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ROUGH $\mathcal{I}_2$-STATISTICAL CONVERGENCE IN CONE METRIC SPACES IN CERTAIN DETAILS

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Abstract. The purpose of this work is to look at rough $\mathcal{I}_2$-statistical convergence as an extension of rough convergence in a cone metric space (briefly CMS). Furthermore, we define the concept of rough $\mathcal{I}_2^*$-statistical convergence of sequences in a CMS and investigate the link between rough $\mathcal{I}_2$-statistical and $\mathcal{I}_2^*$-statistical convergence of sequences.

1. Introduction

Fast introduced the notion concept of statistical convergence of sequences in real numbers in 1951 by in [15]. Pringsheim [36] proposed the convergence of real double sequences initially. Mursaleen and Edely [31] expanded the concept of convergence of real double sequences to statistical convergence. Following that, this idea was explored from a sequence standpoint and linked to the summability theory (see [6, 8, 9, 18, 19, 29, 30, 39, 40, 41, 42, 43]). Das et. al. [11] expanded statistical convergence of double sequences to $\mathcal{I}$-convergence of double sequences using ideals in $\mathbb{N}\times\mathbb{N}$. For further information, read [12, 13, 16, 17, 23, 44, 46]. Belen and Yıldırım [7] recently introduced the concept of ideal statistical convergence of double sequences.

Phu [35] was the first to investigate the notion of rough convergence. Recently, Malik et. al. [26] has examined the idea of rough convergence for double sequences in normed linear spaces. Malik et. al. [27] extended rough convergence of double sequence to rough statistical convergence of double sequence. Dündar et. al. [14] expanded rough statistical convergence of double sequences to rough $\mathcal{I}$-convergence of double sequences. Malik and Ghosh [28] presented the notion of rough $\mathcal{I}$-statistical convergence of double sequences in normed linear spaces.

Huang and Xian [20] pioneered the concept of CMS. In their study, the elements of a real Banach space were used to substitute the distance between two points. CMS is, without a doubt, an extension of the idea of an ordinary metric space. In [4] Banerjee and Mondal investigated and worked the conception of rough convergence of sequences in a CMS. Cone metric spaces were defined many years ago by multiple

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writers and appeared in the literature under various authors (see, for example [1, 2, 3, 5, 10, 20, 21, 25, 34, 37, 38, 45]).

Section 2 of this article will introduce the reader to the fundamental concepts of $\mathcal{I}$-statistical convergence for single and double sequences, as well as some of the consequences of this convergence, definitions and properties of cone metric spaces, and the concept of rough convergence and rough $\mathcal{I}$-convergence of sequences in a CMS. In Section 3, we shall discuss the rough $\mathcal{I}_2$-statistical convergence and rough $\mathcal{I}_2^*$-statistical convergence in CMS for double sequences.

2. Preliminaries

This part will collect all of the relevant outcomes and approaches on which we will rely to achieve our key goals. First, let's define some crucial terms.

Definition 2.1. *([23]) Assume $Y \neq \emptyset$. $\mathcal{I} \subset 2^Y$ is named an ideal on Y provided that (i1) for each $U, V \in \mathcal{I}$ implies $U \cup V \in \mathcal{I}$; (i2) for each $U \in \mathcal{I}$ and $V \subset P$ implies $V \in \mathcal{I}$.*

Definition 2.2. *([23]) Assume $Y \neq \emptyset$. $\mathcal{F} \subset 2^Y$ is named a filter on Y provided that (f1) for all $U, V \in \mathcal{F}$ implies $U \cap V \in \mathcal{F}$; (f2) for all $U \in \mathcal{F}$ and $V \supset P$ implies $V \in \mathcal{F}$.*

An ideal $\mathcal{I}$ is known as non-trivial provided that $Y \notin \mathcal{I}$ and $\mathcal{I} \neq \emptyset$. A non-trivial ideal $\mathcal{I} \subset P(Y)$ is known as an admissible ideal in Y iff $\mathcal{I} \supset \{\{w\} : w \in Y\}$. Afterwards, the filter $F = F(\mathcal{I}) = \{Y - S : S \in \mathcal{I}\}$ is named the filter connected with the ideal.

Utilizing the notion of ideals, Kostyrko et al. [23] determined the notion of $\mathcal{I}$ and $\mathcal{I}^*-$convergence. Also, Kostyrko et al. [23] gave the definition of (AP) condition for admissible ideal, and examined the relation between $\mathcal{I}$ and $\mathcal{I}^*$-convergence under (AP) condition.

See the references in [32, 33] for more information on $\mathcal{I}$-convergent.

Now, we present the notion of $\mathcal{I}_2$-asymptotic density of $\mathbb{N}^2$.

A subset $K \subset \mathbb{N} \times \mathbb{N}$ is named to be have $\mathcal{I}_2$-asymptotic density $d_{\mathcal{I}_2}(K)$ when

$$d_{\mathcal{I}_2}(K) = \mathcal{I}_2 - \lim_{u,v \to \infty} \frac{|K(u,v)|}{u.v},$$

where

$$K(u,v) = \{(s,t) \in \mathbb{N} \times \mathbb{N} : s \leq u, t \leq v; (s,t) \in K\}$$

and $|K(u,v)|$ demonstrates number of elements of the set $K(u,v)$.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N}^2$ is named strongly admissible when $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$.

Throughout the work, we contemplate $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Definition 2.3. *([11]) Presume (Y, ρ) be a metric space. A double sequence $w = (w_{uv})$ is named to be $\mathcal{I}_2$-convergent to w, provided that for any $\sigma > 0$ we acquire*

$$A(\sigma) := \{(u,v) \in \mathbb{N} \times \mathbb{N} : \rho(y_{st}, y^*) \geq \sigma\} \in \mathcal{I}_2.$$

We write

$$\mathcal{I}_2 - \lim_{s,t \to \infty} y_{st} = y^*.$$

A double sequence $y=(y_{st})$ of real numbers is $\mathcal{I}_2$-statistically convergent to y^*, and we show $y_{st} \overset{\mathcal{I}_2-st}{\to} y^*$, provied that for any $\sigma, \delta > 0$

$$\left\{(u,v) \in \mathbb{N}\times\mathbb{N}: \frac{1}{uv}\left|\{(s,t) : \rho(y_{st}, y^*) \geq \sigma,\ s \leq u, t \leq v\}\right| \geq \delta\right\} \in \mathcal{I}_2.$$

Definition 2.4. *([11]) We say that an admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}^2}$ supplies the condition (AP2) provided that for all countable family of mutually disjoint sets $\{U_1, U_2, ...\} \in \mathcal{I}_2$, there exists a countable family of sets $\{V_1, V_2, ...\} \in \mathcal{I}_2$ such that $U_j \Delta V_j \in \mathcal{I}_0$ i.e., $U_j \Delta V_j$ is included in the finite union of rows and columns in $\mathbb{N}^2$ for each $j \in \mathbb{N}$ and $V = \bigcup_{j=1}^{\infty} V_j \in \mathcal{I}_2$ (so $V_j \in \mathcal{I}_2$ for all $j \in \mathbb{N}$).*

A double sequence $y=(y_{st})$ is said to be rough convergent (r-convergent) to y^* with the roughness degree r, denoted by $y_{st} \overset{r}{\longrightarrow} y^*$ provided that

$$\forall \varepsilon > 0\ \exists k_\varepsilon \in \mathbb{N}:\ \ s,t \geq k_\varepsilon \Rightarrow \|y_{st} - y^*\| < r + \varepsilon,$$

or equivalently, if

$$\limsup \|y_{st} - y^*\| \leq r.$$

A double sequence $y=(y_{st})$ is named to be r-$\mathcal{I}_2$-convergent to y^* with the roughness degree r, indicated by $y_{st} \overset{r-\mathcal{I}_2}{\longrightarrow} y^*$ provided that

$$\{(s,t) \in \mathbb{N}\times\mathbb{N} : \|y_{st} - y^*\| \geq r + \varepsilon\} \in \mathcal{I}_2,$$

for all $\varepsilon > 0$; or equivalently, when the condition

$$\mathcal{I}_2 - \limsup \|y_{st} - y^*\| \leq r$$

is supplied. Furthermore, we can signify $y_{st} \overset{r-\mathcal{I}_2}{\longrightarrow} y^*$ iff the inequality $\|y_{st} - y^*\| < r + \varepsilon$ holds for all $\varepsilon > 0$.

Assume $y=(y_{st})$ be a double sequence in a normed linear space $(Y, \|.\|)$ and r be a non negative real number. Then, y is named to be rough $\mathcal{I}_2$-statistical convergent to y^* or r-$\mathcal{I}_2$-statistical convergent to y^* provided that for any $\varepsilon, \delta > 0$

$$\left\{(u,v) \in \mathbb{N}\times\mathbb{N}: \frac{1}{uv}\left|\{(s,t)\,,\ s \leq u, t \leq v : \|y_{st} - y^*\| \geq r + \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}_2.$$

In this case, y^* is called the rough $\mathcal{I}_2$-statistical limit of $y=(y_{st})$ and symbolically, we indicate $y_{st} \overset{r-\mathcal{I}_2-st}{\longrightarrow} y^*$.

We now recall the essential notions from [20, 21] that are required for the remainder of the essay.

Definition 2.5. *Let E be a Hausdorff topological vector space (tvs) with the zero vector 0. A subset P of E is called a (convex) cone if it satisfies the following conditions:*
(i) $P \neq \{0\}, P \neq \emptyset$ and P is closed;
(ii) $\lambda P \subset P$ for $\forall \lambda \geq 0$ and $P + P \subset P$;
(iii) $\{0\} = P \cap (-P)$.

Given a $P \subset E$ cone, we can define a partial ordering $\preceq$ with respect to P by defining $x \preceq y \Longleftrightarrow y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ represent the set of the interior points of P. The sets of the form $[x,y]$ are named *order-intervals* and are defined as the follows:

$$[x,y]=\{z\in E: x\preceq z\preceq y\}.$$

Order-intervals are observed to be convex. If $[x,y]\subset A$ while $x,y\in A$ and $x\preceq y$, then $A\subset E$ is named *order-convex.*

It is order-convex if ordered tvs (E,P) has a neighborhoods' base of 0 that are made up of *order-convex* sets. Accordingly, the cone P is named a normal cone. Considering the normed space, this condition means that the unit ball is *order-convex,* it is equivalent to the condition that $\exists k$ with $x,y\in E$ and $0\preceq x\preceq y\Rightarrow \|x\|\leq k\|y\|$. The smallest constant k is named the normal constant of P [21].

If each of the increasing sequence that is bounded in P is convergent then, we describe to P as a regular cone. To put it another way, if a sequence $\{x_n\}$ exists such that

$$x_1\preceq x_2\preceq ...\preceq x_n\preceq ...\preceq y,\quad \text{for some } y\in E,$$

then $\exists x\in E$ such that $\lim\limits_{n\to\infty}\|x_n-x\|=0$. Similarly, the P cone is regular, if all decreasing sequences that are bounded from below converges. If P is a regular cone, it is known to be a normal cone.

Let E be a tvs, $V\subset E$ is an absolutely convex and absorbent subset, the corresponding Minkowski functional $f_V: E\to\mathbb{R}$ is defined

$$x\mapsto f_V(x)=\inf\{\lambda>0: x\in\lambda V\}.$$

It is a semi-norm on E. If V is an absolutely convex neighborhood of $0\in E$, then f_V is continuos and

$$\{x\in E: f_V(x)<1\}=\operatorname{int}V\subset V\subset\overline{V}=\{x\in E: f_V(x)\leq 1\}.$$

Let $e\in\operatorname{int}P$ and (E,P) be an ordered tvs. After that

$$[-e,e]=(P-e)\cap(e-P)=\{z\in E: -e\preceq z\preceq e\}$$

is an absolutely convex neighborhood of 0. We denote the corresponding Minkowski functional $f_{[-e,e]}$ by f_e. It can be verified that $\operatorname{int}[-e,e]=(\operatorname{int}P-e)\cap(e-\operatorname{int}P)$. If P is normal and solid, then the Minkowski functional f_e is the norm on E. Furthermore, it is an increasing function on P. In fact, for $0\preceq x_1\preceq x_2$ the set $\{\lambda: x_1\in\lambda[-e,e]\}$ is the subset of $\{\lambda: x_2\in\lambda[-e,e]\}$ and it follows that $f_e(x_1)\leq f_e(x_2)$.

Definition 2.6. *([20]) Take $Y\neq\emptyset$. Suppose that $\rho: Y\times Y\to W$ supplies*

(d_1) $\rho(u,v)=0$ iff $u=v$ and $0\preceq\rho(u,v)$ for $\forall u,v\in X$;

(d_2) $\rho(v,u)=\rho(u,v)$ for $\forall u,v\in Y$;

(d_3) $\rho(u,v)\preceq\rho(u,w)+\rho(w,v)$ for $\forall u,v,w\in Y$.

Then ρ is named to be a cone metric on Y. (Y,ρ) is named to be a CMS. Obviously, the notion of CMS generalizes the notion of metric spaces.

Definition 2.7. *([20]) Assume (Y,ρ) be a CMS. $\{y_s\}_{s\in\mathbb{N}}$ be a sequence in CMS Y and assume $y^*\in Y$. If for $\forall c\in W$ with $0\ll c$ there is $N\in\mathbb{N}$ so that for all $s>N$, $\rho(y_s,y^*)\ll c$, then $\{y_s\}_{s\in\mathbb{N}}$ is named to be convergent to y^* and it is named the limit of the sequence $\{y_s\}_{s\in\mathbb{N}}$.*

Definition 2.8. *([20]) Assume (Y,ρ) be a CMS. $\{y_s\}_{s\in\mathbb{N}}$ be a sequence in CMS Y. If for any $c\in W$ with $0\ll c$ there is $N\in\mathbb{N}$ such that for all $s,t>N$, $\rho(y_s,y_t)\ll c$, then $\{y_s\}_{s\in\mathbb{N}}$ is named a Cauchy sequence in Y. All Cauchy sequence in Y is convergent in Y, then Y is named a complete CMS.*

Definition 2.9. *([20]) A sequence $\{y_s\}_{s\in\mathbb{N}}$ in Y is named to be $\mathcal{I}^*$-convergent to $y^* \in Y$ iff there is a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m_1 < m_2 < \cdots < m_j < \ldots\}$ such that $\lim_{j\to\infty} y_{m_j} = y^*$, that is for $\forall c \in W$ with $c \ll 0$, there is $p \in \mathbb{N}$ so that $c - d\left(y_{m_j}, y^*\right) \in \operatorname{int} P$, for $\forall j \geq p$.*

Lemma 2.1. *([22]) Assume (Y, W) be an CMS with $x \in P$ and $y \in intP$. Then, one can find $n \in \mathbb{N}$ such that $x << ny$.*

Theorem 2.2. *([4]) Assume W be a real Banach space and P be a cone in W. When $x_0 \in intP$ and $\alpha\,(>0) \in \mathbb{R}$ then $\alpha x_0 \in intP$.*

Theorem 2.3. *([4]) Assume W be a real Banach space and P be a cone in W. When $x_0 \in P$ and $y_0 \in intP$ then $x_0 + y_0 \in intP$.*

Corollary 2.4. *([4]) When $x_0, y_0 \in intP$ then $x_0 + y_0 \in intP$.*

Theorem 2.5. *([4]) Assume W be a real Banach space and P be a cone in W, then $0 \notin intP$ (0 be the zero element of W).*

Definition 2.10. *([4]) Assume $\{y_s\}_{s\in\mathbb{N}}$ be a sequence in CMS (Y, ρ). A point $c \in Y$ is named to be a cluster point of $\{y_s\}$ if for any $(0 <<)\,\sigma$ in W and for any $p \in \mathbb{N}$, there is a $p_1 \in \mathbb{N}$ so that $p_1 > p$ with $\rho\,(y_{p_1}, c) << \sigma$.*

Definition 2.11. *([34]) Let (Y, ρ) be a CMS. A sequence $\{y_s\}$ in Y is named to be $\mathcal{I}$-convergent to $y^* \in Y$ if for any $c \in W$ with $(0 <<)\,c$ the set*

$$\{s \in \mathbb{N} : c - \rho\,(y_s, y^*) \notin intP\} \in \mathcal{I}.$$

Definition 2.12. *([34]) Let (Y, ρ) be a CMS. A sequence $\{y_s\}$ in Y is named to be $\mathcal{I}^*$-convergent to $y^* \in Y$ iff there is a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m_1 < m_2 < \cdots < m_j < \ldots\}$ such that $\{y_s\}_{s\in M}$ is convergent to y^* i.e., for any $c \in W$ with $(0 <<)\,c$ there exists $p \in \mathbb{N}$ such that $c - \rho\,(y_{m_k}, y^*) \notin intP$ for all $k \geq p$.*

Since it is known [45] that any cone metric space is a first countable Hausdorff topological space with the topology induced by the open balls defined naturally for each element z in X and for each element c in $\operatorname{int} P$. So as in [24] we can show that $\mathcal{I}^*$-convergence always implies $\mathcal{I}$-convergence but the converse is not true. The two concepts are equivalent iff the ideal $\mathcal{I}$ has condition (AP).

Definition 2.13. *([4]) Let (Y, ρ) be a CMS. A sequence $\{y_s\}$ in Y is named to be rough convergent of roughness degree r to $y^* \in Y$ for some $(0 << r) \in W$ or $r = 0$ if for any $\sigma > 0$ with $(0 <<)\,\sigma$ there exists a $m \in \mathbb{N}$ so that $\rho\,(y_s, y^*) << r + \sigma$ for all $s \geq m$.*

Definition 2.14. *([5]) Let (Y, ρ) be a CMS. A sequence $\{y_s\}$ in Y is named to be rough $\mathcal{I}$-convergent of roughness degree r to $y^* \in Y$ for some $(0 << r) \in W$ or $r = 0$ if for any $\sigma > 0$ with $(0 <<)\,\sigma$ the set $A\,(\sigma) = \{s \in \mathbb{N} : (r + \sigma - \rho\,(y_s, y^*)) \notin intP\} \in \mathcal{I}$.*

3. MAIN RESULTS

Throughout our work (Y, ρ) stands for an CMS where $\rho : Y \times Y \to W$ is the cone metric, W being a real Banach space and $\mathcal{I}_2$ stands for a strongly admissible ideal in $\mathbb{N}^2$.

Definition 3.1. *A sequence $y = \{y_{st}\}$ in Y is named to be $\mathcal{I}_2$-statistically convergent to $y^* \in Y$ provided that for any $(0 <<) \, \sigma \in W$ and for all $\kappa > 0$, the set*

$$T(\sigma) := \left\{ (u,v) \in \mathbb{N}^2 : \frac{1}{uv} \left| \{(s,t) : s \le u, t \le v;\ \sigma - \rho(y_{st}, y^*) \notin intP\} \right| \ge \kappa \right\} \in \mathcal{I}_2.$$

Symbolically, we indicate $y_{st} \overset{\mathcal{I}_2 - st}{\to} y^*$.

Definition 3.2. *Assume (Y, ρ) be an CMS. A sequence $\{y_{st}\}$ is said to be rough statistically convergent of roughness degree r to $y^* \in Y$ for some $(0 << r) \in W$ or $r = 0$ i.e., for any $\sigma > 0$ with $(0 <<) \, \sigma$ there exists a $(s,t) \in \mathbb{N}^2$ so that*

$$\lim_{u,v \to \infty} \frac{1}{uv} \left| \left\{ (s,t) \in \mathbb{N}^2 : s \le u, t \le v;\ (r + \sigma - \rho(y_{st}, y^*)) \notin intP \right\} \right| = 0.$$

Symbolically, we denote $r - st_2 - \lim y_{st} = y^*$.

Definition 3.3. *A sequence $\{y_{st}\}$ is called to be rough $\mathcal{I}_2$-convergent of roughness degree r to $y^* \in Y$ for some $r \in W$ with $0 << r$ or $r = 0$ provided that for any $(0 <<) \, \sigma \in W$, the set*

$$T(\sigma) := \left\{ (s,t) \in \mathbb{N}^2 : (r + \sigma - \rho(y_{st}, y^*)) \notin intP \right\} \in \mathcal{I}_2.$$

Symbolically, we demonstrate $y_{st} \overset{r - \mathcal{I}_2}{\to} y^*$ or $r - \mathcal{I}_2 - \lim y_{st} = y^*$.

Definition 3.4. *A sequence $y = \{y_{st}\}$ in Y is named to be rough $\mathcal{I}_2$-statistically convergent of roughness degree r to $y^* \in Y$ for some $r \in W$ with $0 << r$ or $r = 0$ provided that for any $(0 <<) \, \sigma \in W$ and for all $\kappa > 0$, the set*

$$\begin{aligned} T(\sigma) := \{ (u,v) \in \mathbb{N}^2 : \tfrac{1}{uv} \left| \{(s,t) : s \le u, t \le v; \right. \\ \left. (r + \sigma - \rho(y_{st}, y^*)) \notin intP\} \right| \ge \kappa \} \in \mathcal{I}_2. \end{aligned}$$

Symbolically, we indicate $y_{st} \overset{r - \mathcal{I}_2 - st}{\to} y^*$.

For $r = 0$ the description of rough $\mathcal{I}_2$-statistically convergence reduces to the description of $\mathcal{I}_2$-statistically convergence of sequence in an CMS. When a sequence $y = \{y_{st}\}$ is rough $\mathcal{I}_2$-statistically convergent of roughness degree r to $y^* \in Y$ then y^* is named the rough $\mathcal{I}_2$-statistical limit of $y = \{y_{st}\}$. Generally, the rough $\mathcal{I}_2$-statistical limit of a sequence $y = \{y_{st}\}$ is not unique which can be examined from the following example. As a result, the set of all rough $\mathcal{I}_2$-statistical limits of a sequence $y = \{y_{st}\}$ indicated by $\mathcal{I}_2$-st-LIM$^r y$ is known as the rough $\mathcal{I}_2$-statistical limit set of a sequence $y = \{y_{st}\}$ i.e.,

$$\mathcal{I}_2\text{-}st\text{-LIM}^r y := \left\{ y^* \in Y : y_{st} \overset{r - \mathcal{I}_2 - st}{\to} y^* \right\}.$$

Hence, a sequence $y = \{y_{st}\}$ is called to be rough $\mathcal{I}_2$-statistically convergent in an CMS when $\mathcal{I}_2$-st-LIM$^r y \neq \emptyset$.

Example 3.1. *Presume $Y = \mathbb{R}$, $W = \mathbb{R}^2$, $P = \{(u,v) \in W : u, v \ge 0\} \subset W$ and $\rho : Y \times Y \to W$ be a metric. At that time, (Y, ρ) is an CMS. Let us examine the ideal in $\mathbb{N}^2$ which consists of sets whose natural density are zero i.e., $\mathcal{I}_2 = \mathcal{I}_2^d$. Also, let us contemplate the sequence $y = \{y_{st}\}$ in Y identified by*

$$y_{st} = \begin{cases} (-1)^{s+t}, & \text{if } s \neq k^2, t \neq l^2 \text{ (where } k, l \in \mathbb{N}) \\ st, & \text{if not.} \end{cases}$$

Now, we can get that for any $r=(r_1,r_2)\in W$ *with* $0<<r$, *when* $\min(r_1,r_2)=r^*$ *and* $r^*\geq 1$ *then*

$$\mathcal{I}_2-st-LIM^r y=\left[-\left(r^*-1\right),\left(r^*-1\right)\right],$$

as for any $y^*\in\left[-\left(r^*-1\right),\left(r^*-1\right)\right]$ *with* $r^*\geq 1$ *we get*

$$\left\{(u,v)\in\mathbb{N}^2:\tfrac{1}{uv}\left|\left\{(s,t):s\leq u,t\leq v;\ \left(r+\sigma-\rho\left(y_{st},y^*\right)\right)\notin intP\right\}\right|\geq\kappa\right\}$$
$$\subset\left\{1^2,2^2,3^2,...\right\},$$

so

$$\left\{(u,v)\in\mathbb{N}^2:\frac{1}{uv}\left|\left\{(s,t):s\leq u,t\leq v;\ \left(r+\sigma-\rho\left(y_{st},y^*\right)\right)\notin intP\right\}\right|\geq\kappa\right\}\in\mathcal{I}_2,$$

as $\delta\left(\left\{1^2,2^2,3^2,...\right\}\right)=0$ *and when* $r^*<1$ *or* $r=0$ *then* $\mathcal{I}_2$*-st-LIM*$^r y=\emptyset$.

Notation 3.1. *From the above example we can examine that in general* $\mathcal{I}_2$*-st-LIM*$^r y\neq\emptyset$ *does not mean that st-LIM*$^r y\neq\emptyset$. *Hovewer, as* $\mathcal{I}_2$ *is an admissible ideal so st-LIM*$^r y\neq\emptyset$ *gives* $\mathcal{I}_2$*-st-LIM*$^r y\neq\emptyset$. *Namely, when a sequence* $y=\{y_{st}\}$ *in* (Y,ρ) *is rough statistically convergent of roughness degree* r, *where* $r\in W$ *with* $0<<r$ *or* $r=0$, *then it is also rough* $\mathcal{I}_2$*-statistically convergent of similar roughness degree* r. *Hence, when we signify all rough statistically convergence sequences in an CMS* (Y,ρ) *by st-LIM*$^r y$ *and the set of whole rough* $\mathcal{I}_2$*-statistically convergent sequences by* $\mathcal{I}_2$*-st-LIM*$^r y$, *then we obtain st-LIM*$^r y\subseteq\mathcal{I}_2$*-st-LIM*$^r y$.

A sequence $\{y_{st}\}$ in an CMS (Y,ρ) is named to be bounded when there is a $y^*\in Y$ and $r>0$ supplying $\rho\left(y_{st},y^*\right)<r$ for all $s,t\in\mathbb{N}$.

Utilizing this thought we determine $\mathcal{I}_2$-statistically bounded sequence in an CMS as follows:

Definition 3.5. *A sequence* $y=\{y_{st}\}$ *in an CMS* (Y,ρ) *is named to be* $\mathcal{I}_2$*-statistically bounded when there is a* $y^*\in Y$ *and* $Q\in W$ *with* $0<<Q$ *so that*

$$\left\{(u,v)\in\mathbb{N}^2:\frac{1}{uv}\left|\left\{(s,t):s\leq u,t\leq v;\ Q-\rho\left(y_{st},y^*\right)\notin intP\right\}\right|\geq\kappa\right\}\in\mathcal{I}_2.$$

Presume $\{y_{st}\}$ be bounded sequence in an CMS (Y,ρ), then there is a $z^*\in Y$ and $Q\in W$ with $0<<Q$ so that $\rho\left(z^*,y_{st}\right)<<Q$ for all $s,t\in\mathbb{N}$. This means that $Q-\rho\left(z^*,y_{st}\right)\in intP$ for all $s,t\in\mathbb{N}$. Therefore

$$\left\{(u,v)\in\mathbb{N}^2:\frac{1}{uv}\left|\left\{(s,t):s\leq u,t\leq v;\ Q-\rho\left(z^*,y_{st}\right)\notin intP\right\}\right|\geq\kappa\right\}=\emptyset\in\mathcal{I}_2.$$

So, $\{y_{st}\}$ is $\mathcal{I}_2$-statistically bounded. Hovewer, the converse need not to be true as examined in Example 3.1. When we select $y^*=2$ and $(0<<)\,Q=(5,6)$ then we obtain

$$\left\{(u,v)\in\mathbb{N}^2:\tfrac{1}{uv}\left|\left\{(s,t):s\leq u,t\leq v;\ Q-\rho\left(y_{st},y^*\right)\notin intP\right\}\right|\geq\kappa\right\}$$
$$\subset\left\{1^2,2^2,3^2,...\right\}$$

which gives that

$$\left\{(u,v)\in\mathbb{N}^2:\frac{1}{uv}\left|\left\{(s,t):s\leq u,t\leq v;\ Q-\rho\left(y_{st},y^*\right)\notin intP\right\}\right|\geq\kappa\right\}\in\mathcal{I}_2.$$

As a result, the sequence examined here is $\mathcal{I}_2$-statistically bounded.

Theorem 3.1. *Take* $\mathcal{I}_2$ *as an admissible ideal of* $\mathbb{N}^2$. *At that time, a sequence* $y=\{y_{st}\}$ *in an CMS* (Y,ρ) *is* $\mathcal{I}_2$*-statistically bounded iff there exists some* $r\in W$ *with* $0<<r$ *or* $r=0$ *so that* $\mathcal{I}_2$*-st-LIM*$^r y\neq\emptyset$.

Proof. Presume the sequence $y = \{y_{st}\}$ be $\mathcal{I}_2$-statistically bounded. Afterwards, there is a $y^* \in Y$ and $(0 <<)\, r \in W$ so that the set

$$\left\{(u,v) \in \mathbb{N}^2 : \frac{1}{uv} \left|\{(s,t) : s \leq u, t \leq v;\, r - \rho(y_{st}, y^*) \notin intP\}\right| \geq \kappa\right\} \in \mathcal{I}_2.$$

Take $(0 <<)\, \sigma \in W$ (i.e., $\sigma \in intP$). So

$$\begin{aligned} &\left\{(u,v) \in \mathbb{N}^2 : \tfrac{1}{uv} \left|\{(s,t) : s \leq u, t \leq v;\, r + \sigma - \rho(y_{st}, y^*) \notin intP\}\right| \geq \kappa\right\} \\ \subseteq &\left\{(u,v) \in \mathbb{N}^2 : \tfrac{1}{uv} \left|\{(s,t) : s \leq u, t \leq v;\, r - \rho(y_{st}, y^*) \notin intP\}\right| \geq \kappa\right\} \in \mathcal{I}_2. \end{aligned}$$

Let

$$(u,v) \in \left\{(u,v) \in \mathbb{N}^2 : \frac{1}{uv} \left|\{(s,t) : s \leq u, t \leq v;\, r + \sigma - \rho(y_{st}, y^*) \notin intP\}\right| \geq \kappa\right\}.$$

Then, we get $r + \sigma - \rho(y_{st}, y^*) \notin intP$. So, $r - \rho(y_{st}, y^*) \notin intP$, hence we get

$$(u,v) \in \left\{(u,v) \in \mathbb{N}^2 : \frac{1}{uv} \left|\{(s,t) : s \leq u, t \leq v;\, r - \rho(y_{st}, y^*) \notin intP\}\right| \geq \kappa\right\}.$$

As a result, we acquire $y^* \in \mathcal{I}_2$-st-$\text{LIM}^r y$.

Conversely, assume $\mathcal{I}_2$-st-$\text{LIM}^r y \neq \emptyset$ for some $r \in W$ with $0 << r$ or $r = 0$ and $z^* \in \mathcal{I}_2$-st-$\text{LIM}^r y$. So, for any $(0 <<)\, \sigma \in W$ (i.e., $\sigma \in intP$) the set

$$\left\{(u,v) \in \mathbb{N}^2 : \frac{1}{uv} \left|\{(s,t) : s \leq u, t \leq v;\, r + \sigma - \rho(y_{st}, z^*) \notin intP\}\right| \geq \kappa\right\} \in \mathcal{I}_2.$$

Now $r + \sigma \in intP$ for any $\sigma \in intP$. So getting $Q = r + \sigma \in intP$ (i.e., $0 << Q$), we get

$$\left\{(u,v) \in \mathbb{N}^2 : \frac{1}{uv} \left|\{(s,t) : s \leq u, t \leq v;\, Q - \rho(y_{st}, z^*) \notin intP\}\right| \geq \kappa\right\} \in \mathcal{I}_2.$$

As a result, $y = \{y_{st}\}$ is $\mathcal{I}_2$-statistically bounded. □

Theorem 3.2. *An $\mathcal{I}_2$-statistically bounded sequence $y = \{y_{st}\}$ in an CMS (Y, ρ) contains a subsequence that is rough $\mathcal{I}_2$-statistically convergent of roughness degree r for some $(0 <<)\, r \in W$.*

Proof. Presume a sequence $y = \{y_{st}\}$ in a CMS (Y, ρ) is $\mathcal{I}_2$-statistically bounded. So, there is a $x^* \in Y$ and $(0 <<)\, Q \in W$ so that the set

$$A = \left\{(u,v) \in \mathbb{N}^2 : \frac{1}{uv} \left|\{(s,t) : s \leq u, t \leq v;\, Q - \rho(y_{st}, x^*) \notin intP\}\right| \geq \kappa\right\} \in \mathcal{I}_2,$$

i.e.,

$$A^c = \left\{(u,v) \in \mathbb{N}^2 : \frac{1}{uv} \left|\{(s,t) : s \leq u, t \leq v;\, Q - \rho(y_{st}, x^*) \in intP\}\right| < \kappa\right\} \in \mathcal{F}(\mathcal{I}_2).$$

When we contemplate the subsequence $\{y_{st}\}_{s,t \in A^c}$ then this subsequence is statistically bounded. Since for any statistically bounded sequence $y = \{y_{st}\}$, st-$\text{LIM}^r y \neq \emptyset$ for some $(0 <<)\, r \in W$, so the subsequence $\{y_{st}\}_{s,t \in A^c}$ is rough statistically convergent of roughness degree r ($(0 <<)\, r \in W$). As a result, according to the Notation 3.1, $\{y_{st}\}_{s,t \in A^c}$ is also rough $\mathcal{I}_2$-statistically convergent with roughness degree r ($(0 <<)\, r \in W$). □

Theorem 3.3. *Take $y=\{y_{st}\}$ as a sequence in an CMS which is $\mathcal{I}_2$-statistically convergent to y^*. When $z=\{z_{st}\}$ is another sequence in (Y,ρ) so that $\rho(y_{st},z_{st})\leq r$ for some $(0<<)\, r\in W$ and for all $s,t\in\mathbb{N}$. At that time, $z=\{z_{st}\}$ is rough $\mathcal{I}_2$-statistically convergent of roughness degree r to y^*.*

Proof. Take $y=\{y_{st}\}$ as a sequence in an CMS which is $\mathcal{I}_2$-statistically convergent to y^*. For $(0<<)\,\sigma\in W$ the set

$$\left\{(u,v)\in\mathbb{N}^2:\frac{1}{uv}\left|\{(s,t):s\leq u,t\leq v;\ \sigma-\rho(y_{st},y^*)\notin intP\}\right|\geq\kappa\right\}\in\mathcal{I}_2,$$

i.e.,

$$\left\{(u,v)\in\mathbb{N}^2:\frac{1}{uv}\left|\{(s,t):s\leq u,t\leq v;\ \sigma-\rho(y_{st},y^*)\in intP\}\right|<\kappa\right\}\in\mathcal{F}(\mathcal{I}_2).$$

Also

$$\rho(z_{st},y^*)\leq\rho(z_{st},y_{st})+\rho(y_{st},y^*)<r+\rho(y_{st},y^*).$$

This means that $r+\rho(y_{st},y^*)-\rho(z_{st},y^*)\in P$. So, when $\sigma-\rho(y_{st},y^*)\in intP$, then

$$(r+\rho(y_{st},y^*)-\rho(z_{st},y^*))+(\sigma-\rho(y_{st},y^*))=r+\sigma-\rho(z_{st},y^*)\in intP.$$

Hence, the set

$$\left\{(u,v)\in\mathbb{N}^2:\frac{1}{uv}\left|\{(s,t):s\leq u,t\leq v;\ r+\sigma-\rho(z_{st},y^*)\in intP\}\right|<\kappa\right\}\in\mathcal{F}(\mathcal{I}_2).$$

As a result

$$\left\{(u,v)\in\mathbb{N}^2:\frac{1}{uv}\left|\{(s,t):s\leq u,t\leq v;\ r+\sigma-\rho(z_{st},y^*)\notin intP\}\right|\geq\kappa\right\}\in\mathcal{I}_2$$

that means $z=\{z_{st}\}$ is rough $\mathcal{I}_2$-statistically convergent of roughness degree r to y^*. □

Theorem 3.4. *Take $y=\{y_{st}\}$ as a sequence in an CMS which is rough $\mathcal{I}_2$-statistically convergent of roughness degree r for some $(0<<)\,r\in W$. Then, there does not exist x^*, $z^*\in\mathcal{I}_2$-st-LIMy so that $nr<\rho(x^*,z^*)$, where n is a real number grater than 2.*

Proof. Assume on contrary that there exist x^*, $z^*\in\mathcal{I}_2$-*st*-LIMy so that $nr<\rho(x^*,z^*)$, where $n\,(\in\mathbb{R})>2$. Presume $(0<<)\sigma$ be arbitrarily selected in W. Now as x^*, $z^*\in\mathcal{I}_2$-*st*-LIMy, so we have

$$K_1=\left\{(u,v)\in\mathbb{N}^2:\frac{1}{uv}\left|\left\{(s,t):s\leq u,t\leq v;\ r+\frac{\sigma}{2}-\rho(y_{st},x^*)\notin intP\right\}\right|\geq\kappa\right\}\in\mathcal{I}_2$$

and

$$K_2=\left\{(u,v)\in\mathbb{N}^2:\frac{1}{uv}\left|\left\{(s,t):s\leq u,t\leq v;\ r+\frac{\sigma}{2}-\rho(y_{st},z^*)\notin intP\right\}\right|\geq\kappa\right\}\in\mathcal{I}_2.$$

At that time, $K_1^c\in\mathcal{F}(\mathcal{I}_2)$ and $K_2^c\in\mathcal{F}(\mathcal{I}_2)$. Take $(m,n)\in K_1^c\cap K_2^c$. Afterwards,

$$r+\frac{\sigma}{2}-\rho(y_{mn},x^*)\in intP \text{ and } r+\frac{\sigma}{2}-\rho(y_{mn},z^*)\in intP.$$

Hence

$$\begin{aligned}&\left(r+\tfrac{\sigma}{2}-\rho(y_{mn},x^*)\right)+\left(r+\tfrac{\sigma}{2}-\rho(y_{mn},z^*)\right)\\&\quad=2r+\sigma-(\rho(y_{mn},x^*)+\rho(y_{mn},z^*))\in intP.\end{aligned}$$

Now

$$\rho(x^*, z^*) \leq \rho(y_{mn}, x^*) + \rho(y_{mn}, z^*),$$

so

$$\rho(y_{mn}, x^*) + \rho(y_{mn}, z^*) - \rho(x^*, z^*) \in P.$$

As a result, we obtain

$$\begin{aligned}&(2r + \sigma - (\rho(y_{mn}, x^*) + \rho(y_{mn}, z^*))) + (\rho(y_{mn}, x^*) + \rho(y_{mn}, z^*) - \rho(x^*, z^*))\\&= 2r + \sigma - \rho(x^*, z^*) \in intP.\end{aligned}$$

Again by our presumption $\rho(x^*, z^*) - nr \in P$. Hence

$$2r + \sigma - \rho(x^*, z^*) + \rho(x^*, z^*) - nr = 2r + \sigma - nr \in intP.$$

Namely $\sigma - r(n-2) \in intP$. Hovewer, selecting $\sigma = r(n-2)$, we acquire $0 \in intP$, which is a contradiction. So, the result finalizes. □

Theorem 3.5. *Suppose $y = \{y_{st}\}$ be a sequence in an CMS which is rough $\mathcal{I}_2$-statistically convergent of roughness degree r. At that time, $\{y_{st}\}$ is also rough $\mathcal{I}_2$-statistically convergent of roughness degree r_1 for any r_1 with $r < r_1$.*

Proof. The proof is trivial and hence is omitted. □

In the light of previous theorem we get the following corollary.

Corollary 3.6. *Assume $y = \{y_{st}\}$ be a rough $\mathcal{I}_2$-statistically convergent sequence in (Y, ρ) of roughness degree r. At that time, for a $(0 <<) r_1$ with $r < r_1$, $LIM^r y \subset LIM^{r_1} y$.*

Definition 3.6. *An element $\gamma \in Y$ is named to be $\mathcal{I}_2$-statistical cluster point of a double sequence $y = \{y_{st}\}$ in Y provided that for any $(0 <<)\sigma$, the set*

$$\left\{(u, v) \in \mathbb{N}^2 : \frac{1}{uv} |\{(s, t) : s \leq u, t \leq v; \sigma - \rho(y_{st}, \gamma) \in intP\}| < \kappa\right\} \notin \mathcal{I}_2.$$

Theorem 3.7. *Take (Y, ρ) as an CMS. $\gamma \in Y$ and $(0 <<)r$ be such that for any $y^* \in Y$ either $\rho(y^*, \gamma) \leq r$ or $r << \rho(y^*, \gamma)$. When γ is $\mathcal{I}_2$-statistical cluster point of a double sequence $y = \{y_{st}\}$ then $\mathcal{I}_2$-st-$LIM^r y \subset \overline{B_r(\gamma)}$, where $\overline{B_r(\gamma)} = \{y^* \in Y : \rho(y^*, \gamma) \leq r\}$.*

Proof. If possible, presume that there is a $x^* \in \mathcal{I}_2$-st-LIM$^r y$ but $x^* \notin \overline{B_r(\gamma)}$. Now according to our supposition, $r << \rho(x^*, \gamma)$. Take $(0 <<)\sigma_1 = \rho(x^*, \gamma) - r$. At that time, $\rho(x^*, \gamma) = r + \sigma_1$. Assume $(0 <<)\sigma = \frac{\sigma_1}{2}$. Then, we get $\rho(x^*, \gamma) = r + 2\sigma$. In addition, we get $B_{r+\sigma}(x^*) \cap B_\sigma(\gamma) = \emptyset$. For, if $\alpha \in B_{r+\sigma}(x^*) \cap B_\sigma(\gamma)$ then $\rho(\alpha, x^*) << r + \sigma$ and $\rho(\alpha, \gamma) << \sigma$. So $r + \sigma - \rho(\alpha, x^*) \in intP$ and $\sigma - \rho(\alpha, \gamma) \in intP$. Hence

$$(r + \sigma - \rho(\alpha, x^*)) + (\sigma - \rho(\alpha, \gamma)) = r + 2\sigma - (\rho(\alpha, x^*) + \rho(\alpha, \gamma)) \in intP. \quad (3.1)$$

Since $\rho(x^*, \gamma) \leq \rho(x^*, \alpha) + \rho(\alpha, \gamma)$, therefore

$$\rho(x^*, \alpha) + \rho(\alpha, \gamma) - \rho(x^*, \gamma) \in P. \quad (3.2)$$

As a result from (3.1) and (3.2) we obtain

$$\begin{aligned}&r + 2\sigma - (\rho(\alpha, x^*) + \rho(\alpha, \gamma)) + \rho(x^*, \alpha) + \rho(\alpha, \gamma) - \rho(x^*, \gamma)\\&= r + 2\sigma - \rho(x^*, \gamma) = 0 \in intP,\end{aligned}$$

a contradiction. Hence $B_{r+\sigma}(x^*) \cap B_\sigma(\gamma) = \emptyset$. Since $x^* \in \mathcal{I}_2$-st-LIM$^r y$, so the set

$$A = \left\{(u, v) \in \mathbb{N}^2 : \frac{1}{uv} |\{(s, t) : s \leq u, t \leq v; r + \sigma - \rho(y_{st}, x^*) \notin intP\}| \geq \kappa\right\} \in \mathcal{I}_2.$$

So, $A^c = \mathbb{N}^2 \diagdown A \in \mathcal{F}(\mathcal{I}_2)$. Again as γ is a $\mathcal{I}_2$-statistical cluster point of $y = \{y_{st}\}$, so for $(0 <<)\sigma$

$$B = \left\{(u,v) \in \mathbb{N}^2 : \frac{1}{uv} \left|\{(s,t) : s \leq u, t \leq v;\ \sigma - \rho(y_{st}, \gamma) \in intP\}\right| < \kappa\right\} \notin \mathcal{I}_2.$$

It is obvious that B can not be a subset of A. For, if

$$\left\{(u,v) \in \mathbb{N}^2 : \frac{1}{uv} \left|\{(s,t) : s \leq u, t \leq v;\ \sigma - \rho(y_{st}, \gamma) \in intP\}\right| < \kappa\right\} \subset A$$

then we obtain

$$\left\{(u,v) \in \mathbb{N}^2 : \frac{1}{uv} \left|\{(s,t) : s \leq u, t \leq v;\ \sigma - \rho(y_{st}, \gamma) \in intP\}\right| < \kappa\right\} \in \mathcal{I}_2,$$

which contradicts the fact that γ is a $\mathcal{I}_2$-statistical cluster point of $y = \{y_{st}\}$. We contemplate an elemant $(k,l) \in A^c$. So

$$(k,l) \in \left\{(u,v) \in \mathbb{N}^2 : \frac{1}{uv} \left|\{(s,t) : s \leq u, t \leq v;\ \sigma - \rho(y_{st}, \gamma) \in intP\}\right| < \kappa\right\}.$$

Now, $(k,l) \in A^c$ means $r + \sigma - \rho(y_{kl}, x^*) \in intP$. Hence, $\rho(y_{kl}, x^*) << r + \sigma$, which implies $\{y_{kl}\} \in B_{r+\sigma}(x^*)$. Additionally

$$(k,l) \in \left\{(u,v) \in \mathbb{N}^2 : \frac{1}{uv} \left|\{(s,t) : s \leq u, t \leq v;\ \sigma - \rho(y_{st}, \gamma) \in intP\}\right| < \kappa\right\}$$

implies $\sigma - \rho(y_{kl}, \gamma) \in intP$. So $\rho(y_{kl}, \gamma) << \sigma$ which further means that $\{y_{kl}\} \in B_r(\gamma)$. As a result, we obtain $\{y_{kl}\} \in B_{r+\sigma}(x^*) \cap B_r(\gamma)$ which is a contradiction. As a result, we can conclude that our presumption is wrong and $x^* \in \overline{B_r(\gamma)}$. □

Theorem 3.8. *Assume $y = \{y_{st}\}$ be a rough $\mathcal{I}_2$-statistically convergence of roughness degree r in an CMS (Y, ρ) and $q = \{q_{st}\}$ be a $\mathcal{I}_2$-statistically convergent sequence in $\mathcal{I}_2$-st-LIM$^r y$ which is $\mathcal{I}_2$-statistically convergent to x^*. Then $x^* \in \mathcal{I}_2$-st-LIM$^r y$.*

Proof. Suppose $(0 <<)\sigma$ be taken. As the sequence $q = \{q_{st}\}$ is $\mathcal{I}_2$-statistically convergent to x^*, for $(0 <<)\sigma$ the set

$$A = \left\{(u,v) \in \mathbb{N}^2 : \frac{1}{uv} \left|\left\{(s,t) : s \leq u, t \leq v;\ \frac{\sigma}{2} - \rho(q_{st}, x^*) \notin intP\right\}\right| \geq \kappa\right\} \in \mathcal{I}_2.$$

So, $A^c = \mathbb{N}^2 \diagdown A \in \mathcal{F}(\mathcal{I}_2)$. Select a $(k,l) \in A^c$. Then $\frac{\sigma}{2} - \rho(q_{kl}, x^*) \in intP$, and hence

$$\rho(q_{kl}, x^*) << \frac{\sigma}{2}. \tag{3.3}$$

In addition, as $q = \{q_{st}\}$ is a sequence in $\mathcal{I}_2$-st-LIM$^r y$, take $q_{kl} \in \mathcal{I}_2$-st-LIM^r. So, the set

$$B = \left\{(u,v) \in \mathbb{N}^2 : \frac{1}{uv} \left|\left\{(s,t) : s \leq u, t \leq v;\ r + \frac{\sigma}{2} - \rho(y_{st}, q_{kl}) \notin intP\right\}\right| \geq \kappa\right\} \in \mathcal{I}_2.$$

It is clear that its complement $B^c = \mathbb{N}^2 \diagdown B \in \mathcal{F}(\mathcal{I}_2)$. Let us select an element $(h,j) \in B^c\,(\in \mathcal{F}(\mathcal{I}_2))$. So, $r + \frac{\sigma}{2} - \rho(y_{hj}, q_{kl}) \in intP$, and hence

$$\rho(y_{hj}, q_{kl}) << r + \frac{\sigma}{2}. \tag{3.4}$$

Also for all $(s,t) \in \mathbb{N}^2$ we get

$$\rho(y_{st}, x^*) \leq \rho(y_{st}, q_{kl}) + \rho(q_{kl}, x^*).$$

So

$$\rho(y_{st}, q_{kl}) + \rho(q_{kl}, x^*) - \rho(y_{st}, x^*) \in P, \text{ for all } (s,t) \in \mathbb{N}^2.$$

Especially

$$\rho(y_{hj}, q_{kl}) + \rho(q_{kl}, x^*) - \rho(y_{hj}, x^*) \in P. \tag{3.5}$$

According to (3.3) and (3.4) utilizing the Theorem 2.3 we obtain

$$\left(\frac{\sigma}{2} - \rho(q_{kl}, x^*)\right) + \left(r + \frac{\sigma}{2} - \rho(y_{hj}, q_{kl})\right) = r + \sigma - (\rho(q_{kl}, x^*) + \rho(y_{hj}, q_{kl})) \in intP \tag{3.6}$$

Applying again the Theorem 2.3, we get from (3.5) and (3.6)

$$\begin{aligned}&(\rho(y_{hj}, q_{kl}) + \rho(q_{kl}, x^*) - \rho(y_{hj}, x^*)) + (r + \sigma - (\rho(q_{kl}, x^*) + \rho(y_{hj}, q_{kl})))\\ &= r + \sigma - \rho(y_{hj}, x^*) \in intP.\end{aligned}$$

Now as $\mathcal{I}_2$ is selected arbitrarily from B^c, we have

$$\left\{(u,v) \in \mathbb{N}^2 : \frac{1}{uv}\left|\{(h,j) : h \leq u, j \leq v;\ r + \sigma - \rho(y_{hj}, x^*) \notin intP\}\right| \geq \kappa\right\} \subset B$$

and so

$$\left\{(u,v) \in \mathbb{N}^2 : \frac{1}{uv}\left|\{(h,j) : h \leq u, j \leq v;\ r + \sigma - \rho(y_{hj}, x^*) \notin intP\}\right| \geq \kappa\right\} \in \mathcal{I}_2.$$

As a result $x^* \in \mathcal{I}_2$-st-$\text{LIM}^r y$. □

Theorem 3.9. *When $y = \{y_{st}\}$ and $q = \{q_{st}\}$ are two sequences in an CMS (Y, ρ) so that for any $(0 <<)\sigma$ the set*

$$\left\{(u,v) \in \mathbb{N}^2 : \frac{1}{uv}\left|\{(s,t) : s \leq u, t \leq v;\ \rho(y_{st}, q_{st}) > \sigma\}\right| \geq \kappa\right\} \in \mathcal{I}_2.$$

Then, $y = \{y_{st}\}$ is rough $\mathcal{I}_2$-statistically convergent of roughness degree r to x^* iff $q = \{q_{st}\}$ is rough $\mathcal{I}_2$-statistically convergent of roughness degree r to x^*.

Proof. Assume $y = \{y_{st}\}$ be rough $\mathcal{I}_2$-statistically convergent of roughness degree r to x^*. Presume $(0 <<)\sigma$ given. Then, we obtain

$$A = \left\{(u,v) \in \mathbb{N}^2 : \frac{1}{uv}\left|\left\{(s,t) : s \leq u, t \leq v;\ r + \frac{\sigma}{2} - \rho(y_{st}, x^*) \notin intP\right\}\right| \geq \kappa\right\} \in \mathcal{I}_2.$$

Also, by our assumption we get

$$B = \left\{(u,v) \in \mathbb{N}^2 : \frac{1}{uv}\left|\left\{(s,t) : s \leq u, t \leq v;\ \rho(y_{st}, q_{st}) > \frac{\sigma}{2}\right\}\right| \geq \kappa\right\} \in \mathcal{I}_2.$$

$A^c, B^c \in \mathcal{F}(\mathcal{I}_2)$ and hence $A^c \cap B^c \in \mathcal{F}(\mathcal{I}_2)$. Let us select an element $(k,l) \in \mathbb{N}^2$ so that $(k,l) \in A^c \cap B^c$. So

$$r + \frac{\sigma}{2} - \rho(y_{kl}, x^*) \in intP \text{ and } \rho(y_{kl}, q_{kl}) \leq \frac{\sigma}{2} \text{ i.e., } \frac{\sigma}{2} - \rho(y_{kl}, q_{kl}) \in P.$$

Therefore

$$\left(r + \frac{\sigma}{2} - \rho(y_{kl}, x^*)\right) + \left(\frac{\sigma}{2} - \rho(y_{kl}, q_{kl})\right) = r + \sigma - (\rho(y_{kl}, x^*) + \rho(y_{kl}, q_{kl})) \in intP. \tag{3.7}$$

In addition for all $(s,t) \in \mathbb{N}^2$,

$$\rho(q_{st}, x^*) \leq \rho(y_{st}, q_{st}) + \rho(y_{st}, x^*)$$

i.e.,

$$\rho(y_{st}, q_{st}) + \rho(y_{st}, x^*) - \rho(q_{st}, x^*) \in P.$$

Especially

$$\rho(y_{kl}, q_{kl}) + \rho(y_{kl}, x^*) - \rho(q_{kl}, x^*) \in P. \tag{3.8}$$

So from (3.7) and (3.8) we obtain

$$\begin{aligned}&(r + \sigma - (\rho(y_{kl}, x^*) + \rho(y_{kl}, q_{kl}))) + (\rho(y_{kl}, q_{kl}) + \rho(y_{kl}, x^*) - \rho(q_{kl}, x^*))\\ &= r + \sigma - \rho(q_{kl}, x^*) \in intP.\end{aligned}$$

As a result, we get

$$\left\{(u,v) \in \mathbb{N}^2 : \frac{1}{uv} \left|\{(s,t) : s \leq u, t \leq v;\ r + \sigma - \rho(q_{st}, x^*) \notin intP\}\right| \geq \kappa\right\} \in \mathcal{I}_2,$$

which means that $q = \{q_{st}\}$ is rough $\mathcal{I}_2$-statistically convergent of roughness degree r to x^*. □

Definition 3.7. *A sequence $y = \{y_{st}\}$ in an CMS (Y, ρ) is named to be rough $\mathcal{I}_2^*$-statistical convergent of roughness degree r to x^* provided that there is a set $L \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N}^2 \setminus L \in \mathcal{I}_2$) so that the subsequence $\{y_{st}\}_{(s,t)\in L}$ is rough statistically convergent of roughness degree r to x^* for some $(0 << r) \in W$ or $r = 0$ i.e., for any $\sigma > 0$ with $(0 <<)\, \sigma$ there exists a $(s,t) \in \mathbb{N}^2$ such that*

$$\lim_{u,v\to\infty} \frac{1}{uv} \left|\{(s,t) : s \leq u, t \leq v;\ r + \sigma - \rho(y_{st}, y^*) \notin intP\}\right| = 0.$$

We write $\mathcal{I}_2^* - st - \lim_{s,t\to\infty} y_{st} = x^*$.

Notation 3.2. *For $r = 0$ we get the definition of ordinary $\mathcal{I}_2^*$-statistical convergence of sequences in CMS. Obviously the rough $\mathcal{I}_2^*$-statistical limit of a sequence in general not unique. We can indicate the set of all rough $\mathcal{I}_2^*$-statistical limit of a sequence $y = \{y_{st}\}$ by*

$$\mathcal{I}_2^*\text{-}st\text{-}LIM^r y := \left\{y^* \in Y : y_{st} \overset{r-\mathcal{I}_2^*-st}{\to} y^*\right\}.$$

of roughness degree r.

Theorem 3.10. *When a sequence $y = \{y_{st}\}$ is rough $\mathcal{I}_2^*$-statistical convergent of roughness degree r to x^* then it is also rough $\mathcal{I}_2$-statistical convergent of roughness degree r to x^*.*

Proof. Let us presume that $\mathcal{I}_2^* - st - \lim_{s,t\to\infty} y_{st} = x^*$. So, according to the definition there is a set $L \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $Z = \mathbb{N}^2 \setminus L \in \mathcal{I}_2$) so that the subsequence $\{y_{st}\}_{(s,t)\in L}$ is rough statistically convergent of roughness degree r to x^* for some $(0 << r) \in W$ or $r = 0$ i.e., for any $\sigma > 0$ with $(0 <<)\, \sigma$ there exists a $(s,t) \in \mathbb{N}^2$ such that

$$\lim_{u,v\to\infty} \frac{1}{uv} \left|\{(s,t) : s \leq u, t \leq v;\ (r + \sigma - \rho(y_{st}, x^*)) \notin intP\}\right| = 0.$$

Then there is $n_0 \in \mathbb{N}$ such that $\rho(y_{st}, y^*) \ll r + \sigma$ then for all s, t such that $(s,t) \in L$ and $s, t \geq n_0$. Then

$$\begin{aligned}A(\sigma, \gamma) = &\left\{(u,v) \in \mathbb{N}^2 : \frac{1}{uv} \left|\{(s,t) : s \leq u, t \leq v;\ (r + \sigma - \rho(y_{st}, x^*)) \notin intP\}\right| \gg \kappa\right\}\\ &\subset Z \cup (L \cap ((\{1, 2, ..., (n_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, ..., (n_0 - 1)\}))).\end{aligned}$$

Now

$$Z \cup (L \cap ((\{1, 2, ..., (n_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, ..., (n_0 - 1)\}))) \in \mathcal{I}_2.$$

This indicates that $A(\sigma,\gamma)\in\mathcal{I}_2$. Therefore $\mathcal{I}_2-st-\lim_{s,t\to\infty} y_{st}=x^*$. □

Theorem 3.11. *When an ideal $\mathcal{I}_2$ has the property (AP2) then a sequence $y=\{y_{st}\}$ in an CMS (Y,ρ) which is rough $\mathcal{I}_2$-statistical convergent of roughness degree r to x^* is also rough $\mathcal{I}_2^*$-statistical convergent of roughness degree r to x^*.*

Proof. Assume $\mathcal{I}_2$ be an ideal in $\mathbb{N}^2$ which supply the property (AP2). Take a sequence $y=\{y_{st}\}$ be rough $\mathcal{I}_2$-statistical convergent of roughness degree r to x^*. Then, for any $(0<<)\,\sigma\in W$ and for all $\kappa>0$, the set

$$T:=\left\{(u,v)\in\mathbb{N}^2:\frac{1}{uv}\left|\{(s,t):s\leq u,t\leq v;\ (r+\sigma-\rho(y_{st},x^*))\notin intP\}\right|\geq\kappa\right\}\in\mathcal{I}_2.$$

So, we obtain

$$T^c:=\{(u,v)\in\mathbb{N}^2:\tfrac{1}{uv}\left|\{(s,t):s\leq u,t\leq v;\ (r+\sigma-\rho(y_{st},x^*))\in intP\}\right|<\kappa\}\in\mathcal{F}(\mathcal{I}_2).$$

Take $(0<<)\,\eta\in W$. Now determine

$$A_i=\left\{(u,v)\in\mathbb{N}^2:\frac{1}{uv}\left|\left\{(s,t):s\leq u,t\leq v;\rho(y_{st},x^*)<<r+\frac{\eta}{i}\right\}\right|<\kappa\right\}\in\mathcal{F}(\mathcal{I}_2),$$

where $i=1,2,\ldots$. As $\mathcal{I}_2$ has the feature (AP2), so there is a set $B\subset\mathbb{N}$ so that $B\in\mathcal{F}(\mathcal{I}_2)$ and $B\diagdown A_i$ is finite for $i=1,2,\ldots$. Now take $(0<<)\,\sigma\in W$, then there is a $j\in\mathbb{N}$ so that $\frac{\eta}{j}<<\sigma$. Since $B\diagdown A_j$ is finite, so there is a $t=t(j)\in\mathbb{N}$ so that $(u,v)\in B\cap A_j$ for all $u,v\geq t$. Hence $\rho(y_{st},x^*)<<r+\frac{\eta}{j}<<r+\sigma$ for all $(u,v)\in B$ and $u,v\geq t$. As a result, the subsequence $\{y_{st}\}_{s,t\in B}$ is rough statistically convergent of roughness degree r to x^*, i.e,

$$\lim_{u,v\to\infty}\frac{1}{uv}\left|\{(s,t):s\leq u,t\leq v;(r+\sigma-\rho(y_{st},x^*))\notin intP\}\right|=0.$$

Hence, the sequence $y=\{y_{st}\}$ is rough $\mathcal{I}_2^*$-statistical convergent of roughness degree r to x^*. □

Theorem 3.12. *If $y'=\{y_{s_pt_q}\}_{p,q\in\mathbb{N}}$ be a subsequence of the sequence $y=\{y_{st}\}$, then $\mathcal{I}_2$-st-LIM$^r y\subset\mathcal{I}_2$-st-LIM$^r y'$.*

Proof. If possible assume $x^*\in\mathcal{I}_2$-st-LIM$^r y$. Then, for any $(0<<)\,\sigma\in W$ and for all $\kappa>0$, the set

$$T:=\left\{(u,v)\in\mathbb{N}^2:\frac{1}{uv}\left|\{(s,t):s\leq u,t\leq v;\ (r+\sigma-\rho(y_{st},x^*))\notin intP\}\right|\geq\kappa\right\}\in\mathcal{I}_2.$$

Now for the subsequence $y'=\{y_{s_pt_q}\}_{p,q\in\mathbb{N}}$, since

$$\{(u,v)\in\mathbb{N}^2:\tfrac{1}{uv}\left|\{(s_p,t_q):s_p\leq u,t_q\leq v;\ (r+\sigma-\rho(y_{s_pt_q},x^*))\notin intP\}\right|\geq\kappa\}$$
$$\subset\{(u,v)\in\mathbb{N}^2:\tfrac{1}{uv}\left|\{(s,t):s\leq u,t\leq v;\ (r+\sigma-\rho(y_{st},x^*))\notin intP\}\right|\geq\kappa\}$$

and

$$\left\{(u,v)\in\mathbb{N}^2:\frac{1}{uv}\left|\{(s_p,t_q):s_p\leq u,t_q\leq v;\ (r+\sigma-\rho(y_{st},x^*))\notin intP\}\right|\geq\kappa\right\}\in\mathcal{I}_2,$$

so

$$\left\{(u,v)\in\mathbb{N}^2:\frac{1}{uv}\left|\{(s,t):s\leq u,t\leq v;\ (r+\sigma-\rho(y_{s_pt_q},x^*))\notin intP\}\right|\geq\kappa\right\}\in\mathcal{I}_2.$$

Hence, the set

$$W := \{(u,v) \in \mathbb{N}^2 : \tfrac{1}{uv} |\{(s_p,t_q) : s_p \leq u, t_q \leq v; \\ (r+\sigma-\rho(y_{st},x^*)) \in intP\}| < \kappa\} \in \mathcal{F}(\mathcal{I}_2).$$

Take $\{y_{s_pt_q}\}_{s_p,t_q \in W}$. At that time, we get

$$\lim_{u,v\to\infty} \frac{1}{uv} \left|\left\{(s,t) : s \leq u, t \leq v; r+\sigma-\rho\left(y_{s_pt_q},x^*\right) \notin intP\right\}\right| = 0.$$

and so $y' = \{y_{s_pt_q}\}$ is rough statistically convergent of roughness degree r to x^*. Therefore, the subsequence $y' = \{y_{s_pt_q}\}$ is rough $\mathcal{I}_2^*$-statistical convergent of roughness degree r to x^*. So, we obtain $x^* \in \mathcal{I}_2$-st-$\text{LIM}^r y'$. As a result, we get $\mathcal{I}_2$-st-$\text{LIM}^r y \subset \mathcal{I}_2$-$st$-$\text{LIM}^r y'$. □

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