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FIXED POINT THEOREMS IN STRONG PARTIAL B-METRIC SPACES

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Abstract. The objective of this research is to provide some fixed point findings for kannan type contractions in strong partial b-metric spaces (SPbMS), which is a generalisation of strong b-metric space and metric space. Exmaples are also given to support our results.

1. Introduction

Many fields of mathematics use the Banach contraction principle [7] as a common tool to solve existence issues (for eg, [13],[21]). Extensions of this concept were achieved by either expanding the mapping's domain or the contractive condition on the mappings [[5],[9],[17],[25],[28]]. Ran and Reurings [26] are the first to examine and apply the presence of fixed points in ordered metric spaces. After that, a number of findings have been supported by evidence in the context of ordered metric spaces ([1], [3],[4],[10],[16],[18],[24],[27]). As an extension of metric spaces, Bakhtin [6] and Czerwik [11] invented b-metric spaces. They developed the contraction mapping principle in b-metric spaces, which expanded on the well-known Banach contraction principle in those spaces. After that, a number of studies have addressed fixed point theory in b-metric spaces for single-valued and multivalued operators ([2],[8],[12][14],[15],[19],[20]). In 2021, Doan [14] proved some generalisation of fixed point results for single valued and multivalued mappings on complete strong b-metric spaces. In [23], the notion of SPbMSs was introduced. They also discussed the relationship between strong b-metric and SPbMSs.
In this paper, we prove some fixed point results for kannan type contractions in SPbMS, which is generalisation of strong b-metric space and metric space. Some examples are also included to support the results.

2. Preliminaries

Here, we provide the relevant definitions and findings for different spaces that will be helpful for further explanation.

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Definition 2.1. [22] *"A partial metric on a set E is a function $d : E \times E \to \mathbb{R}_0^+$ such that for all $u, v, \mathfrak{c} \in E$, the following conditions hold:*
(PM1) $u = v \Leftrightarrow d(u,u) = d(v,v) = d(u,v)$;
(PM2) $d(u,u) \leq d(u,v)$;
(PM3) $d(u,v) = d(v,u)$;
(PM4) $d(u,v) \leq d(u,\mathfrak{c}) + d(\mathfrak{c},v) - d(\mathfrak{c},\mathfrak{c})$.
Then (E,d) is called a partial metric space."

Definition 2.2. [20] *"A map $d : E \times E \to \mathbb{R}_0^+$ is a strong b-metric on a non empty set E if for all $u, v, \mathfrak{c} \in E$ and $\alpha \geq 1$ the following conditions hold:*
(SB1) $u = v$ *iff* $d(u,v) = 0$;
(SB2) $d(u,v) = d(v,u)$;
(SB3) $d(u,v) \leq d(u,\mathfrak{c}) + \alpha d(\mathfrak{c},v)$.
The triple (E,d,α) is called a strong b-metric space."

Definition 2.3. [23] *"A map $d : E \times E \to \mathbb{R}_0^+$ is a strong partial b-metric on a non empty set E if for all $u, v, \mathfrak{c} \in E$ and $\alpha \geq 1$ the following conditions hold:*
(SPbMS1) $u = v \Leftrightarrow d(u,u) = d(v,v) = d(u,v)$;
(SPbMS2) $d(u,u) \leq d(u,v)$;
(SPbMS3) $d(u,v) = d(v,u)$;
(SPbMS4) $d(u,v) \leq d(u,\mathfrak{c}) + \alpha d(\mathfrak{c},v) - d(\mathfrak{c},\mathfrak{c})$.
The triple (E,d,α) is called a Strong Partial b-Metric Space (SPbMS)."

***Remark* 2.4.** [23] *"Every metric space is a strong b-metric space but converse is not neccessarily true. Every strong b-metric space is a SPbMS but not conversely."*

Definition 2.5. [23] *" Let (E,d,α) be a SPbMS. Then*
(i) A sequence $\{u_n\}$ in (E,d,α) converges to a point $u \in E$ if $d(u,u) = \lim_n d(u_n,u) = \lim_n d(u_n,u_n)$.
(ii) A sequence $\{u_n\}$ in (E,d,α) is Cauchy if the $\lim_{n,m} d(u_n,u_m)$ *exists and finite."*

3. Main Results

Let $\mathcal{F}$ be the class of functions which satisfy

$$\mathcal{F} = f : (0,\infty) \to [0,\frac{1}{2}) : f(z_n) \to \frac{1}{2} \implies z_n \to 0 \text{ as } n \to \infty.$$

Theorem 3.1. Let (E,d,α) be a complete SPbMS and $S : E \to E$ be a self map. Suppose, $\exists\ f \in \mathcal{F}$ such that for all $u, v \in E$, with $u \neq v$,

$$d(Su,Sv) \leq f(d(u,v))\{d(u,Su) + d(v,Sv)\}. \tag{3.1}$$

Then, S has a unique fixed point in E.

Proof. Let $u_0 \in E$ be any arbitrary point. We introduce a sequence $\{u_n\}$ in E,

$$u_{n+1} = Su_n\ \forall\ n \geq 0.$$

Suppose, there exists $n \geq 0$ such that $u_{n+1} = u_n$, then obviously u_n is a fixed point of S. So, assume that $u_{n+1} \neq u_n\ \forall\ n \geq 0$.

Now, define $D_n = d(u_{n+1}, u_n)\ \forall\ n \geq 0$. By inequality (3.1), we get

$$\begin{aligned} D_{n+1} &= d(u_{n+2}, u_{n+1}) \\ &= d(Su_{n+1}, Su_n) \\ &\leq f(d(u_{n+1}, u_n))\{d(u_{n+1}, Su_{n+1}) + d(u_n, Su_n\} \\ &< \frac{1}{2}\{d(u_{n+1}, Su_{n+1}) + d(u_n, Su_n\} \\ &= \frac{1}{2}\{D_n + D_{n+1}\}. \end{aligned}$$

Clearly, $D_{n+1} < D_n\ \forall\ n \geq 0$. Hence, $\{D_n\}$ is a monotonically decreasing and bounded below sequence. So, $\exists\ \beta \geq 0$ so as

$$\lim_{n\to\infty} D_n = \beta.$$

Now, assume $\beta > 0$. Then, by inequality (3.1), we get

$$d(u_{n+2}, u_{n+1}) \leq f(d(u_{n+1}, u_n))\{d(u_{n+1}, u_{n+2}) + d(u_n, u_{n+1})\};$$

that is

$$D_{n+1} \leq f(D_n)\{D_{n+1} + D_n\}.$$

That implies

$$\frac{D_{n+1}}{D_{n+1} + D_n} \leq f(D_n)\ \forall\ n \geq 0.$$

Applying $n \to \infty$, we get $\frac{1}{2} \leq \lim_{n\to\infty} f(D_n)$, but $\frac{1}{2} > \lim_{n\to\infty} f(D_n)$, because $f \in \mathcal{F}$. Which is a contradiction. So, $\lim_{n\to\infty}(D_n) = \beta = 0$.

We demonstrate that $\{u_n\}$ is a Cauchy sequence in E. Let $m < n$. So, by inequality (3.1), we get

$$\begin{aligned} d(u_{m+1}, u_{n+1}) &\leq f(d(u_m, u_n))\{d(u_m, Su_m) + d(u_n, Su_n)\} \\ &\leq \frac{1}{2}\{d(u_m, u_{m+1}) + d(u_n, u_{n+1})\}. \end{aligned}$$

As $m, n \to \infty, d(u_m, u_{m+1})$ and $d(u_n, u_{n+1}) \to 0$. So, $d(u_{m+1}, u_{n+1}) \to 0$ as $n \to \infty$.

So, $\{u_n\}$ is a cauchy sequence. Now, $\{S^n u_0\}$ is a Cauchy sequence and by hypothesis E is complete. So, $\exists\ u^* \in E$ such that

$$\lim_{n\to\infty} S^n u_0 = u^*.$$

Now, by (SPbMS4)

$$\begin{aligned} d(Su^*, u^*) &\leq d(Su^*, Su_n) + \alpha d(Su_n, u^*) - d(Su_n, Su_n) \\ &\leq f(d(u^*, u_n))\{d(u^*, Su^*) + d(u_n, Su_n)\} + \alpha d(u_{n+1}, u^*) - d(u_{n+1}, u_{n+1}) \\ &\leq f(d(u^*, u_n))\{d(u^*, Su^*) + d(u_n, Su_n)\} + \alpha d(u_{n+1}, u^*). \end{aligned}$$

So,

$$d(Su^*, u^*)(1 - f(d(u^*, u_n))) \leq f(d(u^*, u_n))d(u_n, Su_n) + \alpha d(u_{n+1}, u^*),$$

which implies

$$d(Su^*, u^*) \leq \frac{f(d(u^*, u_n))}{1 - f(d(u^*, u_n))} D_n + \frac{\alpha}{1 - f(d(u^*, u_n))} d(u_{n+1}, u^*) \tag{3.2}$$

As $n \to \infty$ right hand side of (3.2) is zero. So,

$$d(Su^*, u^*) = 0. \tag{3.3}$$

Now, by (SPbMS2) $d(Su^*, Su^*) \leq d(Su^*, u^*)$.

Since, $S : E \times E \to [0, \infty)$ and $d(Su^*, u^*) = 0$. So, $d(Su^*, Su^*) = 0$. Similarly, we can show that $d(u^*, u^*) = 0$. Thus, we get $d(u^*, u^*) = d(Su^*, u^*) = d(Su^*, Su^*)$.

So, by (SPbMS1) $Su^* = u^*$. Hence $u^* \in E$ is a fixed point of S.
Uniqueness: If possible, let v^* be any other fixed point of S. So, $Sv^* = v^*$.
Using inequality (3.1), we get

$$\begin{aligned} d(u^*, v^*) &= d(Su^*, Sv^*) \\ &\leq f(d(u^*, v^*))\{d(u^*, Su^*) + d(v^*, Sv^*)\}. \end{aligned}$$

By using equation (3.3), we have $d(u^*, v^*) = 0$.
Now, $d(u^*, u^*) = d(v^*, v^*) = 0$. [$\because d(u^*, u^*) \leq d(u^*, v^*)$ and $d(v^*, v^*) \leq d(u^*, v^*)$.]
So, $d(u^*, u^*) = d(u^*, v^*) = d(v^*, v^*)$. Hence $u^* = v^*$. Thus, S has unique fixed point $u^* \in E$. □

Corollary 3.2. [14] "Let (E, d, α) be a complete strong b-metric space and $S : E \to E$ be a self map. Suppose, $\exists\ f \in \mathcal{F}$ such that for all $u, v \in E$ with $u \neq v$,

$$d(Su, Sv) \leq f(d(u, v))\{d(u, Su) + d(v, Sv)\}.$$

Then, S has a unique fixed point in E and for any $u \in E$, the sequence of iterates $\{S^n(u)\}$ converges to u^*."

Corollary 3.3. [15] "Let (E, d, α) be a complete metric space and $S : E \to E$ be a self map. Suppose, $\exists\ f \in \mathcal{F}$ such that for all $u, v \in E$ with $u \neq v$,

$$d(Su, Sv) \leq f(d(u, v))\{d(u, Su) + d(v, Sv)\}.$$

Then, S has a unique fixed point in E and for any $u \in E$, the sequence of iterates $\{S^n(u)\}$ converges to u^*."

***Example* 3.4.** Let $E = \{0, 1, 2\}$ and $d : E \times E \to [0, \infty)$ be defined by

$$\begin{aligned} &d(0,0) = d(2,2) = 0, d(1,1) = \frac{1}{4}, \\ &d(1,0) = \frac{1}{2} = d(0,1), \\ &d(1,2) = 6 = d(2,1), \\ &d(2,0) = 8 = d(0,2). \end{aligned}$$

Here $d(u, u) \leq d(u, v)\ \forall\ u, v \in E$. And

$$\begin{aligned} &d(0,1) \leq d(0,2) + \alpha d(2,1) - d(2,2),\ \forall\ \alpha \geq 1, \\ &d(1,0) \leq d(1,2) + \alpha d(2,0) - d(2,2),\ \forall\ \alpha \geq 1, \\ &d(0,2) \leq d(0,1) + \alpha d(1,2) - d(1,1),\ \forall\ \alpha \geq \frac{31}{24}, \\ &d(2,0) \leq d(2,1) + \alpha d(1,0) - d(1,1),\ \forall\ \alpha \geq \frac{9}{2}, \\ &d(1,2) \leq d(1,0) + \alpha d(0,2) - d(0,0),\ \forall\ \alpha \geq 1, \\ &d(2,1) \leq d(2,0) + \alpha d(0,1) - d(2,2),\ \forall\ \alpha \geq 1. \end{aligned}$$

So, (E, α, d) is a SPbMS, for $\alpha = 5$ but it is neither metric nor strong b-metric space, because $d(1,1) = \frac{1}{4} \neq 0$.
So, above corollary (3.2) and corollary (3.3) can't be apply.
Let $S : E \to E$ be a self map defined by $S0 = 0, S1 = 0, S2 = 1$ and $f \in \mathcal{F}$ defined by

$$f(z) = \frac{1}{2}e^{-\frac{z}{6}} \text{ for } z > 0 \text{ and } f(0) \in [0, \frac{1}{2}).$$

Then

$$d(S0, S1) = d(0,0) = 0 < \frac{1}{4}e^{-\frac{1}{12}} = 0.23 = f(d(0,1))\{d(0, S0) + d(1, S1)\},$$

$$d(S1,S2)=d(0,1)=\frac{1}{2}<\frac{13}{4}e^{-1}=1.1927=f(d(1,2))\{d(1,S1)+d(2,S2)\},$$

$$d(S0,S2)=d(0,1)=\frac{1}{2}<3e^{-\frac{8}{6}}=0.7908=f(d(0,2))\{d(0,S0)+d(2,S2)\}.$$

Therefore, S meets all the condition of theorem (3.1). Here S has unique fixed point $u^*=0$.

***Example* 3.5.** Let $E=\{0,2,4\}$ and $d:E\times E\to[0,\infty)$ be defined by

$$\begin{aligned}
&d(0,0)=d(2,2)=d(4,4)=0,\\
&d(0,2)=\frac{1}{2}=d(2,0),\\
&d(0,4)=8=d(4,0),\\
&d(2,4)=5=d(4,2).
\end{aligned}$$

Here, we can see that

$$\begin{aligned}
&d(0,2)\le d(0,4)+\alpha d(4,2),\ \forall\ \alpha\ge 1,\\
&d(2,0)\le d(2,4)+\alpha d(4,0),\ \forall\ \alpha\ge 1,\\
&d(0,4)\le d(0,2)+\alpha d(2,4),\ \forall\ \alpha\ge \frac{3}{2},\\
&d(4,0)\le d(4,2)+\alpha d(2,0),\ \forall\ \alpha\ge 6,\\
&d(2,4)\le d(2,0)+\alpha d(0,4),\ \forall\ \alpha\ge 1,\\
&d(4,2)\le d(4,0)+\alpha d(0,2),\ \forall\ \alpha\ge 1.
\end{aligned}$$

So, $(E,d,\alpha=6)$ is a strong b-metric space, but is not a metric space, because $d(0,4)>d(0,2)+d(2,4)$. Thus, corollary (3.3) can not be applied. Let $S:E\to E$ be a self map defined by $S0=0, S2=0, S4=2$ and $f\in\mathcal{F}$ defined by

$$f(z)=\frac{1}{2}e^{-\frac{z}{8}}\text{ for } z>0 \text{ and } f(0)\in[0,\frac{1}{2}).$$

Then

$$d(S0,S2)=d(0,0)=0<\frac{1}{4}e^{-\frac{1}{16}}=f(d(0,2))\{d(0,S0)+d(2,S2)\},$$

$$d(S0,S4)=d(0,2)=\frac{1}{2}<0.9196=\frac{5}{2e}=f(d(0,4))\{d(0,S0)+d(4,S4)\},$$

$$d(S2,S4)=d(0,2)=\frac{1}{2}<1.4719=\frac{11}{4}e^{\frac{-5}{8}}=f(d(2,4))\{d(2,S2)+d(4,S4)\}.$$

Therefore, S meets all the condition of corollary (3.2). Here S has unique fixed point $u^*=0$.

Now, we consider $\mathcal{G}$ is the class of functions which satisfy

$$\mathcal{G}=g:(0,\infty)\to[0,\frac{1}{3}):g(z_n)\to\frac{1}{3}\implies z_n\to 0 \text{ as } n\to\infty.$$

Theorem 3.6. Let (E,d,α) be a complete SPbMS and $S:E\to E$ be a self map. Assume, $\exists\ g\in\mathcal{G}$ such that $\forall\ u,v\in E$, with $u\ne v$,

$$d(Su,Sv)\le g(d(u,v))\{d(u,Su)+d(v,Sv)+d(u,v)\}. \tag{3.4}$$

Then, S has a unique fixed point in E.

Proof. Let $u_0 \in E$ be any random point. We introduce a sequence $\{u_n\}$ in E,

$$u_{n+1} = Su_n \;\forall\; n \geq 0.$$

Suppose, there exists $n \geq 0$ such that $u_{n+1} = u_n$, then obviously u_n is a fixed point of S. So, assume that $u_{n+1} \neq u_n \;\forall\; n \geq 0$.
Now, define $D_n = d(u_{n+1}, u_n) \;\forall\; n \geq 0$. By inequality (3.4), we get

$$\begin{aligned} D_{n+1} &= d(u_{n+2}, u_{n+1}) \\ &= d(Su_{n+1}, Su_n) \\ &\leq g(d(u_{n+1}, u_n))\{d(u_{n+1}, Su_{n+1}) + d(u_n, Su_n + d(u_{n+1}, u_n)\} \\ &< \frac{1}{3}\{d(u_{n+1}, Su_{n+1}) + d(u_n, Su_n) + d(u_n, u_{n+1})\} \\ &= \frac{1}{3}\{2D_n + D_{n+1}\}. \end{aligned}$$

Clearly, $D_{n+1} < D_n \;\forall\; n \geq 0$. Hence, $\{D_n\}$ is a monotonically decreasing and bounded below sequence. So, $\exists\; \beta \geq 0$ so as

$$\lim_{n\to\infty} D_n = \beta.$$

Now, suppose $\beta > 0$. Then, by inequality (3.4), we have

$$d(u_{n+2}, u_{n+1}) \leq g(d(u_{n+1}, u_n))\{d(u_{n+1}, u_{n+2}) + d(u_n, u_{n+1}) + d(u_{n+1}, u_n)\};$$

that is

$$D_{n+1} \leq g(D_n)\{D_{n+1} + 2D_n\}.$$

That implies

$$\frac{D_{n+1}}{D_{n+1} + 2D_n} \leq g(D_n) \;\forall\; n \geq 0.$$

Applying $n \to \infty$, we get $\frac{1}{3} \leq \lim_{n\to\infty} g(D_n)$, but $\frac{1}{3} > \lim_{n\to\infty} g(D_n)$, because $g \in \mathcal{G}$. Which is a contradiction. So, $\lim_{n\to\infty}(D_n) = \beta = 0$.
We demonstrate that $\{u_n\}$ is a Cauchy sequence in E. Let $m < n$. So, by inequality (3.4), we have

$$\begin{aligned} d(u_{m+1}, u_{n+1}) &\leq g(d(u_m, u_n))\{d(u_m, Su_m) + d(u_n, Su_n) + d(u_m, u_n)\} \\ &\leq \frac{1}{3}\{d(u_m, Su_m) + d(u_n, Su_n) + d(u_m, u_{n+1}) + \alpha d(u_{n+1}, u_n) - d(u_{n+1}, u_{n+1})\} \\ &\leq \frac{1}{3}\{d(u_m, u_{m+1}) + d(u_n, u_{n+1}) + d(u_m, u_{n+1}) + \alpha d(u_{n+1}, u_n)\} \\ &\leq \frac{1}{3}\{d(u_m, u_{m+1}) + \alpha d(u_m, u_{m+1}) + d(u_{m+1}, u_{n+1}) - d(u_{m+1}, u_{m+1}) + (1+\alpha)d(u_{n+1}, u_n)\} \\ &\leq \frac{1}{3}\{d(u_m, u_{m+1}) + \alpha d(u_m, u_{m+1}) + d(u_{m+1}, u_{n+1}) + (1+\alpha)d(u_{n+1}, u_n)\}, \end{aligned}$$

which means

$$d(u_{m+1}, u_{n+1}) \leq \frac{\alpha + 1}{2}\{D_m + D_n\}.$$

As $m, n \to \infty, d(u_m, u_{m+1})$ and $d(u_n, u_{n+1}) \to 0$. So, $d(u_{m+1}, u_{n+1}) \to 0$ as $n \to \infty$.
Hence, $\{u_n\}$ is a Cauchy sequence. Now, $\{S^n u_0\}$ is a Cauchy sequence and by hypothesis, E is complete. So, $\exists\; u^* \in E$ so that

$$\lim_{n\to\infty} S^n u_0 = u^*.$$

Now, by (SPbMS4)

$$\begin{aligned}
d(Su^*, u^*) &\leq d(Su^*, Su_n) + \alpha d(Su_n, u^*) - d(Su_n, Su_n) \\
&\leq g(d(u^*, u_n))\{d(u^*, Su^*) + d(u_n, Su_n) + d(u^*, u_n)\} + \alpha d(u_{n+1}, u^*) - d(u_{n+1}, u_{n+1}) \\
&\leq g(d(u^*, u_n))\{d(u^*, Su^*) + d(u_n, Su_n)\} + d(u^*, u_n) + \alpha d(u_{n+1}, u^*).
\end{aligned}$$

So,

$$d(Su^*, u^*)(1 - g(d(u^*, u_n))) \leq g(d(u^*, u_n))d(u_n, Su_n) + g(d(u^*, u_n))d(u_n, Su^*) + \alpha d(u_{n+1}, u^*).$$

This implies

$$d(Su^*, u^*) \leq \frac{g(d(u^*, u_n))}{1 - g(d(u^*, u_n))}D_n + \frac{g(d(u^*, u_n))}{1 - g(d(u^*, u_n))}d(u_n, u^*) + \frac{\alpha}{1 - g(d(u^*, u_n))}d(u_{n+1}, u^*). \tag{3.5}$$

As $n \to \infty$ right hand side of (3.5) is zero. So,

$$d(Su^*, u^*) = 0. \tag{3.6}$$

Now, by (SPbMS2) $d(Su^*, Su^*) \leq d(Su^*, u^*)$.
Since, $S : E \times E \to [0, \infty)$ and $d(Su^*, u^*) = 0$. So, $d(Su^*, Su^*) = 0$. Similarly, we can show that $d(u^*, u^*) = 0$. Thus, we get $d(u^*, u^*) = d(Su^*, u^*) = d(Su^*, Su^*)$. So, by (SPbMS1) $Su^* = u^*$. Hence $u^* \in E$ is a fixed point of S.
Uniqueness: Let if possible v^* is another fixed point of S. So, $Sv^* = v^*$.
Using inequality (3.4), we get

$$\begin{aligned}
d(u^*, v^*) &= d(Su^*, Sv^*) \\
&\leq g(d(u^*, v^*))\{d(u^*, Su^*) + d(v^*, Sv^*) + d(u^*, v^*)\} \\
&\leq \frac{1}{3}\{d(u^*, Su^*) + d(v^*, Sv^*) + d(u^*, v^*)\}.
\end{aligned}$$

and

$$\frac{2}{3}d(u^*, v^*) \leq \frac{1}{3}\{d(u^*, Su^*) + d(v^*, Sv^*).$$

By using equation (3.6), we have $d(u^*, v^*) = 0$.
Now, $d(u^*, u^*) = d(v^*, v^*) = 0$. [$\because d(u^*, u^*) \leq d(u^*, v^*)$ and $d(v^*, v^*) \leq d(u^*, v^*)$.]
So, $d(u^*, u^*) = d(u^*, v^*) = d(v^*, v^*)$. Hence $u^* = v^*$. Thus, S has exactly one fixed point $u^* \in E$. □

Corollary 3.7. [14] "Let (E, d, α) be a complete strong b-metric space and $S : E \to E$ be a self map. Suppose, $\exists\, g \in \mathcal{G}$ such that for all $u, v \in E$ with $u \neq v$,

$$d(Su, Sv) \leq g(d(u, v))\{d(u, Su) + d(v, Sv) + d(u, v)\}.$$

Then, S has a unique fixed point in E and for any $u \in E$, the sequence of iterates $\{S^n(u)\}$ converges to u^*."

Corollary 3.8. [15] "Let (E, d, α) be a complete metric space and $S : E \to E$ be a self map. Suppose, $\exists\, g \in \mathcal{G}$ such that for all $u, v \in E$ with $u \neq v$,

$$d(Su, Sv) \leq g(d(u, v))\{d(u, Su) + d(v, Sv) + d(u, v)\}.$$

Then, S has a unique fixed point in E and for any $u \in E$, the sequence of iterates $\{S^n(u)\}$ converges to u^*."

***Example* 3.9.** Let $E = \{1, 2, 3\}$ and $d : E \times E \to [0, \infty)$ be defined by

$$d(1,1) = d(3,3) = 0, d(2,2) = \frac{1}{3},$$
$$d(1,2) = \frac{1}{2} = d(2,1),$$
$$d(1,3) = 3 = d(3,1),$$
$$d(2,3) = 1 = d(3,2).$$

Here $d(u,u) \leq d(u,v)\ \forall\ u, v \in E$. And

$$d(1,2) \leq d(1,3) + \alpha d(3,2) - d(3,3),\ \forall\ \alpha \geq 1,$$
$$d(2,1) \leq d(2,3) + \alpha d(3,1) - d(3,3),\ \forall\ \alpha \geq 1,$$
$$d(1,3) \leq d(1,2) + \alpha d(2,3) - d(2,2),\ \forall\ \alpha \geq \frac{17}{6},$$
$$d(3,1) \leq d(3,2) + \alpha d(2,1) - d(2,2),\ \forall\ \alpha \geq \frac{14}{3},$$
$$d(2,3) \leq d(2,1) + \alpha d(1,3) - d(1,1),\ \forall\ \alpha \geq 1,$$
$$d(3,2) \leq d(3,1) + \alpha d(1,2) - d(1,1),\ \forall\ \alpha \geq 1.$$

So, (E, d, α), for $\alpha = 5$ is a partial strong b-metric space, but it is neither metric nor strong b-metric space, because $d(1,1) = \frac{1}{4} \neq 0$ and $d(1,3) > d(1,2) + d(2,3)$. So, above corollary (3.7) and corollary (3.8) can't be apply.
Let $S : E \to E$ be a self map defined by $S1 = 1, S2 = 1, S3 = 2$ and $g \in \mathcal{G}$ defined by

$$g(z) = \frac{1}{3}e^{-\frac{z}{4}} \text{ for } z > 0 \text{ and } g(0) \in [0, \frac{1}{3}).$$

Then

$$d(S1, S2) = d(1,1) = 0 < \frac{1}{3}e^{-\frac{1}{8}} = 0.2914 = g(d(1,2))\{d(1, S1) + d(2, S2) + d(1,2)\},$$

$$d(S1, S3) = d(1,2) = \frac{1}{2} < \frac{4}{3}e^{\frac{-3}{4}} = 0.6298 = g(d(1,3))\{d(1, T1) + d(3, T3) + d(1,3)\},$$

$$d(S2, S3) = d(1,2) = \frac{1}{2} < 3e^{-\frac{8}{6}} = 0.6490 = g(d(2,3))\{d(2, S2) + d(3, S3) + d(2,3)\}.$$

Therefore, S meets all the condition of theorem (3.6). Here S has unique fixed point $u^* = 1$.

Now, we consider Φ is the class of functions which satisfy

$$\Phi_k = \{\phi : (0, \infty) \to [0, k) : \phi(z_n) \to k \implies z_n \to 0 \text{ as } n \to \infty\},\ \text{where } k \in (0, \frac{1}{2}).$$

Proposition 3.10. Let $\{u_n\}$ be a sequence in a SPbMS and suppose

$$\sum_{j=1}^{\infty} d(u_j, u_{j+1}) < \infty.$$

Then $\{u_n\}$ is a Cauchy sequence.

Proof. If for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq \mathbb{N}$

$$\sum_{j=n}^{n+k} d(u_j, u_{j+1}) < \epsilon.$$

So, $\forall\ k, n \in \mathbb{N}$ with n sufficiently large

$$d(u_n, u_{n+k+1}) \leq \alpha \sum_{j=n}^{n+k} d(u_j, u_{j+1}) - \sum_{j=n+1}^{n+k-1} u_j \leq \alpha \sum_{j=n}^{n+k} d(u_j, u_{j+1}) \leq \alpha\epsilon.$$

Hence $\{u_n\}$ is a Cauchy sequence. □

Theorem 3.11. Let (E, d, α) be a complete SPbMS and $S : E \to E$ be a self map. Assume, $\exists\ \phi \in \Phi_k$ such as for all $u, v \in E$, with $u \neq v$,

$$\frac{1}{\alpha+1} d(u, Su) \leq d(u, v)$$

implies

$$d(Su, Sv) \leq \phi(d(u, v))\{d(u, Su) + d(v, Sv)\}. \tag{3.7}$$

Then, S has a unique fixed point in E.

Proof. Let $u_0 \in E$ be any arbitrary point. We define a sequence $\{u_n\}$ in E,

$$u_{n+1} = Su_n\ \forall\ n \geq 0.$$

Suppose, there exists $n \geq 0$ such that $u_{n+1} = u_n$, then obviously u_n is a fixed point of S. So, assume that $u_{n+1} \neq u_n\ \forall\ n \geq 0$.
Now, define $D_n = d(u_{n+1}, u_n)\ \forall\ n \geq 0$. Since,

$$\frac{1}{\alpha+1} d(u_n, Su_n) = \frac{1}{\alpha+1} d(u_n, u_{n+1}) \leq d(u_n, u_{n+1}),$$

and according to our assumption, we get

$$\begin{aligned} D_{n+1} &= d(u_{n+2}, u_{n+1}) \\ &= d(Su_{n+1}, Su_n) \\ &\leq \phi(d(u_{n+1}, u_n))\{d(u_{n+1}, v_{n+1}) + d(u_n, Su_n\} \\ &< k\{d(u_{n+1}, Su_{n+1}) + d(u_n, Su_n\} \\ &= k\{D_n + D_{n+1}\}. \end{aligned}$$

So, $D_{n+1} < \frac{k}{1-k} D_n = bD_n$, where $b = \frac{k}{1-k} \in (0, 1)$. Clearly, $D_n < b^n D_0\ \forall\ n \geq 0$. Hence,

$$\sum_{n=1}^{\infty} D_n \leq \sum_{n=1}^{\infty} b^n < +\infty$$

By above proposition, we have $\{u_n\}$ is Cauchy sequence in E and by hypothesis, E is complete. So, $\exists\ u^* \in E$ such that $\lim_{n\to\infty} u_n = u^* \in E$. We prove that for all $n \geq 0$ either

$$\frac{1}{\alpha+1} d(u_n, Su_n) \leq d(u_n, u^*), \text{ or } \frac{1}{\alpha+1} d(Su_n, Su_{n+1}) \leq d(Su_n, u^*), \tag{3.8}$$

Let if possible $\exists\ n \in \{0, 1, 2, ...\}$ such that

$$d(u_n, u^*) < \frac{1}{\alpha+1} d(u_n, Su_n),$$

and

$$d(Su_n, u^*) < \frac{1}{\alpha+1} d(Su_n, Su_{n+1}).$$

By (SPbMS4)

$$\begin{aligned} D_n &= d(u_n, u_{n+1}) \\ &\leq d(u_n, u^*) + \alpha d(u_{n+1}, u^*) - d(u^*, u^*) \\ &\leq \frac{1}{\alpha+1} d(u_n, Su_n) + \frac{\alpha}{1+\alpha} d(Su_n, Su_{n+1}) - d(u^*, u^*) \\ &= \frac{1}{\alpha+1} D_n + \frac{\alpha}{1+\alpha} D_{n+1} \\ &\leq D_n. \end{aligned}$$

This is impossible. Hence, inequality (3.8) will hold. That means

$$d(Su_n, Su^*) \leq \phi(d(u_n, u^*))\{d(u_n, Su_n) + d(u^*, Su^*)\}. \tag{3.9}$$

or

$$d(Su_{n+1}, Su^*) \leq \phi(d(u_{n+1}, u^*))\{d(u_{n+1}, Su_{n+1}) + d(u^*, Su^*)\}. \tag{3.10}$$

Let inequality (3.9) holds, then we can take an infinite sequence $\{u_{n_j}\}$, which is subsequence of $\{x_n\}$. We have

$$\begin{aligned} d(u_{n_j+1}, Su^*) &\leq \phi(d(u_{n_j}, u^*))\{d(u_{n_j}, Su_{n_j}) + d(u^*, Su^*)\} \\ &< k\{(d(u_{n_j}, u^*)) + \alpha d(u^*, Su_{n_j}) - d(u^*, u^*) + d(u_{n_j+1}, u^*)) + \alpha d(Su^*, u_{n_j+1}) - d(u_{n_j+1}, u_{n_j+1}) \\ &\leq k\{(d(u_{n_j}, u^*)) + \alpha d(u^*, Su_{n_j}) + d(u_{n_j+1}, u^*)) + \alpha d(Su^*, u_{n_j+1}) \\ &< \frac{k}{1-k}\{d(u_{n_j}, u^*) + 2\alpha d(u_{n_j+1}, u^*)\} \end{aligned}$$

Now, $u_{n_j+1} \to u^*$ when $j \to \infty$, so, $\lim_{j\to\infty} u_{n_j+2} = Su^*$. Thus $Su^* = u^*$. If inequality (3.10) holds then by using similar argument we have u^* is fixed point of S.

Uniquness: Let if possible v^* is any other fixed point of S.

$$\frac{1}{\alpha} d(u^*, Su^*) = 0 \leq d(u^*, v^*)$$

and

$$d(u^*, v^*) = d(Su^*, Sv^*) \leq \phi(d(u^*, v^*))\{d(u^*, Su^*) + d(v^*, Sv^*)\} = 0.$$

Thus, $d(u^*, v^*) = 0$, which implies $u^* = v^*$. Hence, S has only one fixed point u^*. □

Corollary 3.12. [14] *"Let (E, d, α) be a complete strong b-metric space and $S : E \to E$ be a self map. Assume, $\exists\ \phi \in \Phi_k$ such that for each $u, v \in E$, with $u \neq v$,*

$$\frac{1}{\alpha+1} d(u, Su) \leq d(u, u)$$

implies

$$d(Su, Sv) \leq \phi(d(u, u))\{d(u, Su) + d(v, Sv)\}.$$

Then, S has a unique fixed point in E."

Corollary 3.13. [15] *"Let (E, d, α) be a complete metric space and $S : E \to E$ be a self map. Assume, $\exists\ \phi \in \Phi_k$ such that for each $u, v \in E$, with $u \neq v$,*

$$\frac{1}{\alpha+1} d(u, Su) \leq d(u, v)$$

implies

$$d(Su, Sv) \leq \phi(d(u, v))\{d(u, Su) + d(v, Sv)\}.$$

Then, S has a unique fixed point in E."

***Example* 3.14.** Let $E = \{-1, 0, 1\}$ and $d : E \times E \to [0, \infty)$ be defined by

$$d(1,1) = d(-1,-1) = 0, d(0,0) = \frac{1}{4},$$
$$d(0,1) = d(1,0) = \frac{1}{2},$$
$$d(0,-1) = d(-1,0) = 4,$$
$$d(1,-1) = d(-1,1) = 1.$$

Here $d(u,u) \leq d(u,v)\ \forall\ u, v \in E$. And

$$d(1,-1) \leq d(1,0) + \alpha d(0,-1) - d(0,0),\ \forall\ \alpha \geq 1,$$
$$d(-1,1) \leq d(-1,0) + \alpha d(0,1) - d(0,0),\ \forall\ \alpha \geq 1,$$
$$d(1,0) \leq d(1,-1) + \alpha d(-1,0) - d(-1,-1),\ \forall\ \alpha \geq 1,$$
$$d(0,1) \leq d(0,-1) + \alpha d(-1,1) - d(-1,-1),\ \forall\ \alpha \geq 1,$$
$$d(0,-1) \leq d(0,1) + \alpha d(1,-1) - d(1,1),\ \forall\ \alpha \geq \frac{7}{2},$$
$$d(-1,0) \leq d(-1,1) + \alpha d(1,0) - d(1,1),\ \forall\ \alpha \geq 6.$$

So, $(E, d, \alpha = 6)$ is a SPbMS, but it is neither strong b-metric nor metric space, because $d(0,0) = \frac{1}{4} \neq 0$ and $d(0,-1) > d(0,1) + d(1,-1)$.
So, above corollary (3.12) and corollary (3.13) can't be apply.
Let $S : E \to E$ be a self map defined by $S(-1) = 0, S0 = 1, S1 = 1$ and $\phi \in \Phi$ defined by

$$\phi(z) = \frac{1}{4}e^{-\frac{z}{8}} \text{ for } z > 0 \text{ and } \phi(0) \in [0, \frac{1}{4}).$$

So, $\phi \in \Phi_{\frac{1}{4}}$. Now,

$$\frac{4}{7} = \frac{1}{7}d(-1, S(-1)) \leq d(-1, v),$$

for $v \in E - \{-1\}$ and

$$d(S(-1), S0) = d(0,1) = \frac{1}{2} < \frac{9}{8}e^{-\frac{1}{2}} = 0.6823 = \phi(d(-1,0))\{d(-1, S(-1)) + d(0, S0)\},$$

$$d(S(-1), S1) = d(0,1) = \frac{1}{2} < e^{\frac{-1}{8}} = 0.8824 = \phi(d(-1,1))\{d(-1, S(-1)) + d(1, S1)\}.$$

So, we get

$$\frac{1}{7}d(-1, S(-1)) \leq d(-1, v) \Rightarrow d(S(-1), Sv) \leq \phi(d(-1, v))\{d(-1, S(-1)) + d(v, Sv)\},$$

for all $v \in E - \{-1\}$. Again, since

$$\frac{1}{14} = \frac{1}{7}d(0, S(0)) \leq d(0, v),$$

for $v \in E - \{0\}$ and

$$d(S(0), S(-1)) = d(1,0) = \frac{1}{2} < \frac{9}{8}e^{-\frac{1}{2}} = 0.6823 = \phi(d(0, (-1)))\{d(0, S0) + d(-1, S(-1))\},$$

$$d(S0, S1) = d(1,1) = 0 < 0.1174 = \frac{1}{8}e^{\frac{-1}{16}} = 0.8824 = \phi(d(0,1))\{d(0, S0) + d(1, S1)\}.$$

So, we get

$$\frac{1}{7}d(0, S(0)) \leq d(0, v) \Rightarrow d(S0, Sv) \leq \phi(d(0, v))\{d(0, S(0)) + d(v, v)\},$$

for all $v \in E - \{0\}$. Similarly

$$0 = \frac{1}{7}d(1, S(1)) \leq d(1, v),$$

for $v \in E - \{1\}$ and

$$d(S(1), S0) = d(1, 1) = 0 < \frac{1}{8}e^{-\frac{1}{16}} = 0.1174 = \phi(d(1, 0))\{d(1, S(1)) + d(0, S0)\},$$

$$d(S1, S(-1)) = d(1, 0) = \frac{1}{2} < e^{\frac{-1}{8}} = 0.8824 = \phi(d(1, -1))\{d(1, S1) + d(-1, S(-1))\}.$$

So, we get

$$\frac{1}{7}d(1, S(1)) \leq d(1, v) \Rightarrow d(S(1), Sv) \leq \phi(d(1, v))\{d(1, S(1)) + d(v, Sv)\},$$

for all $v \in E - \{1\}$. Therefore, S meets all the condition of theorem (3.6). Thus, S has unique fixed point $u^* = 1$.

Theorem 3.15. Let (E, d, α) be a complete SPbMS with coefficient $\alpha \geq 1$ and $S : E \to E$ be a self map, which satisfy

$$d(Su, Sv) \leq \beta \max\{d(u, Su), d(v, Sv), d(u, v)\}, \tag{3.11}$$

where $\beta \in [0, \frac{1}{\alpha})$. Then, S has a unique fixed point in E.

Proof. Let $u_0 \in E$ be any arbitrary point. We define a sequence $\{u_n\}$ in E,

$$u_{n+1} = Su_n \ \forall \ n \geq 0.$$

Suppose, $\exists \ n \geq 0$ so as $u_{n+1} = u_n$, then obviously u_n is a fixed point of S. So, assume that $u_{n+1} \neq u_n \ \forall \ n \geq 0$.
Now, define $D_n = d(u_{n+1}, u_n) \ \forall \ n \geq 0$. By inequality (3.11), we get

$$\begin{aligned}
D_{n+1} &= d(u_{n+2}, u_{n+1}) \\
&= d(Su_{n+1}, Su_n) \\
&\leq \beta \max\{d(u_{n+1}, Su_{n+1}), d(u_n, Su_n), d(u_{n+1}, u_n)\} \\
&= \beta \max\{d(u_{n+1}, u_{n+2}), d(u_n, u_{n+1}), d(u_n, u_{n+1})\} \\
&= \beta \max\{d(u_{n+1}, u_{n+2}), d(u_n, u_{n+1})\}.
\end{aligned}$$

If, $\max\{d(u_{n+1}, u_{n+2}), d(u_n, u_{n+1})\} = \{d(u_{n+1}, u_{n+2})$. Then,

$$D_{n+1} = d(u_{n+2}, u_{n+1}) \leq \beta d(u_{n+2}, u_{n+1}) < d(u_{n+2}, u_{n+1}).$$

This contradiction means $\max\{d(u_{n+1}, u_{n+2}), d(u_n, u_{n+1})\} = d(u_n, u_{n+1})$. Thus we get

$$D_{n+1} = d(u_{n+2}, u_{n+1}) \leq \beta d(u_n, u_{n+1}).$$

By repeating same process, we have

$$d(u_{n+1}, u_{n+2}) \leq \beta^{n+1} d(u_0, u_1),$$

that is

$$d(u_n, u_{n+1}) \leq \beta^n d(u_0, u_1) \tag{3.12}$$

Let $n, m \in \mathbb{N}$, where $n > m$, then

$$\begin{aligned}
d(u_m, u_n) &\leq d(u_m, u_{m+1}) + \alpha d(u_{m+1}, u_n) - d(u_{m+1}, u_{m+1}) \\
&\leq d(u_m, u_{m+1}) + \alpha\{d(u_{m+1}, u_{m+2}) + \alpha d(u_{m+2}, u_n) - d(u_{m+2}, u_{m+2})\} - d(u_{m+1}, u_{m+1}) \\
&\leq d(u_m, u_{m+1}) + \alpha d(u_{m+1}, u_{m+2}) + \alpha^2 d(u_{m+2}, u_n) \\
&\leq d(u_m, u_{m+1}) + \alpha d(u_{m+1}, u_{m+2}) + \alpha^2 d(u_{m+2}, u_{m+3}) + ... + \alpha^{n-m-1} d(u_{n-1}, u_n).
\end{aligned}$$

By using inequality (3.12)

$$\begin{aligned} d(u_m,u_n) &\le \beta^m d(u_0,u_1)+\alpha\beta^{m+1}d(u_0,u_1)+\alpha^2\beta^{m+2}d(u_0,u_1)+...+\alpha^{n-m}\beta^{n-1}d(u_0,u_1)\\ &\le \beta^m\{1+\alpha\beta+\alpha^2\beta^2+...+\alpha^{n-m-1}\beta^{n-m-1}\}d(u_0,u_1)\\ &\le \frac{\beta^m}{1-\alpha\beta}d(u_0,u_1). \end{aligned}$$

As $\beta\in[0,\frac{1}{\alpha})$ and $\alpha\ge 1$, so

$$d(u_m,u_n)\to 0 \text{ as } m,n\to\infty.$$

So, $\{u_n\}$ is a Cauchy sequence in E. Now, E is complete. So, $\exists\, u^*\in E$ such that

$$\lim_{n\to\infty} d(u_n,u^*)=\lim_{n,m\to\infty} d(u_n,u_m)=d(u^*,u^*)=0. \tag{3.13}$$

Now, we prove u^* is a fixed point in E. So, for any $n\in\mathbb{N}$, we have

$$\begin{aligned} d(u^*,Su^*) &\le d(u^*,u_{n+1})+\alpha d(u_{n+1},Su^*)-d(u_{n+1},u_{n+1})\\ &\le d(u^*,u_{n+1})+\alpha d(u_{n+1},Su^*)\\ &\le d(u^*,u_{n+1})+\alpha\beta\max\{d(u_n,u^*),d(u_n,Su_n),d(u^*,Su^*)\}\\ &\le d(u^*,u_{n+1})+\alpha\beta d(u_n,u^*) \end{aligned}$$

Using equation (3.13) we get $d(u^*,Su^*)=0$. So, $Su^*=u^*$. Thus, u^* is a fixed point of S.

Uniqueness: Let if possible v^* is another fixed point of S. So, $Sv^*=v^*$.

Using inequality (3.11), we have

$$\begin{aligned} d(u^*,v^*) &= d(Su^*,Sv^*)\\ &\le \beta\max\{d(u^*,v^*),d(u^*,Su^*),d(v^*,Sv^*)\}\\ &= \beta\max\{d(u^*,v^*),d(u^*,u^*),d(v^*,v^*)\}\\ &= \beta d(u^*,v^*)\\ &< d(u^*,v^*). \end{aligned}$$

This contradiction implies $d(u^*,v^*)=0$. Hence $u^*=v^*$. Thus, S has exactly one fixed point $u^*\in E$. □

Corollary 3.16. *"Let (E,d,α) be a complete strong b-metric space with coefficient $\alpha\ge 1$ and $S:E\to E$ be a self map, which satisfy*

$$d(Su,Sv)\le\beta\max\{d(u,Su),d(v,Sv),d(u,v)\},$$

where $\beta\in[0,\frac{1}{\alpha})$. Then, S has a unique fixed point in E."

Corollary 3.17. *"Let (E,d,α) be a complete metric space with coefficient $\alpha\ge 1$ and $S:E\to E$ be a self map, which satisfy*

$$d(Su,Sv)\le\beta\max\{d(u,Su),d(v,Sv),d(u,v)\},$$

where $\beta\in[0,\frac{1}{\alpha})$. Then, S has a unique fixed point in E."

4. Application

Here, we apply theorem (3.1) to the first order initial value problem

$$\begin{cases} u'(t) &= g(t,u(t)),\\ u(t_0) &= u_0 \end{cases} \tag{4.1}$$

where g is a continuous function from $I=[t_0-(\frac{1}{b})^{p-1},t_0+(\frac{1}{b})^{p-1}]\times[u_0-\frac{b}{2},u_0+\frac{b}{2}]$ to $\mathbb{R}$ and $b>1,\ p>2,\ t_0,\ u_0\in\mathbb{R}$.

Theorem 4.1. Consider $R = \{(t,u) : |t - t_0| \leq (\frac{1}{b})^{p-1}, |u - u_0| \leq \frac{b}{2}\}$, the first order initial value problem (4.1) and

(i) g satisfies
$$|g(t,u(t)) - g(t,v(t))| \leq b[\frac{1}{2}\{|(|u(t) - Su|) - (|v(t) - Sv|)|\}] \tag{4.2}$$
for all $(t,u),(t,v) \in R$.
(ii) g is bounded as
$$|g(t,u)| \leq \frac{b^p}{2}. \tag{4.3}$$

Then intial value problem (4.1) has a unique solution on I. The solution is further demonstrated as follows
$$u(t) = u_0 + \lim_{n\to\infty} \int_{t_0}^{t} g(x, u_n(x))dx,$$
where $u_0(t) = u_0$ and for $n = 1, 2, ...$
$$u_n(t) = u_0 + \int_{t_0}^{t} g(x, u_{n-1}(x))dx.$$

Proof. Let $X = \{u \in C(I) : |u(t) - u_0| \leq \frac{b}{2}\}$, where $C(I)$ be the set of all continuous functions on I. Let us define a mapping $d : X \times X \to \mathbb{R}^+$ by
$$d(u,v) = \begin{cases} |u-v|^2 + \max\{u,v\} & \text{if } u \neq v \\ u & \text{if } u = v \neq 1 \;. \\ 0 & \text{if } u = v = 1 \end{cases} \tag{4.4}$$
Here, $(C(I), d, \alpha)$ is a complete strong partial b-metric space with $\alpha = 3$. $(X, d, 3)$ is also complete strong partial b-metric space, because X is a closed subspace of $C(I)$.
Let f is a function from $(0,\infty)$ to $[0, \frac{1}{2})$, defined as
$$f(z) = \frac{1}{2}e^{-z/6}.$$
By integrating (4.1), we get
$$u(t) = u_0 + \int_{t_0}^{t} g(x, u(x))dx. \tag{4.5}$$
As an outcome, solving (4.1) is equivalent to finding the fixed point of $S : X \times X$ defined as
$$Su(t) = u_0 + \int_{t_0}^{t} g(x, u(x))dx. \tag{4.6}$$
Since, g is continuous on R, then the integral (4.6) exists and S is well defined for all $u \in X$.
The conclusion we reach is that S is a self mapping on X. Utilizing (4.3) and (4.6),

we get

$$\begin{aligned}|Su(t)-u_0| &= |\int_{t_0}^{t} g(x,u(x))dx| \\ &\leq \int_{t_0}^{t} |g(x,u(x))dx| \\ &\leq \frac{b^p}{2}|t-t_0| \\ &\leq \frac{b^p}{2}(\frac{1}{b})^{p-1} \\ &= \frac{b}{2}.\end{aligned}$$

From (4.2),(4.4) and (4.6), we get

$$\begin{aligned}|Su-Sv|^2 &= |\int_{t_0}^{t}[g(x,u(x))-g(x,v(x))]dx|^2 \\ &\leq [\int_{t_0}^{t}|g(x,u(x))-g(x,v(x))|dx]^2 \\ &\leq [\int_{t_0}^{t} b|\frac{1}{2}\{|u(x)-Su|-|v(x)-Sv|\}dx|]^2 \\ &\leq b^2[\int_{t_0}^{t}\frac{1}{2}\{|u(x)-Su|-|v(x)-Sv|\}dx]^2 \\ &= b^2|t-t_0|^2\frac{1}{4}\{|u(x)-Su|-|v(x)-Sv|\}^2 \\ &\leq b^2(\frac{1}{b})^{2p-2}\frac{1}{4}\{|u(x)-Su|-|v(x)-Sv|\}^2.\end{aligned}$$

Thus,

$$\begin{aligned}|Su-Sv|^2+\max\{Su,Sv\} &\leq (\frac{1}{b})^{2p-4}\frac{1}{4}\{|u(x)-Su|^2+|v(x)-Sv|^2\}+\max\{u(x),Su\}+\max\{v(x),Sv\} \\ &\leq \frac{1}{4}(\frac{1}{b})^{2p-4}|u(x)-Su|^2+\max\{u(x),Su\}+|v(x)-Sv|^2+\max\{v(x),Sv\},\end{aligned}$$

which means

$$d(Su,Sv) \leq \frac{1}{4}(\frac{1}{b})^{2p-4}[d(u(x),Su)+d(v(x),Sv)].$$

Here $f(z)=\frac{1}{4}(\frac{1}{b})^{2p-4}$, where $z=6(\log b)^{2p-4}$.
We can see that $z>0$ for $b>1$ and $p>2$.
Choose b and p such as $z\neq 1$ and take $u=v=z\neq 1$. Hence,

$$d(Su,Sv) \leq f(d(u,v))[d(u,Su)+d(v,Sv)].$$

Thus all the conditions of theorem (3.1) are satisfied. So, S has unique fixed point. Using successive approximation method, we find the unique solution of problem (4.1). For that take $u_0(t)=u_0$ and

$$u_n(t) = u_0 + \int_{t_0}^{t} g(x,u_{n-1}(x))dx. \tag{4.7}$$

Thus,

$$u_{n-1}(t) = u_0 + \int_{t_0}^{t} g(x,u_{n-2}(x))dx. \tag{4.8}$$

By equations (4.6) and (4.7), we have

$$u_n(t) - u_{n-1}(t) = \int_{t_0}^{t} [g(x, u_{n-1}(x)) - g(x, u_{n-2}(x))]dx. \tag{4.9}$$

Let us assume

$$v_n(t) = u_n(t) - u_{n-1}(t) \text{ and } v_0(t) = u_0. \tag{4.10}$$

So, we get

$$u_n(t) = \sum_{j=0}^{n} v_j(t).$$

By using inequality (4.2), equations (4.9) and (4.10), we get

$$\begin{aligned}
|v_n(t)| &\leq \int_{t_0}^{t} b[\frac{1}{2}\{|u_{n-1}(x) - Su_{n-1}| - |u_{n-2}(x) - Su_{n-2}|\}]dx \\
&= \frac{b}{2}\int_{t_0}^{t} |u_{n-1}(x) - u_n(x)|dx - \frac{b}{2}\int_{t_0}^{t} |u_{n-2}(x) - u_{n-1}(x)|dx \\
&= \frac{b}{2}\int_{t_0}^{t} |v_n(x)|dx - \frac{b}{2}\int_{t_0}^{t} |v_{n-1}(x)|dx
\end{aligned}$$

Since these series are convergent on I. So, $\sum_{j=0}^{n} v_n(t)$ is convergent to some function $u(t)$ as $n \to \infty$.

Now, we show that $u(t) = \sum_{n=0}^{\infty} v_n(t)$ is the solution of equation (4.5). It means that $u(t) = \sum_{n=0}^{\infty} v_n(t)$ is the solution of problem (4.1) also. Now, assume

$$u(t) = u_n(t) + \Delta_n(t). \tag{4.11}$$

From equations (4.7) and (4.11), we get

$$u(t) - \Delta_n(t) = u_0 + \int_{t_0}^{t} g((x,\ u(x) - \Delta_{n-1}(x))dx$$

Thus,

$$u(t) - u_0 - \int_{t_0}^{t} g(x, u(x))dx = \Delta_n(t) + \int_{t_0}^{t} [g((x,\ u(x) - \Delta_{n-1}(x))]dx \tag{4.12}$$

Using inequality (4.2) and equation (4.12), we get

$$\begin{aligned}
|u(t) - u_0 - \int_{t_0}^{t} g(x, u(x))|dx &= |\Delta_n(t) + \int_{t_0}^{t} [g((x,\ u(x) - \Delta_{n-1}(x)) - g(x, u(x))]dx| \\
&\leq |\Delta_n(t)| + \int_{t_0}^{t} |g((x,\ u(x) - \Delta_{n-1}(x)) - g(x, u(x))|dx \\
&\leq |\Delta_n(t)| + \frac{b}{2}\int_{t_0}^{t} [|u(x) - \Delta_n(x), S(u(x) - \Delta_n(x))]| - |u(x) - Su|
\end{aligned}$$

As $n \to \infty$, $\lim_{n\to\infty} |\Delta_n(t) = 0$. So,

$$|u(t) - u_0 - \int_{t_0}^{t} g(x, u(x))|dx = 0.$$

Hence,

$$u(t) = u_0 + \int_{t_0}^{t} g(x, u(x))dx.$$

Thus, $u(t) = \sum_{n=0}^{\infty} v_n(t)$ is the solution of equation (4.5) and so of equation (4.1) also.
Finally, we seek the mathematical formulation of the answer to the first problem. To achieve this, utilising equations (4.9) and (4.10), we get

$$\begin{aligned}
u(t) &= \sum_{n=0}^{\infty} v_n(t) \\
&= v_0(t) + v_1(t) + \sum_{n=2}^{\infty} v_n(t) \\
&= v_0(t) + v_1(t) + \sum_{n=2}^{\infty} [u_n(t) - u_{n-1}(t)] \\
&= u_1(t) + \sum_{n=2}^{\infty} \int_{t_0}^{t} [g(x, u_{n-1}(x) - g(x, u_{n-2}(x))] dx \\
&= u_0 + \int_{t_0}^{t} g(x, u_0) dx + \int_{t_0}^{t} \sum_{n=2}^{\infty} [g(x, u_{n-1}(x)) - g(x, u_{n-2}(x))] dx \\
&= u_0 + \int_{t_0}^{t} g(x, u_0) dx + \int_{t_0}^{t} \lim_{n\to\infty} g(x, u_{n-1}(x)) dx - \int_{t_0}^{t} g(x, u_0(x)) dx \\
&= u_0 + \int_{t_0}^{t} \lim_{n\to\infty} g(x, u_{n-1}(x)) dx
\end{aligned}$$

□

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