

where

$$\Delta_1 = \sqrt{2k_1^2 + (ak_2 + bk_3)^2}\sqrt{\vartheta_1^2 + \vartheta_2^2 + \vartheta_3^2}.$$

Now, from Eq. (3.4) we have

$$\ddot{\varphi}(\varsigma^*) = \frac{1}{\sqrt{2}}\Big\{-\big[a\kappa^2 + b\dot{k}_1 + bk_2k_3\big]T(\varsigma) + \big[a\dot{k}_1 - bk_1^2 - k_3(ak_2 + bk_3)\big]G(\varsigma) + \big[a(\dot{k}_2 + k_1k_3) + b(\dot{k}_3 - k_1k_2)\big]P(\varsigma)\Big\},$$

similarly

$$\dddot{\varphi}(\varsigma^*) = \frac{1}{\sqrt{2}}\Big(\lambda_1 T(\varsigma) + \lambda_2 G(\varsigma) + \lambda_3 P(\varsigma)\Big),$$

where

$$\begin{aligned}
\lambda_1 &= -\Big[\big[a\kappa^2 + b\dot{k}_1 + bk_2k_3\big]_\varsigma - k_1\big[a\dot{k}_1 bk_1^2 - k_3(ak_2 + bk_3)\big] \\
&\quad + k_2\big[a(\dot{k}_2 + k_1k_3) + b(\dot{k}_3 - k_1k_2)\big]\Big],\\
\lambda_2 &= \big[a\dot{k}_1 - bk_1^2 - k_3(ak_2 + bk_3)\big]_\varsigma + k_1\big[a\kappa^2 + b\dot{k}_1 + bk_2k_3\big] \\
&\quad - k_3\big[a(\dot{k}_2 + k_1k_3) + b(\dot{k}_3 - k_1k_2)\big],\\
\lambda_3 &= \big[a(\dot{k}_2 + k_1k_3) + b(\dot{k}_3 - k_1k_2)\big]_\varsigma + k_2\big[a\kappa^2 + b\dot{k}_1 + bk_2k_3\big] \\
&\quad + k_3\big[a\dot{k}_1 - bk_1^2 - k_3(ak_2 + bk_3)\big].
\end{aligned}$$

The trajectory torsion of is then calculated using equations

$$\begin{aligned}
\tau_\varphi =& \frac{\sqrt{2}}{\Delta_1^*}\Big\{bk_1\Big[\lambda_2\big[a(\dot{k}_2 + k_1k_3) + b(\dot{k}_3 - k_1k_2)\big] - \lambda_3\big[a\dot{k}_1 - bk_1^2 - k_3(ak_2 + bk_3)\big]\Big] \\
&+ ak_1\Big[\lambda_1\big[a(\dot{k}_2 + k_1k_3) + b(\dot{k}_3 - k_1k_2)\big] + \lambda_3\big[a\kappa^2 + b\dot{k}_1 + bk_2k_3\big]\Big] \\
&- (ak_2 + bk_3)\Big[\lambda_2\big[a\kappa^2 + b\dot{k}_1 + bk_2k_3\big] + \lambda_1\big[a\dot{k}_1 - bk_1^2 - k_3(ak_2 + bk_3)\big]\Big]\Big\},
\end{aligned}$$

where

$$\begin{aligned}
\Delta_1^* =& \Big[ak_1\big[a(\dot{k}_2 + k_1k_3) + b(\dot{k}_3 - k_1k_2)\big] - (ak_2 + bk_3)\big[a\dot{k}_1 - bk_1^2 - k_3(ak_2 + bk_3)\big]\Big]^2 \\
&+ \Big[bk_1\big[a(\dot{k}_2 + k_1k_3) + b(\dot{k}_3 - k_1k_2)\big] + (ak_2 + bk_3)\big[a\kappa^2 + b\dot{k}_1 + bk_2k_3\big]\Big]^2 \\
&+ \Big[bk_1\big[a\dot{k}_1 - bk_1^2 - k_3(ak_2 + bk_3)\big] - ak_1\big[a\kappa^2 + b\dot{k}_1 + bk_2k_3\big]\Big]^2.
\end{aligned}$$

□

Corollary 3.2. Let $\zeta = \zeta(\varsigma)$ be a trajectory unit speed curve of moving point particle of constant mass m in space E^3 via to PAF (2.2). If $\varphi : I \subset R \to E^3$ is the TG-Smarandache trajectory curve of ζ. If $\Theta(\varsigma)=\dfrac{\pi}{2}$, then the natural trajectory curvature functions of the TG-Smarandache trajectory curve can therefore be defined as follows in terms of κ and τ:

$$\begin{aligned}\kappa_{\varphi}(\varsigma^{*})&=\frac{\sqrt{2}\sqrt{(\tau^{2}+\kappa^{2})(a\kappa+b\tau)^{2}+4(a\dot{\kappa}+b\dot{\tau})^{2}}}{(a\kappa+b\tau)^{2}},\\ \tau_{\varphi}(\varsigma^{*})&=-\frac{\sqrt{2}\big[\tau\dot{\kappa}(5a\kappa+b\tau)+\dot{\tau}\kappa(a\kappa+5b\tau)\big]}{(\tau^{2}+\kappa^{2})(a\kappa+b\tau)^{2}}.\end{aligned}\tag{3.7}$$

Corollary 3.3. Let $\zeta=\zeta(\varsigma)$ be a trajectory unit speed curve of moving point particle of constant mass m in space E^3 via to PAF (2.2). If $\varphi: I\subset R\to E^3$ is the TG-Smarandache trajectory curve of ζ. If $\Theta(\varsigma)=-\dfrac{\pi}{2}$, then the natural trajectory curvature functions of the TG-Smarandache trajectory curve can therefore be defined as follows in terms of κ and τ:

$$\begin{aligned}\kappa_{\varphi}(\varsigma^{*})&=\frac{\sqrt{2}\sqrt{(\tau^{2}+\kappa^{2})(a\kappa-b\tau)^{2}+4(a\dot{\kappa}-b\dot{\tau})^{2}}}{(a\kappa-b\tau)^{2}},\\ \tau_{\varphi}(\varsigma^{*})&=\frac{\sqrt{2}\big[\dot{\tau}\kappa(a\kappa+3b\tau)-\tau\dot{\kappa}(3a\kappa+b\tau)\big]}{(\tau^{2}+\kappa^{2})(a\kappa-b\tau)^{2}}.\end{aligned}\tag{3.8}$$

Definition 3.2. Let $\zeta=\zeta(\varsigma)$ be a trajectory unit speed curve in E^3. The TP-Smarandache trajectory curve via to PAF (2.2) of $\zeta(\varsigma)$ defined by

$$\psi=\psi(\varsigma^{*})=\frac{1}{\sqrt{2}}\Big(a\,T(\varsigma)+b\,P(\varsigma)\Big),\quad a^{2}+b^{2}=2.\tag{3.9}$$

Theorem 3.4. Let $\zeta=\zeta(\varsigma)$ be a trajectory unit speed curve of moving point particle of constant mass m in space E^3 via to PAF (2.2). If $\psi: I\subset R\to E^3$ is the TP-Smarandache trajectory curve of ζ with non-zero curvature function, then its Frenet frame $\{T_\psi, N_\psi, B_\psi\}$ is given by

$$\begin{bmatrix}T_{\psi}\\ N_{\psi}\\ B_{\psi}\end{bmatrix}=\begin{bmatrix}\frac{-bk_{2}}{\sqrt{2k_{2}^{2}+(ak_{1}-bk_{3})^{2}}} & \frac{ak_{1}-bk_{3}}{\sqrt{2k_{2}^{2}+(ak_{1}-bk_{3})^{2}}} & \frac{ak_{2}}{\sqrt{2k_{2}^{2}+(ak_{1}-bk_{3})^{2}}}\\ \frac{\varepsilon_{1}}{\sqrt{\varepsilon_{1}^{2}+\varepsilon_{2}^{2}+\varepsilon_{3}^{2}}} & \frac{\varepsilon_{2}}{\sqrt{\varepsilon_{1}^{2}+\varepsilon_{2}^{2}+\varepsilon_{3}^{2}}} & \frac{\varepsilon_{3}}{\sqrt{\varepsilon_{1}^{2}+\varepsilon_{2}^{2}+\varepsilon_{3}^{2}}}\\ \frac{a(\varepsilon_{3}k_{1}-\varepsilon_{2}k_{2})-b\varepsilon_{3}k_{3}}{\Delta_{2}} & \frac{k_{2}(a\varepsilon_{1}+b\varepsilon_{3})}{\Delta_{2}} & \frac{-a\varepsilon_{1}k_{1}+b(\varepsilon_{1}k_{3}-\varepsilon_{2}k_{2})}{\Delta_{2}}\end{bmatrix}\begin{bmatrix}T\\ G\\ P\end{bmatrix},\tag{3.10}$$

where

$$\begin{aligned}\varepsilon_{1}&=-\big[a\kappa^{2}+b\dot{k}_{2}-bk_{1}k_{3}\big]\big[2k_{2}^{2}+(ak_{1}-bk_{3})^{2}\big]+bk_{2}\big[2k_{2}\dot{k}_{2}+(ak_{1}-bk_{3})(a\dot{k}_{1}\\ &\quad-b\dot{k}_{3})\big],\\ \varepsilon_{2}&=\big[a(\dot{k}_{1}+k_{2}k_{3})-b(\dot{k}_{3}+k_{1}k_{2})\big]\big[2k_{2}^{2}+(ak_{1}-bk_{3})^{2}\big]-(ak_{1}-bk_{3})\big[2k_{2}\dot{k}_{1}\\ &\quad+(ak_{1}-bk_{3})(a\dot{k}_{1}-b\dot{k}_{3})\big],\\ \varepsilon_{3}&=\big[a\dot{k}_{2}-bk_{2}^{2}+k_{3}(ak_{1}-bk_{3})\big]\big[2k_{2}^{2}+(ak_{1}-bk_{3})^{2}\big]+ak_{2}\big[2k_{2}\dot{k}_{1}\\ &\quad+(ak_{1}-bk_{3})(a\dot{k}_{1}-b\dot{k}_{3})\big],\\ \Delta_{2}&=\sqrt{2k_{2}^{2}+(ak_{1}-bk_{3})^{2}}\sqrt{\varepsilon_{1}^{2}+\varepsilon_{2}^{2}+\varepsilon_{3}^{2}}.\end{aligned}\tag{3.11}$$

Proof. Differentiate (3.9) to ς and using (2.2), we get

$$\dot{\psi}(\varsigma^*) = \frac{1}{\sqrt{2}}\Big(- bk_2\, T(\varsigma) + (ak_1 - bk_3)G(\varsigma) + ak_2\, P(\varsigma)\Big). \tag{3.12}$$

So

$$T_\psi(\varsigma^*) = \frac{-bk_2\, T(\varsigma) + (ak_1 - bk_3)G(\varsigma) + ak_2\, P(\varsigma)}{\sqrt{2k_2^2 + (ak_1 - bk_3)^2}}, \tag{3.13}$$

such that

$$\frac{d\varsigma^*}{d\varsigma} = \frac{\sqrt{2k_2^2 + (ak_1 - bk_3)^2}}{\sqrt{2}}. \tag{3.14}$$

Then, we have

$$\dot{T}_\psi(\varsigma^*) = \frac{\sqrt{2}\Big(\varepsilon_1 T(\sigma) + \varepsilon_2 B_1(\varsigma) + \varepsilon_3 B_2(\varsigma)\Big)}{\big[2k_2^2 + (ak_1 - bk_3)^2)^2\big]^2}.$$

where

$$\begin{aligned}
\varepsilon_1 &= -\big[a\kappa^2 + b\dot{k}_2 - bk_1k_3\big]\big[2k_2^2 + (ak_1 - bk_3)^2\big] + bk_2\big[2k_2\dot{k}_2 + (ak_1 - bk_3)(a\dot{k}_1 \\
&\quad - b\dot{k}_3)\big],\\
\varepsilon_2 &= \big[a(\dot{k}_1 + k_2k_3) - b(\dot{k}_3 + k_1k_2)\big]\big[2k_2^2 + (ak_1 - bk_3)^2\big] - (ak_1 - bk_3)\big[2k_2\dot{k}_1 \\
&\quad + (ak_1 - bk_3)(a\dot{k}_1 - b\dot{k}_3)\big],\\
\varepsilon_3 &= \big[a\dot{k}_2 - bk_2^2 + k_3(ak_1 - bk_3)\big]\big[2k_2^2 + (ak_1 - bk_3)^2\big] + ak_2\big[2k_2\dot{k}_1 \\
&\quad + (ak_1 - bk_3)(a\dot{k}_1 - b\dot{k}_3)\big].
\end{aligned}$$

Rather, the trajectory curvature and, as a result, the principal normal vector field of ψ are

$$\kappa_\psi(\varsigma^*) = \frac{\sqrt{2}\sqrt{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2}}{\big[2k_2^2 + (ak_1 - bk_3)^2\big]^2},$$

and

$$N_\psi(\varsigma^*) = \frac{\varepsilon_1\, T(\varsigma) + \varepsilon_2\, G(\varsigma) + \varepsilon_3\, P(\varsigma)}{\sqrt{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2}}.$$

So, we have

$$\begin{aligned}
B_\psi(\varsigma^*) = \frac{1}{\Delta_2}\Big\{&\big[a(\varepsilon_3k_1 - \varepsilon_2k_2) - b\varepsilon_2k_3\big]T(\varsigma) + k_2(a\varepsilon_1 + b\varepsilon_3)G(\varsigma)\\
&+ \big[-a\varepsilon_1k_2 + b(\varepsilon_1k_3 - \varepsilon_2k_2)\big]P(\varsigma)\Big\},
\end{aligned}$$

where

$$\Delta_2 = \sqrt{2k_2^2 + (ak_1 - bk_3)^2}\sqrt{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2}.$$

Now, from Eq. (3.12) we have

$$\begin{aligned}
\ddot{\psi}(\varsigma^*) = \frac{1}{\sqrt{2}}\Big\{&-\big[a\kappa^2 + b\dot{k}_2 - bk_1k_3\big]T(\varsigma) + \big[a(\dot{k}_1 + k_2k_3) - b(\dot{k}_3 + k_1k_2)\big]G(\varsigma)\\
&+ \big[a\dot{k}_2 - bk_2^2 + k_3(ak_1 - bk_3)\big]P(\varsigma)\Big\},
\end{aligned}$$

similarly

$$\dddot{\psi}(\varsigma^*) = \frac{1}{\sqrt{2}}\Big(\omega_1 T(\varsigma) + \omega_2 G(\varsigma) + \omega_3 P(\varsigma)\Big),$$

where

$$\begin{aligned}
\omega_1 = & -\Big[\big[a\kappa^2 + b\dot{k}_2 - bk_1k_3\big]_\varsigma - k_1\big[a(\dot{k}_1 + k_2k_3) - b(\dot{k}_3 + k_1k_2)\big] \\
& + k_2\big[a\dot{k}_2 - bk_2^2 + k_3(ak_1 - bk_3)\big]\Big], \\
\omega_2 = & \big[a(\dot{k}_1 + k_2k_3) - b(\dot{k}_3 + k_1k_2)\big]_\varsigma + k_1\big[a\kappa^2 + b\dot{k}_2 - bk_1k_3\big] \\
& - k_3\big[a\dot{k}_2 - bk_2^2 + k_3(ak_1 - bk_3)\big], \\
\omega_3 = & \big[a\dot{k}_2 - bk_2^2 + k_3(ak_1 - bk_3)\big]_\varsigma + k_2\big[a\kappa^2 + b\dot{k}_2 - bk_1k_3\big] \\
& + k_3\big[a(\dot{k}_1 + k_2k_3) - b(\dot{k}_3 + k_1k_2)\big].
\end{aligned}$$

The trajectory torsion of is then calculated using equations

$$\begin{aligned}
\tau_\psi = & \frac{\sqrt{2}}{\Delta_2^*}\Big\{bk_2\Big[\omega_2\big[a\dot{k}_2 - bk_2^2 + k_3(ak_1 - bk_3)\big] - \omega_3\big[a(\dot{k}_1 + k_2k_3) - b(\dot{k}_3 + k_1k_2)\big]\Big] \\
& + (ak_1 - bk_3)\Big[\omega_1\big[a\dot{k}_2 - bk_2^2 + k_3(ak_1 - bk_3)\big] + \omega_3\big[a\kappa^2 + b\dot{k}_2 - bk_1k_3\big]\Big] \\
& - ak_2\Big[\omega_1\big[a(\dot{k}_1 + k_2k_3) - b(\dot{k}_3 + k_1k_2)\big] - \omega_2\big[a\kappa^2 + b\dot{k}_2 - bk_1k_3\big]\Big]\Big\},
\end{aligned}$$

where

$$\begin{aligned}
\Delta_2^* = & \Big[(ak_1 - bk_3)\big[a\dot{k}_2 - bk_2^2 + k_3(ak_1 - bk_3)\big]\Big]^2 + \Big[ak_2\big[a\kappa^2 + b\dot{k}_2 - bk_1k_3\big] \\
& + bk_2\big[a\dot{k}_2 - bk_2^2 + k_3(ak_1 - bk_3)\big]\Big]^2 + \Big[bk_2\big[a(\dot{k}_1 + k_2k_3) - b(\dot{k}_3 + k_1k_2)\big] \\
& + (ak_1 - bk_3)\big[a\kappa^2 + b\dot{k}_2 - bk_1k_3\big]\Big]^2.
\end{aligned}$$

□

Corollary 3.5. Let $\zeta = \zeta(\varsigma)$ be a trajectory unit speed curve of moving point particle of constant mass m in space E^3 via to PAF (2.2). If $\psi : I \subset R \to E^3$ is the TP-Smarandache trajectory curve of ζ. If $\Theta(\varsigma) = \frac{\pi}{2}$, then the trajectory curvature of the TP-Smarandache trajectory curve can therefore be defined as follows in terms of κ and τ:

$$\begin{aligned}
\kappa_\psi(\varsigma^*) = & \frac{\sqrt{2}}{(2\kappa^2 + b^2\tau^2)^2}\Big\{\big[b^3\tau(\dot{\tau}\kappa - \tau\dot{\kappa}) - a\kappa^2(2\kappa^2 + b^2\tau^2)\big]^2 \\
& + \big[(a\tau\kappa - b\dot{\tau})(2\kappa^2 + b^2\tau^2)^2 + b\tau(2\kappa\dot{\kappa} + b^2\tau\dot{\tau})\big]^2 \\
& + \big[(2\kappa^2 + b^2\tau^2)[a\dot{\kappa} - b(\tau^2 + \kappa^2)] + a\kappa(2\kappa\dot{\kappa} + b^2\tau\dot{\tau})\big]^2\Big\}^{\frac{1}{2}}.
\end{aligned} \tag{3.15}$$

Corollary 3.6. Let $\zeta = \zeta(\varsigma)$ be a trajectory unit speed curve of moving point particle of constant mass m in space E^3 via to PAF (2.2). If $\psi : I \subset R \to E^3$ is the TP-Smarandache trajectory curve of ζ. If $\Theta(\varsigma) = -\frac{\pi}{2}$, then the trajectory curvature of the TP-Smarandache trajectory curve can therefore be defined as follows in terms of κ and τ:

$$\kappa_\psi(\varsigma^*) = \frac{\sqrt{2}}{(2\kappa^2+b^2\tau^2)^2}\Big\{\big[b^3\tau(\dot{\tau}\kappa-\tau\dot{\kappa})-a\kappa^2(2\kappa^2+b^2\tau^2)\big]^2 + \big[b\tau(2\kappa\dot{\kappa}+b^2\tau\dot{\tau})-(a\tau\kappa-b\dot{\tau})(2\kappa^2+b^2\tau^2)^2\big]^2 + \big[(2\kappa^2+b^2\tau^2)[a\dot{\kappa}+b(\tau^2+\kappa^2)]+a\kappa(2\kappa\dot{\kappa}+b^2\tau\dot{\tau})\big]^2\Big\}^{\frac{1}{2}}. \tag{3.16}$$

Definition 3.3. Let $\zeta=\zeta(\varsigma)$ be a trajectory unit speed curve in E^3. The *GP*-Smarandache trajectory curve via to PAF (2.2) of $\zeta(\varsigma)$ defined by

$$\mu=\mu(\varsigma^*)=\frac{1}{\sqrt{2}}\Big(a\,G(\varsigma)+b\,P(\varsigma)\Big), \quad a^2+b^2=2. \tag{3.17}$$

Theorem 3.7. Let $\zeta=\zeta(\varsigma)$ be a trajectory unit speed curve of moving point particle of constant mass m in space E^3 via to PAF (2.2). If $\mu: I\subset \mathrm{R}\to\mathrm{E}^3$ is the *GP*-Smarandache trajectory curve of ζ with non-zero curvature function, then its Frenet frame $\{T_\mu, N_\mu, B_\mu\}$ is given by

$$\begin{bmatrix} T_\mu \\ N_\mu \\ B_\mu \end{bmatrix} = \begin{bmatrix} \frac{-(ak_1+bk_2)}{\sqrt{2k_3^2+(ak_1+bk_2)^2}} & \frac{-bk_3}{\sqrt{2k_3^2+(ak_1+bk_2)^2}} & \frac{ak_3}{\sqrt{2k_3^2+(ak_1+bk_2)^2}} \\ \frac{\beta_1}{\sqrt{\beta_1^2+\beta_2^2+\beta_3^2}} & \frac{\beta_2}{\sqrt{\beta_1^2+\beta_2^2+\beta_3^2}} & \frac{\beta_3}{\sqrt{\beta_1^2+\beta_2^2+\beta_3^2}} \\ \frac{-k_3(a\beta_2+b\beta_3)}{\Delta_3} & \frac{a\beta_1k_3+\beta_3(ak_1+bk_2)}{\Delta_3} & \frac{b\beta_1k_3-\beta_2(ak_1-bk_3)}{\Delta_3} \end{bmatrix} \begin{bmatrix} T \\ G \\ P \end{bmatrix}, \tag{3.18}$$

where

$$\begin{aligned} \beta_1 &= \big[k_3(bk_1-ak_2)-a\dot{k}_1-b\dot{k}_2\big]\big[2k_3^2+(ak_1+bk_2)^2\big]+(ak_1+bk_2)\big[2k_3\dot{k}_3 + (ak_1+bk_2)(a\dot{k}_1+b\dot{k}_2)\big], \\ \beta_2 &= -\big[ak_3^2+b\dot{k}_3+k_1(ak_1+bk_2)\big]\big[2k_3^2+(ak_1+bk_2)^2\big]+bk_3\big[2k_3\dot{k}_3 + (ak_1+bk_2)(a\dot{k}_1+b\dot{k}_2)\big], \\ \beta_3 &= \big[a\dot{k}_3-bk_3^2-k_2(ak_1+bk_2)\big]\big[2k_3^2+(ak_1+bk_2)^2\big]-ak_3\big[2k_3\dot{k}_3 + (ak_1+bk_2)(a\dot{k}_1+b\dot{k}_2)\big], \\ \Delta_3 &= \sqrt{2k_3^2+(ak_1+bk_2)^2}\sqrt{\beta_1^2+\beta_2^2+\beta_3^2}. \end{aligned} \tag{3.19}$$

Proof. Differentiate (3.17) to ς and using (2.2), we get

$$\dot{\mu}(\varsigma^*)=\frac{1}{\sqrt{2}}\Big(-(ak_1+bk_2)T(\varsigma)-bk_3\,G(\varsigma)+ak_3\,P(\varsigma)\Big). \tag{3.20}$$

Then

$$T_\mu(\varsigma^*)=\frac{-(ak_1+bk_2)T(\varsigma)-bk_3\,G(\varsigma)+ak_3\,P(\varsigma)}{\sqrt{2k_3^2+(ak_1+bk_2)^2}}, \tag{3.21}$$

such that

$$\frac{d\varsigma^*}{d\varsigma}=\frac{\sqrt{2k_3^2+(ak_1+bk_2)^2}}{\sqrt{2}}. \tag{3.22}$$

Then, we have

$$\dot{T}_\mu(\varsigma^*)=\frac{\sqrt{2}\Big(\beta_1T(\sigma)+\beta_2B_1(\varsigma)+\beta_3B_2(\varsigma)\Big)}{\big[2k_3^2+(ak_1+bk_2)^2\big]^2}.$$

where

$$\begin{aligned}\beta_1 &= \big[k_3(bk_1-ak_2)-a\dot{k}_1-b\dot{k}_2\big]\big[2k_3^2+(ak_1+bk_2)^2\big]+(ak_1+bk_2)\big[2k_3\dot{k}_3\\&\quad+(ak_1+bk_2)(a\dot{k}_1+b\dot{k}_2)\big],\\ \beta_2 &= -\big[ak_3^2+b\dot{k}_3+k_1(ak_1+bk_2)\big]\big[2k_3^2+(ak_1+bk_2)^2\big]+bk_3\big[2k_3\dot{k}_3\\&\quad+(ak_1+bk_2)(a\dot{k}_1+b\dot{k}_2)\big],\\ \beta_3 &= \big[a\dot{k}_3-bk_3^2-k_2(ak_1+bk_2)\big]\big[2k_3^2+(ak_1+bk_2)^2\big]-ak_3\big[2k_3\dot{k}_3\\&\quad+(ak_1+bk_2)(a\dot{k}_1+b\dot{k}_2)\big].\end{aligned}$$

The trajectory curvature and, as a result, the principal normal vector field of μ are

$$\kappa_\mu(\varsigma^*)=\frac{\sqrt{2}\sqrt{\beta_1^2+\beta_2^2+\beta_3^2}}{\big[2k_3^2+(ak_1+bk_2)^2\big]^2},$$

and

$$N_\mu(\varsigma^*)=\frac{\beta_1\,T(\varsigma)+\beta_2\,G(\varsigma)+\beta_3\,P(\varsigma)}{\sqrt{\beta_1^2+\beta_2^2+\beta_3^2}}.$$

So, we have

$$\begin{aligned}B_\mu(\varsigma^*)=\frac{1}{\Delta_3}\Big\{&-k_3(a\beta_2+b\beta_3)T(\varsigma)+\big[a\beta_1k_3+\beta_3(ak_1+bk_2)\big]G(\varsigma)\\&+\big[b\beta_1k_3-\beta_2(ak_1+bk_2)\big]P(\varsigma)\Big\},\end{aligned}$$

where

$$\Delta_3=\sqrt{2k_3^2+(ak_1+bk_2)^2}\sqrt{\beta_1^2+\beta_2^2+\beta_3^2}.$$

Now, from Eq. (3.18) we have

$$\begin{aligned}\ddot{\mu}(\varsigma^*)=\frac{1}{\sqrt{2}}\Big\{&\big[k_3(bk_1-ak_2)-a\dot{k}_1-b\dot{k}_2\big]T(\varsigma)-\big[ak_3^2+b\dot{k}_3+k_1(ak_1+bk_2)\big]G(\varsigma)\\&+\big[a\dot{k}_3-bk_3^2-k_2(ak_1+bk_2)\big]P(\varsigma)\Big\},\end{aligned}$$

similarly

$$\dddot{\mu}(\varsigma^*)=\frac{1}{\sqrt{2}}\Big(\gamma_1T(\varsigma)+\gamma_2G(\varsigma)+\gamma_3P(\varsigma)\Big),$$

where

$$\begin{aligned}\gamma_1 &= \big[k_3(bk_1-ak_2)-a\dot{k}_1-b\dot{k}_2\big]_\varsigma+k_1\big[ak_3^2+b\dot{k}_3+k_1(ak_1+bk_2)\big]\\&\quad-k_2\big[a\dot{k}_3-bk_3^2-k_2(ak_1+bk_2)\big],\\ \gamma_2 &= -\big[ak_3^2+b\dot{k}_3+k_1(ak_1+bk_2)\big]_\varsigma+k_1\big[k_3(bk_1-ak_2)-a\dot{k}_1-b\dot{k}_2\big]\\&\quad-k_3\big[a\dot{k}_3-bk_3^2-k_2(ak_1+bk_2)\big],\\ \gamma_3 &= \big[a\dot{k}_3-bk_3^2-k_2(ak_1+bk_2)\big]_\varsigma+k_2\big[k_3(bk_1-ak_2)-a\dot{k}_1-b\dot{k}_2\big]\\&\quad-k_3\big[ak_3^2+b\dot{k}_3+k_1(ak_1+bk_2)\big].\end{aligned}$$

The trajectory torsion of is then calculated using equations

$$\begin{aligned}\tau_\mu =&\frac{\sqrt{2}}{\Delta_3^*}\Big\{(ak_1+bk_2)\Big(\gamma_2\big[a\dot{k}_3-bk_3^2-k_2(ak_1+bk_2)\big]-\gamma_3\big[ak_3^2+b\dot{k}_3\\&+k_1(ak_1+bk_2)\big]\Big)+bk_3\Big(\gamma_3\big[k_3(bk_1-ak_2)-a\dot{k}_1-b\dot{k}_2\big]-\gamma_1\big[a\dot{k}_3-bk_3^2\\&-k_2(ak_1+bk_2)\big]\Big)+ak_3\Big(\gamma_2\big[k_3(bk_1-ak_2)-a\dot{k}_1-b\dot{k}_2\big]+\gamma_1\big[ak_3^2+b\dot{k}_3\\&+k_1(ak_1+bk_2)\big]\Big)\Big\},\end{aligned}$$

where

$$\begin{aligned}\Delta_3^* =& \Big[bk_3\big[a\dot{k}_3-bk_3^2-k_2(ak_1+bk_2)\big]+ak_3\big[ak_3^2+b\dot{k}_3+k_1(ak_1+bk_2)\big]\Big]^2\\&+\Big[ak_3\big[k_3(bk_1-ak_2)-a\dot{k}_1-b\dot{k}_2\big]+(ak_1+bk_2)\big[a\dot{k}_3-bk_3^2-k_2(ak_1+bk_2)\big]\Big]^2\\&+\Big[bk_3\big[k_3(bk_1-ak_2)-a\dot{k}_1-b\dot{k}_2\big]-(ak_1+bk_2)\big[ak_3^2+b\dot{k}_3+k_1(ak_1+bk_2)\big]\Big]^2.\end{aligned}$$

□

Definition 3.4. Let $\zeta=\zeta(\varsigma)$ be a trajectory unit speed curve in E^3. The TGP-Smarandache trajectory curve via to PAF (2.2) of $\zeta(\varsigma)$ defined by

$$\phi=\phi(\varsigma^*)=\frac{1}{\sqrt{3}}\Big(a\,T(\varsigma)+b\,G(\varsigma)+c\,P(\varsigma)\Big),\quad a^2+b^2+c^2=3. \tag{3.23}$$

Theorem 3.8. Let $\zeta=\zeta(\varsigma)$ be a trajectory unit speed curve of moving point particle of constant mass m in space E^3 via to PAF (2.2). If $\phi: I\subset R\to E^3$ is the TGP-Smarandache trajectory curve of ζ with non-zero curvature function, then its Frenet frame $\{T_\phi, N_\phi, B_\phi\}$ is given by

$$\begin{aligned}T_\phi =& \frac{-(bk_1+ck_2)T+(ak_1-ck_3)G+(ak_2+bk_3)P}{\sqrt{(ak_1+ck_2)^2+(ak_1-ck_3)^2+(ak_2+bk_3)^2}},\\N_\phi =& \frac{\delta_1\,T+\delta_2\,G+\delta_3\,P}{\sqrt{\delta_1^2+\delta_2^2+\delta_3^2}},\\B_\phi =& \frac{1}{\Delta_4}\Big\{\big[a(\delta_3k_1-\delta_2k_2)-k_3(b\delta_2+c\delta_3)\big]T+\big[b(\delta_3k_1+\delta_1k_3)+k_2(a\delta_1+c\delta_3)\big]G\\&+\big[c(\delta_1k_3-\delta_2k_2)-k_1(a\delta_1+b\delta_2)\big]P\Big\}.\end{aligned} \tag{3.24}$$

where

$$\begin{aligned}
\delta_1 &= -\big[k_1(ak_1-ck_3)+k_2(ak_2+bk_3)+b\dot{k}_1+c\dot{k}_3\big]\big[(ak_1+ck_2)^2+(ak_1-ck_3)^2\\
&\quad+(ak_2+bk_3)^2\big]+(bk_1+ck_2)\big[(bk_1+ck_2)(b\dot{k}_1+c\dot{k}_2)+(ak_1-ck_3)(a\dot{k}_1-c\dot{k}_3)\\
&\quad+(ak_2+bk_3)(a\dot{k}_2+b\dot{k}_3)\big],\\
\delta_2 &= \big[a\dot{k}_1-c\dot{k}_3-k_1(bk_1+ck_2)-k_3(ak_2+bk_3)\big]\big[(ak_1+ck_2)^2+(ak_1-ck_3)^2\\
&\quad+(ak_2+bk_3)^2\big]-(ak_1-ck_3)\big[(bk_1+ck_2)(b\dot{k}_1+c\dot{k}_2)+(ak_1-ck_3)(a\dot{k}_1-c\dot{k}_3)\\
&\quad+(ak_2+bk_3)(a\dot{k}_2+b\dot{k}_3)\big],\\
\delta_3 &= \big[a\dot{k}_2-b\dot{k}_3+k_3(ak_1-ck_3)-k_2(bk_1+ck_2)\big]\big[(ak_1+ck_2)^2+(ak_1-ck_3)^2\\
&\quad+(ak_2+bk_3)^2\big]-(ak_2+bk_3)\big[(bk_1+ck_2)(b\dot{k}_1+c\dot{k}_2)+(ak_1-ck_3)(a\dot{k}_1-c\dot{k}_3)\\
&\quad+(ak_2+bk_3)(a\dot{k}_2+b\dot{k}_3)\big],\\
\Delta_4 &= \sqrt{(ak_1+ck_2)^2+(ak_1-ck_3)^2+(ak_2+bk_3)^2}\sqrt{\delta_1^2+\delta_2^2+\delta_3^2}.
\end{aligned} \tag{3.25}$$

Proof. Differentiate (3.23) to ς and using (2.2), we get

$$\dot{\phi}(\varsigma^*) = \frac{1}{\sqrt{3}}\Big(-(bk_1+ck_2)T+(ak_1-ck_3)G+(ak_2+bk_3)P\Big). \tag{3.26}$$

Then

$$T_\phi(\varsigma^*) = \frac{-(bk_1+ck_2)T+(ak_1-ck_3)G+(ak_2+bk_3)P}{\sqrt{(ak_1+ck_2)^2+(ak_1-ck_3)^2+(ak_2+bk_3)^2}}, \tag{3.27}$$

such that

$$\frac{d\varsigma^*}{d\varsigma} = \frac{\sqrt{(ak_1+ck_2)^2+(ak_1-ck_3)^2+(ak_2+bk_3)^2}}{\sqrt{3}}. \tag{3.28}$$

Then, we have

$$\dot{T}_\phi(\varsigma^*) = \frac{\sqrt{3}\Big(\delta_1 T(\sigma)+\delta_2 B_1(\varsigma)+\delta_3 B_2(\varsigma)\Big)}{\big[(ak_1+ck_2)^2+(ak_1-ck_3)^2+(ak_2+bk_3)^2\big]^2}.$$

where

$$\begin{aligned}
\delta_1 &= -\big[k_1(ak_1-ck_3)+k_2(ak_2+bk_3)+b\dot{k}_1+c\dot{k}_3\big]\big[(ak_1+ck_2)^2+(ak_1-ck_3)^2\\
&\quad+(ak_2+bk_3)^2\big]+(bk_1+ck_2)\big[(bk_1+ck_2)(b\dot{k}_1+c\dot{k}_2)+(ak_1-ck_3)(a\dot{k}_1-c\dot{k}_3)\\
&\quad+(ak_2+bk_3)(a\dot{k}_2+b\dot{k}_3)\big],\\
\delta_2 &= \big[a\dot{k}_1-c\dot{k}_3-k_1(bk_1+ck_2)-k_3(ak_2+bk_3)\big]\big[(ak_1+ck_2)^2+(ak_1-ck_3)^2\\
&\quad+(ak_2+bk_3)^2\big]-(ak_1-ck_3)\big[(bk_1+ck_2)(b\dot{k}_1+c\dot{k}_2)+(ak_1-ck_3)(a\dot{k}_1-c\dot{k}_3)\\
&\quad+(ak_2+bk_3)(a\dot{k}_2+b\dot{k}_3)\big],\\
\delta_3 &= \big[a\dot{k}_2-b\dot{k}_3+k_3(ak_1-ck_3)-k_2(bk_1+ck_2)\big]\big[(ak_1+ck_2)^2+(ak_1-ck_3)^2\\
&\quad+(ak_2+bk_3)^2\big]-(ak_2+bk_3)\big[(bk_1+ck_2)(b\dot{k}_1+c\dot{k}_2)+(ak_1-ck_3)(a\dot{k}_1-c\dot{k}_3)\\
&\quad+(ak_2+bk_3)(a\dot{k}_2+b\dot{k}_3)\big].
\end{aligned}$$

The trajectory curvature and, as a result, the principal normal vector field of ϕ are

$$\kappa_\phi(\varsigma^*)=\frac{\sqrt{3}\sqrt{\delta_1^2+\delta_2^2+\delta_3^2}}{\big[(ak_1+ck_2)^2+(ak_1-ck_3)^2+(ak_2+bk_3)^2\big]^2},$$

and

$$N_\phi(\varsigma^*)=\frac{\delta_1\,T(\varsigma)+\delta_2\,G(\varsigma)+\delta_3\,P(\varsigma)}{\sqrt{\delta_1^2+\delta_2^2+\delta_3^2}}.$$

So, we have

$$\begin{aligned}B_\phi=&\frac{1}{\Delta_4}\Big\{\big[a(\delta_3k_1-\delta_2k_2)-k_3(b\delta_2+c\delta_3)\big]T+\big[b(\delta_3k_1+\delta_1k_3)+k_2(a\delta_1+c\delta_3)\big]G\\&+\big[c(\delta_1k_3-\delta_2k_2)-k_1(a\delta_1+b\delta_2)\big]P\Big\},\end{aligned}$$

where

$$\Delta_4=\sqrt{(ak_1+ck_2)^2+(ak_1-ck_3)^2+(ak_2+bk_3)^2}\sqrt{\delta_1^2+\delta_2^2+\delta_3^2}.$$

Now, from Eq. (3.26) we have

$$\begin{aligned}\ddot{\phi}(\varsigma^*)=&\frac{1}{\sqrt{3}}\Big\{-\big[k_1(ak_1-ck_3)+k_2(ak_2+bk_3)+b\dot{k}_1+c\dot{k}_3\big]T(\varsigma)+\big[a\dot{k}_1-c\dot{k}_3\\&-k_1(bk_1+ck_2)-k_3(ak_2+bk_3)\big]G(\varsigma)+\big[a\dot{k}_2-b\dot{k}_3+k_3(ak_1-ck_3)\\&-k_2(bk_1+ck_2)\big]P(\varsigma)\Big\},\end{aligned}$$

similarly

$$\dddot{\phi}(\varsigma^*)=\frac{1}{\sqrt{3}}\Big(\eta_1T(\varsigma)+\eta_2G(\varsigma)+\eta_3P(\varsigma)\Big),$$

where

$$\begin{aligned}\eta_1=&-\big[k_1(ak_1-ck_3)+k_2(ak_2+bk_3)+b\dot{k}_1+c\dot{k}_3\big]_\varsigma-k_1\big[a\dot{k}_1-c\dot{k}_3-k_1(bk_1\\&+ck_2)-k_3(ak_2+bk_3)\big]-k_2\big[a\dot{k}_2-b\dot{k}_3+k_3(ak_1-ck_3)-k_2(bk_1+ck_2)\big],\\ \eta_2=&\big[a\dot{k}_1-c\dot{k}_3-k_1(bk_1+ck_2)-k_3(ak_2+bk_3)\big]_\varsigma-k_1\big[k_1(ak_1-ck_3)+k_2(ak_2\\&+bk_3)+b\dot{k}_1+c\dot{k}_3\big]-k_3\big[a\dot{k}_2-b\dot{k}_3+k_3(ak_1-ck_3)-k_2(bk_1+ck_2)\big],\\ \eta_3=&\big[a\dot{k}_2-b\dot{k}_3+k_3(ak_1-ck_3)-k_2(bk_1+ck_2)\big]_\varsigma-k_2\big[k_1(ak_1-ck_3)+k_2(ak_2\\&+bk_3)+b\dot{k}_1+c\dot{k}_3\big]+k_3\big[a\dot{k}_1-c\dot{k}_3-k_1(bk_1+ck_2)-k_3(ak_2+bk_3)\big].\end{aligned}$$

The trajectory torsion calculated as:

$$\begin{aligned}\tau_\phi=\frac{\sqrt{3}}{\Delta_4^*}\Big\{&(\eta_2\delta_3-\eta_3\delta_2)(bk_1+ck_2)+(\eta_1\delta_3-\eta_3\delta_1)(ak_1-ck_3)\\&+(\eta_2\delta_1-\eta_1\delta_2)(ak_2+bk_3)\Big\},\end{aligned}$$

where

$$\begin{aligned}\Delta_4^* =& \Big[(ak_1 - ck_3)\big[a\dot{k}_2 - b\dot{k}_3 + k_3(ak_1 - ck_3) - k_2(bk_1 + ck_2)\big] - (ak_2 + bk_3)\big[a\dot{k}_1 \\ & - c\dot{k}_3 - k_1(bk_1 + ck_2) - k_3(ak_2 + bk_3)\big]\Big]^2 + \Big[(bk_1 + ck_2)\big[a\dot{k}_2 - b\dot{k}_3 + k_3(ak_1 \\ & - ck_3) - k_2(bk_1 + ck_2)\big] - (ak_2 + bk_3)\big[k_1(ak_1 - ck_3) + k_2(ak_2 + bk_3) + b\dot{k}_1 \\ & + c\dot{k}_3\big]\Big]^2 + \Big[(ak_1 - ck_3)\big[k_1(ak_1 - ck_3) + k_2(ak_2 + bk_3) + b\dot{k}_1 + c\dot{k}_3\big] \\ & + (bk_1 + ck_2)\big[a\dot{k}_1 - c\dot{k}_3 - k_1(bk_1 + ck_2) - k_3(ak_2 + bk_3)\big]\Big]^2.\end{aligned}$$

□

4. Example

We build a computerized example of Smarandache trajectory curves of a trajectory unit speed curve of a moving point particle of constant mass m in space E^3 in this section using PAF. Assume that a constant-mass point particle p follows the track $\zeta(\varsigma) = \left(\frac{5\cos\varsigma}{13}, \frac{8}{13} - \sin\varsigma, -\frac{12\cos\varsigma}{13}\right)$ (see Figure 1). This trajectory's Frenet apparatus is written as

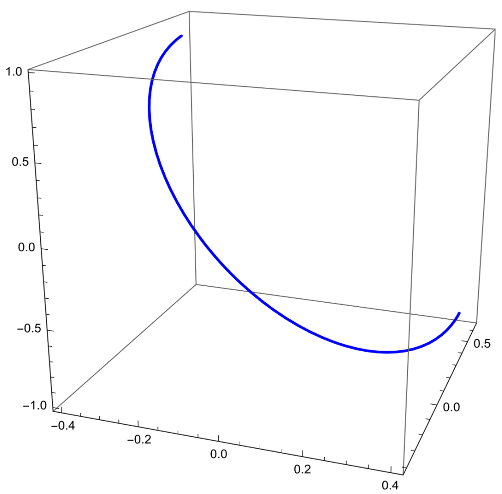

FIGURE 1. Trajectory curve $\zeta = \zeta(\varsigma)$.

$$\begin{aligned}T(\varsigma) &= \left(-\frac{5\sin\varsigma}{13}, -\cos\varsigma, \frac{12\sin\varsigma}{13}\right), \\ N(\varsigma) &= \left(-\frac{5\cos\varsigma}{13}, \sin\varsigma, \frac{12\cos\varsigma}{13}\right), \\ B(\varsigma) &= \left(-\frac{12}{13}, 0, -\frac{5}{13}\right), \\ \kappa = 1, &\qquad \tau = 0.\end{aligned}$$

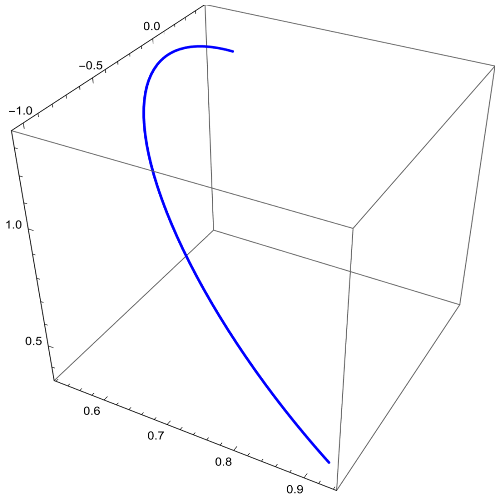

FIGURE 2. TG-Smarandache trajectory curve.

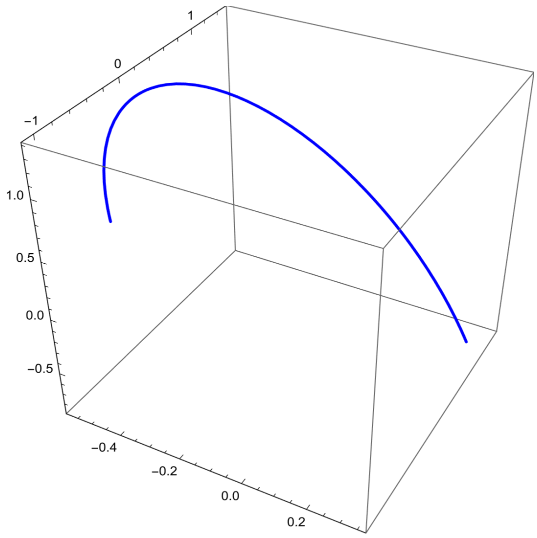

FIGURE 3. TP-Smarandache trajectory curve.

Since $\langle \zeta(\varsigma), B(\varsigma)\rangle = 0$ and $\langle \zeta(\varsigma), N(\varsigma)\rangle < 0$, then we get $\Theta(\varsigma) = \frac{\pi}{2}$. As a result of the given knowledge, we may construct the PAF apparatus as follows:

$$\begin{aligned}
&T(\varsigma) = \left(-\frac{5\sin\varsigma}{13}, -\cos\varsigma, \frac{12\sin\varsigma}{13}\right),\\
&G(\varsigma) = \left(\frac{12}{13}, 0, \frac{5}{13}\right),\\
&P(\varsigma) = \left(-\frac{5\cos\varsigma}{13}, \sin\varsigma, \frac{12\cos\varsigma}{13}\right),\\
&k_1 = 0, \qquad k_2 = 1, \qquad k_3 = 0.
\end{aligned}$$

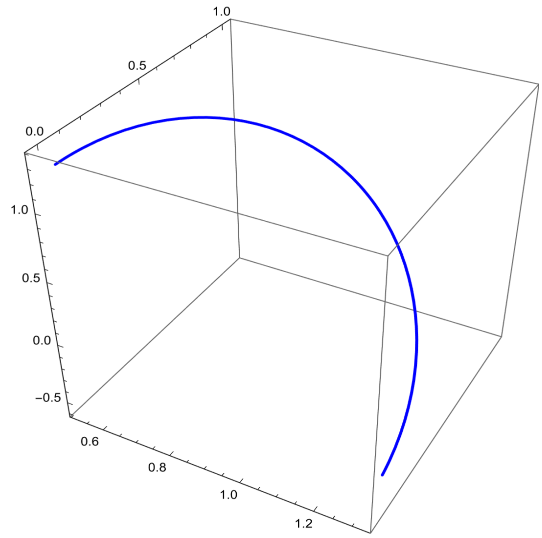

FIGURE 4. GP-Smarandache trajectory curve.

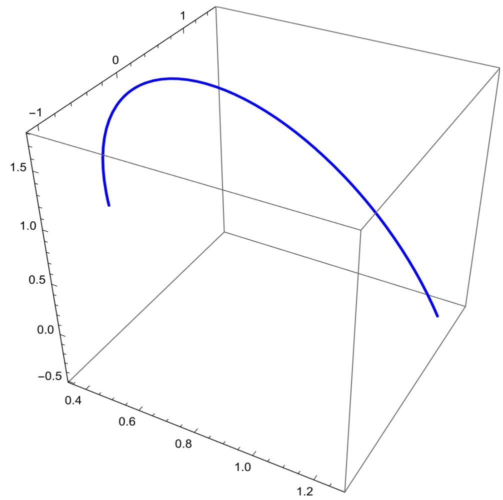

FIGURE 5. TGP-Smarandache trajectory curve.

In this article, we look at the Smarandache trajectory curves for the first time in terms of definitions (see Figures 2–5).

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