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INFINITE PRODUCTS, SERIES WITH LOGARITHMS, AND SERIES WITH ZETA VALUES

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Abstract. In this note, we point out an interesting connection between series with zeta values, series with logarithm values, and certain infinite products. Using this connection, we give a closed-form evaluation of various series with zeta values in the coefficients.

1. Introduction

In [3] the author studied the special constant

$$M = \int_0^1 \frac{\psi(t+1)+\gamma}{t}\, dt \quad \approx 1.257746 \tag{1.1}$$

and proved, among other things, the identity [7, p.142].

$$M = \sum_{n=1}^{\infty} \frac{1}{n} \ln\left(1+\frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\zeta(n+1)}{n} \tag{1.2}$$

where $\psi(s) = \frac{d}{ds}\ln\Gamma(s)$ is the digamma function and $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (Res > 1)$ is Riemann's zeta function.

In this note, we will extend equation (1.2) to the identity with parameters

$$\sum_{n=1}^{\infty} \frac{1}{n^a} \ln\left(1+\frac{\lambda}{n^z}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\lambda^{n-1}\zeta(nz+a)}{n} \tag{1.3}$$

and provide several explicit evaluations of such series.

When $\lambda = z = a = 1$ equation (1.3) turns into (1.2).

The results in this paper complement those in [4].

2. Results and proofs

We start by considering series of the form

$$\sum_{p=1}^{\infty} \frac{1}{p^a(\lambda+p^z)}, \quad Re\,(z) > 1, \;\; |\lambda| < 1, \;\; a \geq 0.$$

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They will be related to series with zeta values.

Let $H_m^{(s)}$ be the generalized harmonic numbers

$$H_m^{(s)} = 1 + \frac{1}{2^s} + ... + \frac{1}{m^s}, \;\; H_0^{(s)} = 0$$

which are partial sums of the Riemann zeta function $\zeta(s)$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \;, \quad Res > 1.$$

We prove the theorem:

Theorem 2.1. *For every integer $m \geq 1$, $|\lambda| < 1$, $a \geq 0$, $Re\,(z) > 1$*

$$\sum_{p=1}^{m} \frac{1}{p^a(\lambda + p^z)} = \sum_{n=1}^{\infty} (-1)^{n-1} \lambda^{n-1} H_m^{(n\,z+a)} \tag{2.1}$$

and also,

$$\sum_{p=1}^{m} \frac{1}{p^a} \ln\left(1 + \frac{\lambda}{p^z}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \lambda^n H_m^{(n\,z+a)}. \tag{2.2}$$

Changing λ to $-\lambda$ we have as well

$$\sum_{p=1}^{m} \frac{1}{p^a} \ln\left(1 - \frac{\lambda}{p^z}\right) = -\sum_{n=1}^{\infty} \frac{\lambda^n}{n} H_m^{(n\,z+a)}.$$

Proof. Using geometric series, we write

$$\sum_{p=1}^{m} \frac{1}{p^a(\lambda + p^z)} = \sum_{p=1}^{m} \frac{1}{p^{z+a}} \left(1 - (-\lambda p^{-z})\right)^{-1} = \sum_{p=1}^{m} \frac{1}{p^{z+a}} \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k}{p^{k\,z}} \right\}$$

$$= \sum_{p=1}^{m} \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k}{p^{(k+1)z+a}} \right\} = \sum_{k=0}^{\infty} (-1)^k \lambda^k \left\{ \sum_{p=1}^{m} \frac{1}{p^{(k+1)z+a}} \right\} = \sum_{k=0}^{\infty} (-1)^k \lambda^k H_m^{((k+1)z+a)}$$

Changing the index in the last sum $k + 1 = n$, we obtain equation (2.1). Next, we integrate both sides in (2.1) with respect to λ. This gives

$$\sum_{p=1}^{m} \frac{1}{p^a} \ln(\lambda + p^z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \lambda^n H_m^{(n\,z+a)} + C.$$

Setting $\lambda = 0$ we find $C = \sum_{p=1}^{m} \frac{\ln(p^z)}{p^a}$, so that

$$\sum_{p=1}^{m} \frac{1}{p^a} \ln(\lambda + p^z) - \sum_{p=1}^{m} \frac{1}{p^a} \ln(p^z) = \sum_{p=1}^{m} \frac{1}{p^a} \ln\left(1 + \frac{\lambda}{p^z}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \lambda^n H_m^{(n\,z+a)}$$

and the theorem is proved. □

For example, for $a = 0,\ z = 1$ in (2.2) we have from [8]

$$\prod_{p=1}^{m} \left(1 + \frac{\lambda}{p}\right) = \frac{\Gamma(m + \lambda + 1)}{m!\Gamma(\lambda + 1)}.$$

This gives

$$\sum_{p=1}^{m} \ln\left(1 + \frac{\lambda}{p}\right) = \ln \prod_{p=1}^{m} \left(1 + \frac{\lambda}{p}\right) = \ln \frac{\Gamma(m + \lambda + 1)}{m!\Gamma(\lambda + 1)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \lambda^n H_m^{(n\,z)}.$$

Corollary 2.2. *With a, z, λ as in Theorem 2.1,*

$$\sum_{p=1}^{\infty} \frac{1}{p^a(\lambda + p^z)} = \sum_{n=1}^{\infty} (-1)^{n-1} \lambda^{n-1} \zeta(n\,z + a) \tag{2.3}$$

$$\sum_{p=1}^{\infty} \frac{1}{p^a} \ln\left(1 + \frac{\lambda}{p^z}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \lambda^n \zeta(n\,z + a) \tag{2.4}$$

$\sum_{p=1}^{\infty} \frac{1}{p^a} \ln\left(1 - \frac{\lambda}{p^z}\right) = -\sum_{n=1}^{\infty} \frac{\lambda^n}{n} \zeta(n\,z + a)$ *(changing λ to −λ)*

Proof. The result follows from Theorem 2.1 by letting $m \to \infty$. The limit can go through the sum because the series is absolutely convergent. □

For $a = \lambda = z = 1$ in (2.4) we get equation (1.2).
With $a = 1$ we find from (2.4) the series identity

$$\sum_{p=1}^{\infty} \frac{1}{p} \ln\left(1 + \frac{\lambda}{p^z}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \lambda^n \zeta(n\,z + 1).$$

The series are convergent also for $\lambda = 1$ (see argument below after equation (2.6)).
The case $z = \lambda = 1$ in (2.4) appeared in the papers [2, 5, 6]

$$\sum_{p=1}^{\infty} \frac{1}{p^a} \ln\left(1 + \frac{1}{p}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \zeta(n + a).$$

When $a > 1$ we can write

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \zeta(n + a) = \sum_{p=1}^{\infty} \frac{1}{p^a} \ln\left(1 + \frac{1}{p}\right) = \sum_{p=1}^{\infty} \frac{1}{p^a} \ln\left(\frac{p+1}{p}\right) = \sum_{p=1}^{\infty} \frac{\ln(p+1)}{p^a} - \sum_{p=1}^{\infty} \frac{\ln(p)}{p^a}$$

and since $-\sum_{p=1}^{\infty} \frac{\ln(p)}{p^a} = \zeta'(a)$ this becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\zeta(n + a)}{n} = \sum_{p=1}^{\infty} \frac{1}{p^a} \ln\left(1 + \frac{1}{p}\right) = \sum_{p=1}^{\infty} \frac{\ln(p+1)}{p^a} + \zeta'(a)$$

([2, Theorem 4] and [6, equation 4]).

The above series resist evaluation in closed form. Anyway, we want to mention one interesting identity from [5, Theorem 10] related to the above result. First, following the notations in [5], let

$$\lambda_1 = \frac{1}{2},\ \lambda_{n+1} = \int_0^1 x(1-x)...(n-x)dx$$

be the non-alternating Cauchy numbers. Let also $H_m^{(1)} = H_n = 1 + \frac{1}{2} + ... + \frac{1}{n}$ be the ordinary harmonic numbers. Then for integers $a > 1$, we have the representation

$$\sum_{p=1}^{\infty} \frac{1}{p^a} \ln\left(1 + \frac{1}{p}\right) = \zeta'(a) - \gamma\zeta(a) - \zeta(a+1) + \sum_{n=1}^{\infty} \frac{H_n}{n^a} - \sum_{k=1}^{a-1} \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{(n+1)^k n^{a-k}}$$

$$+ \sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^2} P_{a-1}(H_n,, H_n^{(2)}, ...H_n^{(a-1)})$$

where P_m are the modified Bell polynomials defined by the generating function

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{z^k}{k}\right) = \sum_{m=0}^{\infty} P_m(x_1, x_2, ..., x_m)\, z^m.$$

Corollary 2.3. *With $a = 0$ in (2.2) we have*

$$\sum_{p=1}^{m} \ln\left(1 + \frac{\lambda}{p^z}\right) = \ln \prod_{p=1}^{m}\left(1 + \frac{\lambda}{p^z}\right) = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n} \lambda^n H_m^{(n\,z)} \tag{2.5}$$

and with $m \to \infty$

$$\sum_{p=1}^{\infty} \ln\left(1 + \frac{\lambda}{p^z}\right) = \ln \prod_{p=1}^{\infty}\left(1 + \frac{\lambda}{p^z}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \lambda^n \zeta(n\,z). \tag{2.6}$$

Note that the series with zeta values in (2.6) converges also for $\lambda = 1$, that is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \zeta(n\,z) = \sum_{p=1}^{\infty} \ln\left(1 + \frac{1}{p^z}\right) = \ln \prod_{p=1}^{\infty}\left(1 + \frac{1}{p^z}\right)$$

as $\lim_{n\to\infty} |\zeta(n\,z)| = 1$ and the series is alternating.

With $\lambda = x^2$ and $z = 2$ in (2.6) we come to the known identity

$$\sum_{p=1}^{\infty} \ln\left(1 + \frac{x^2}{p^2}\right) = \ln \prod_{p=1}^{\infty}\left(1 + \frac{x^2}{p^2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^{2n} \zeta(2n) = \ln \frac{\sinh(\pi x)}{\pi x} \tag{2.7}$$

by using the classical representation

$$\frac{\sinh(\pi x)}{\pi x} = \prod_{p=1}^{\infty}\left(1 + \frac{x^2}{p^2}\right).$$

In particular, with $x = 1$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \zeta(2n) = \ln \frac{\sinh \pi}{\pi},$$

(see also [10, p. 161]) while the series $\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n}$ is divergent.

With $x = 1/\mu,\ \mu > 1$ identity (2.7) implies

$$\ln \prod_{p=1}^{\infty}\left(1 + \frac{1}{\mu^2 p^2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\mu^{2n} n} \zeta(2n) = \ln \frac{\mu \sinh(\pi/\mu)}{\pi}. \tag{2.8}$$

In particular, with $\mu = 2$,

$$\ln \prod_{p=1}^{\infty}\left(1 + \frac{1}{4p^2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4^n n} \zeta(2n) = \ln \frac{2\sinh(\pi/2)}{\pi}$$

From equation (2.6) and the above examples, we can make the following

Conclusion. *When the infinite product $\prod_{p=1}^{\infty}\left(1 + \frac{\lambda}{p^z}\right)$ can be evaluated in explicit closed form, then the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \lambda^n \zeta(n\,z)$ can be evaluated in closed form.*

We will show here some more examples following this observation. First, we will use a formula for infinite products from Hansen's table [9] to evaluate explicitly certain series with zeta values.

[9, Entry 89.6.8] reads (in corrected form)

$$\prod_{p=1}^{\infty}\left(1+\frac{x^3}{p^3}\right)=\frac{1}{\Gamma(1+x)\Gamma\left(1-\frac{x}{2}-\frac{x\sqrt{3}}{2}i\right)\Gamma\left(1-\frac{x}{2}+\frac{x\sqrt{3}}{2}i\right)}.$$

With $\lambda = x^3,\ z = 3$ in (2.6) we find

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^{3n}\zeta(3n)=-\ln\left(\Gamma(1+x)\Gamma(1-\frac{x}{2}-\frac{x\sqrt{3}}{2}i)\Gamma(1-\frac{x}{2}+\frac{x\sqrt{3}}{2}i)\right) \tag{2.9}$$

(this is the alternating variant of [4, equation (11)]). For $x = 1$ this comes to

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\zeta(3n)=\ln\left(\frac{1}{\pi}\cosh\frac{\pi\sqrt{3}}{2}\right)$$

([4, equation (13)].

The case $z = 4$ was considered in [4]. For $z = 5$ we use [12, equation (33)]

$$\prod_{p=1}^{\infty}\left(1+\frac{1}{p^5}\right)=\left|\Gamma\left[\exp(2\pi i/5)\right]\Gamma\left[\exp(6\pi i/5)\right]\right|^{-2}$$

which provides the evaluation

$$\sum_{p=1}^{\infty}\ln\left(1+\frac{1}{p^5}\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\zeta(5\,n)=\ln\left(\left|\Gamma\left[\exp(2\pi i/5)\right]\Gamma\left[\exp(6\pi i/5)\right]\right|^{-2}\right). \tag{2.10}$$

For $z = 6$ we use [12, equation 34] that says

$$\prod_{p=1}^{\infty}\left(1+\frac{1}{p^6}\right)=\frac{\sinh\pi(\cosh(\pi)-\cos(\pi\sqrt{3}))}{2\pi^3}$$

and it gives

$$\sum_{p=1}^{\infty}\ln\left(1+\frac{1}{p^6}\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\zeta(6\,n)=\ln\frac{\sinh\pi(\cosh(\pi)-\cos(\pi\sqrt{3}))}{2\pi^3}. \tag{2.11}$$

It is appropriate to mention here [10, Proposition 3.2] where it was shown by a different method that

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}[\zeta(k\,n)-1]=\ln\prod_{j=1}^{k-1}\Gamma(2-(-1)^{(2j+1)/k}) \tag{2.12}$$

(a result previously obtained by Adamchik and Srivastava [1, Proposition 1, p. 135]; see also [11, Proposition 3.5, p. 262]). The series on the left-hand side can be split into two series, the second one of which represents $-\ln 2$. This way equation (2.12) can be written in the form

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\zeta(k\,n)=\ln 2+\ln\prod_{j=1}^{k-1}\Gamma(2-(-1)^{(2j+1)/k})=\ln 2\prod_{j=1}^{k-1}\Gamma(2-(-1)^{(2j+1)/k}), \tag{2.13}$$

that is,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \zeta(k\,n) = -\ln 2 \prod_{j=1}^{k-1} \Gamma(2 - (-1)^{(2j+1)/k}).$$

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