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m-QUASI-$(n, \mathbf{A})$-PARANORMAL OPERATORS IN SEMI-HILBERTIAN SPACES

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Abstract. The study of semi-Hilbert spaces operators is motivated by what are called pseudo-Hermitian quantum mechanics. In this paper, we introduce the concept of m-quasi-n-paranormal of a bounded linear operators on a complex Hilbert space with a semi-inner product induced by a positive operator $\mathbf{A}$. This generalizes the classical m-quasi-n-paranormality of operators on Hilbert spaces to semi-Hilbert space. We investigate some basic properties of this new class. Product and tensor product results were also investigated.

1. Introduction

Assume that $\big(\mathcal{Z},\ \|.\|\big)$ is a complex Hilbert space with associated norm $\|.\|$. Let $\mathcal{B}\big[\mathcal{Z}\big]$ denotes the C^*-algebra of all bounded linear operators acting on $\mathcal{Z}$. The identity operator on $\mathcal{Z}$ is denoted simply by $\mathbf{I}$. For every $\mathbf{N} \in \mathcal{B}\big[\mathcal{Z}\big]$, $\mathbf{Null}(\mathbf{N})$, $\mathbf{Ran}(\mathbf{N})$, $\overline{\mathbf{Ran}(\mathbf{N})}$ and $\mathbf{P}_{\overline{\mathbf{Ran}(\mathbf{N})}}$ (or $\mathbf{P}$) denote, the null space, the range, the closure of the range of $\mathbf{N}$ and the orthogonal projection onto $\overline{\mathbf{Ran}(\mathbf{N})}$ respectively Let $A \in \mathcal{B}\big[\mathcal{Z}\big]$ be a positive operator. Set $\langle \varphi \mid \psi\rangle_{\mathbf{A}} = \langle \mathbf{A}\varphi \mid \psi\rangle$. It was observed that $\langle . \mid .\rangle_A : \mathcal{Z}\times\mathcal{Z} \longrightarrow \mathbb{C}$, is a positive semidefinite sesquilinear form which yield a seminorm $\|.\|_{\mathbf{A}}$ as $\|\varphi\|_{\mathbf{A}} = \langle \varphi \mid \varphi\rangle_{\mathbf{A}}^{\frac{1}{2}}$ for any $\varphi \in \mathcal{Z}$. Moreover $\|\varphi\|_{\mathbf{A}} = 0$ if and only if $\varphi \in \mathbf{Null}(\mathbf{A})$. The study of these concepts goes back to the papers [1, 2, 3].

From [1], we recall that for $\mathbf{N} \in \mathcal{B}\big[\mathcal{Z}\big]$, an element $\mathbf{T} \in \mathcal{B}\big[\mathcal{Z}\big]$ is said to be an $\mathbf{A}$-adjoint operator of $\mathbf{N}$ if $\langle \mathbf{N}\varphi \mid \psi\rangle_{\mathbf{A}} = \langle \varphi \mid \mathbf{T}\psi\rangle_{\mathbf{A}}$ for every $\varphi, \psi \in \mathcal{Z}$, which can be view as $\mathbf{N}^*\mathbf{A} = \mathbf{A}\mathbf{T}$ where $\mathbf{N}^*$ is the adjoint of $\mathbf{N}$. According to [8, Theorem 1], it follows that $\mathbf{N}$ admits an $\mathbf{A}$-adjoint operator if and only if $\mathbf{Ran}\big(\mathbf{N}^*\mathbf{A}\big) \subseteq \mathbf{Ran}(\mathbf{A})$. The unique solution of the operator equation $\mathbf{A}\mathbf{X} = \mathbf{N}^*\mathbf{A}$ for $\mathbf{X} \in \mathcal{B}\big[\mathcal{Z}\big]$ such that $\mathbf{Ran}(\mathbf{X}) \subseteq \overline{\mathbf{Ran}(\mathbf{A})}$ is denoted by $\mathbf{N}^\sharp$ and is called the distinguished $\mathbf{A}$-adjoint operator of $\mathbf{N}$. The set of all operators in $\mathcal{B}\big[\mathbf{Z}\big]$ which admitting A-adjoint is

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denoted by $\mathcal{B}_{\mathbf{A}}\big[\mathcal{Z}\big]$. An operator $\mathbf{N} \in \mathcal{B}\big[\mathbf{Z}\big]$ is called $\mathbf{A}$-positive if $\mathbf{AN}$ is positive and it symbols by $\mathbf{N} \geq_{\mathbf{A}} 0$. Notice that for $\mathbf{N}, \mathbf{T} \in \mathcal{B}_{\mathbf{A}^{\frac{1}{2}}}\big[\mathcal{Z}\big]$ we have $\mathbf{N} \geq_{\mathbf{A}} \mathbf{T}$ if $\mathbf{N} - \mathbf{T} \geq_A 0$.

We mention here some properties of the members of $\mathcal{B}_{\mathbf{A}}\big[\mathcal{Z}\big]$ that we use in this work and which are extracted from [1, 2, 3].

For $\mathbf{N} \in \mathcal{B}_{\mathbf{A}}\big[\mathcal{Z}\big]$, the following properties are met.

(1) $\mathbf{A}\mathbf{N}^{\sharp} = \mathbf{N}^{*}\mathbf{A}, \quad \mathbf{Ran}\left(\mathbf{N}^{\sharp}\right) \subset \overline{\mathbf{Ran}\left(\mathbf{A}\right)}, \quad \mathbf{Null}\left(\mathbf{N}^{\sharp}\right) = \mathbf{Null}\left(\mathbf{N}^{*}\mathbf{A}\right),$

(2) $\mathbf{N}^{\sharp} \in \mathcal{B}_{\mathbf{A}}\big[\mathcal{Z}\big], \quad \left(\mathbf{N}^{\sharp}\right)^{\sharp} = \mathbf{P}_{\overline{\mathbf{Ran}(\mathbf{A})}}\mathbf{N}\mathbf{P}_{\overline{\mathbf{Ran}(\mathbf{A})}},$

(3) $\mathbf{N}^{\sharp}\mathbf{N}$ and $\mathbf{N}\mathbf{N}^{\sharp}$ are $\mathbf{A}-$ selfadjoint and $\mathbf{A}-$ positive.

(4) If $\mathbf{S} \in \mathcal{B}_{\mathbf{A}}\big[\mathcal{Z}\big]$, then $\mathbf{NS} \in \mathcal{B}_{\mathbf{A}}\big[\mathcal{Z}\big]$ and $(\mathbf{NS})^{\sharp} = \mathbf{S}^{\sharp}\mathbf{N}^{\sharp}$,

(5) $\|\mathbf{N}\|_{\mathbf{A}}^{2} = \left\|\mathbf{N}^{\sharp}\right\|_{\mathbf{A}}^{2} = \left\|\mathbf{N}^{\sharp}\mathbf{N}\right\|_{\mathbf{A}} = \left\|\mathbf{N}\mathbf{N}^{\sharp}\right\|_{A}.$

An operator $N \in \mathcal{B}\big[\mathcal{Z}\big]$ is said to be A-bounded if there exists $k > 0$ such that $\|\mathbf{N}\varphi\|_{\mathbf{A}} \leq k\|\varphi\|_{\mathbf{A}}$ for all $\varphi \in \mathcal{Z}$. The set of all operators in $\mathcal{B}\big[\mathcal{Z}\big]$ admitting $\mathbf{A}^{\frac{1}{2}}$-adjoint is denoted by $\mathcal{B}_{\mathbf{A}^{\frac{1}{2}}}\big[\mathcal{Z}\big]$. We note from

$$\mathcal{B}_{\mathbf{A}^{\frac{1}{2}}}\big[\mathcal{Z}\big] = \left\{\mathbf{N} \in \mathcal{B}\big[\mathcal{Z}\big] : \exists\, k > 0; \|\mathbf{N}\varphi\|_{\mathbf{A}} \leq k\,\|\varphi\|_{\mathbf{A}}, \forall\, \varphi \in \mathcal{Z}\right\}.$$

The A-norm of $\mathbf{N} \in \mathcal{B}_{\mathbf{A}^{\frac{1}{2}}}\big[\mathcal{Z}\big]$ is given by

$$\|\mathbf{N}\|_{\mathbf{A}} := \sup_{\varphi \notin \mathbf{Null}(\mathbf{A})} \frac{\|\mathbf{N}\varphi\|_{\mathbf{A}}}{\|\varphi\|_{\mathbf{A}}} = \sup_{\|\varphi\|_{\mathbf{A}}=1} \|\mathbf{N}\varphi\|_{\mathbf{A}} = \sup_{\|\varphi\|_{\mathbf{A}}\leq 1} \|\mathbf{N}\varphi\|_{\mathbf{A}}.$$

(see [3]). Observe that if $\mathbf{N}$ is $\mathbf{A}$-bounded, then

$$\|\mathbf{N}\varphi\|_{\mathbf{A}} \leq \|\mathbf{N}\|_{\mathbf{A}}\,\|\varphi\|_{\mathbf{A}}, \forall \varphi \in \mathcal{Z}.$$

This implies that, for $\mathbf{N}, \mathbf{T} \in \mathcal{B}_{\mathbf{A}^{\frac{1}{2}}}\big[\mathcal{Z}\big]$ we have $\|\mathbf{NT}\|_{\mathbf{A}} \leq \|\mathbf{N}\|_{A}\|\mathbf{T}\|_{\mathbf{A}}$ and $\mathbf{N}\left(\mathbf{Null}\left(\mathbf{A}\right)\right) \subseteq \mathbf{Null}\left(\mathbf{A}\right)$. Note that $\mathcal{B}_{\mathbf{A}}\big[\mathcal{Z}\big] \subset \mathcal{B}_{\mathbf{A}^{\frac{1}{2}}}\big[\mathcal{Z}\big]$.

An operator $\mathbf{N} \in \mathbf{B}_{\mathbf{A}}\big[\mathcal{Z}\big]$ is called ([1])

(1) $\mathbf{A}$-isometry if $\mathbf{N}^{\sharp}\mathbf{N} = \mathbf{P} \quad \Big(\|\mathbf{N}\varphi\|_{\mathbf{A}} = \|\varphi\|_{\mathbf{A}} \;\; \forall\, \varphi \in \mathcal{Z}\Big),$

(2) $\mathbf{A}$-unitary if $\mathbf{N}^{\sharp}\mathbf{N} = \left(\mathbf{N}^{\sharp}\right)^{\sharp}\mathbf{N}^{\sharp} = \mathbf{P} \quad \Big(\|\mathbf{N}\varphi\|_{\mathbf{A}} = \|\mathbf{N}^{\sharp}\varphi\|_{\mathbf{A}} = \|\varphi\|_{\mathbf{A}} \;\; \forall\, \varphi \in \mathcal{Z}\Big).$

For more details on semi-Hilbertian space operators can be found in [1, 2, 3, 4, 5, 9, 14, 15, 16, 17, 18, 20, 21, 23, 24, 25] and references therein.

The concepts of paranormal, n-paranormal, k-quasi-paranormal and m-quasi-k-paranormal for Hilbert space operators where introduced and investigated in [6, 7, 12, 13, 19, 26]. An operator $\mathbf{N} \in \mathcal{B}\big[\mathcal{Z}\big]$ is said to be

(i) hyponormal if $\|\mathbf{N}\varphi\| \geq \|\mathbf{N}^{*}\varphi\| \quad \forall\, \varphi \in \mathcal{Z}$,

(ii) paranormal if $\|\mathbf{N}^{2}\varphi\|\|\varphi\| \geq \|\mathbf{N}\varphi\|^{2} \quad \forall\, \varphi \in \mathcal{Z}$ ([11]),

(iii) n-paranormal if $\|\mathbf{N}^{n+1}\varphi\|\|\varphi\|^{n} \geq \|\mathbf{N}\varphi\|^{n+1} \quad \forall\, \varphi \in \mathcal{Z}$ ([7]),

(iv) k-quasi-paranormal if $\|\mathbf{N}^{k+2}\varphi\|\|\mathbf{N}^{k}\varphi\| \geq \|\mathbf{N}^{k+1}\varphi\|^2$, for all $\varphi \in \mathcal{Z}$ and for some positive integer k ([13])

(v) m-quasi-n-paranormal if $\|\mathbf{N}^{m+n+1}\varphi\|\|\mathbf{N}^{m}\varphi\|^n \geq \|\mathbf{N}^{m+1}\varphi\|^{n+1} \quad \forall\, \varphi \in \mathcal{Z}$ for some positive integers n and m ([26]).

Here and henceforth, suppose that m is a nonnegative integer, and n is a positive integer.

Many authors has extended some of these concepts to the semi-Hilbertian operators. An operator $\mathbf{N} \in \mathcal{B}_{\mathbf{A}}\big[\mathcal{Z}\big]$ is said to be

(i) A-hyponormal if $\|\mathbf{N}\varphi\|_{\mathbf{A}} \geq \|\mathbf{N}^{\#}\varphi\|_{\mathbf{A}}$ ([24],

(ii) k-quasi-$\mathbf{A}$-hyponormal if $\|N^{k+1}\varphi\|_{\mathbf{A}} \geq \|\mathbf{N}^{\#}\mathbf{N}^{k}\varphi\|_{\mathbf{A}}$ ([24]),

(iii) $\mathbf{A}$-paranormal if $\|\mathbf{N}^2\varphi\|_{A}\|\varphi\|_{\mathbf{A}} \geq \|\mathbf{N}\varphi\|_{\mathbf{A}}^2$, for all $\varphi \in \mathcal{Z}$ ([15]),

(iv) (n, A)-paranormal if $\|\mathbf{N}^{n+1}\varphi\|_{\mathbf{A}}\|\varphi\|_{\mathbf{A}}^n \geq \|\mathbf{N}\varphi\|_{\mathbf{A}}^{n+1} \quad \forall\, \varphi \in \mathcal{Z}$, ([22])

(v) k-quasi-$\mathbf{A}$-paranormal if $\|\mathbf{N}^{k+2}\varphi\|_{\mathbf{A}}\|\mathbf{N}\varphi\|_{\mathbf{A}} \geq \|\mathbf{N}^{k+1}\varphi\|_{\mathbf{A}}^2 \quad \forall\, \varphi \in \mathcal{Z}$ ([14]).

Following our work in [21], in the present paper we introduce and study a class of operators on the semi-Hilbertian space $(\mathcal{Z}, \langle\, .\, \rangle_{\mathbf{A}})$ which is a common generalization of $(n, \mathbf{A})$-paranormal and k-quasi-$\mathbf{A}$-paranormal operators. More precisely, which is called the class of m-quasi-$(n, \mathbf{A})$-paranormal operator. It is proved in Example 2.1 that there is an operator which is m-quasi-$(n, \mathbf{A})$- paranormal but not $(n, \mathbf{A})$-paranormal for some positive integers m and n, and thus, the proposed new class of operators contains the class of $(n, \mathbf{A})$-paranormal operators as a proper subclass. This paper consists of two parts as follows. In Section 2, we show some properties of m-quasi-(n, A)-paranormal operators via an equivalent condition for an operator $\mathbf{N} \in \mathcal{B}_{\mathbf{A}}\big[\mathcal{Z}\big]$ to be m-quasi-$(n, \mathbf{A})$-paranormal (Theorem 2.2). Several properties are proved by exploiting this characterization (Proposition 2.3, Proposition 2.4, Proposition 2.5, Theorem 2.7, Lemma 3.1). In particular, we prove that if $\mathbf{N} \in \mathcal{B}_{\mathbf{A}}\big[\mathcal{H}\big]$ is an m-quasi-$(n, \mathbf{A})$-paranormal and $\mathbf{T} \in \mathcal{B}_{\mathbf{A}}\big[\mathcal{H}\big]$ is an $\mathbf{A}$-isometry or an $\mathbf{A}$-unitary operator then $\mathbf{N}.\mathbf{T}$ is an m-quasi-$(m, \mathbf{A})$-paranormal under suitable conditions (Theorem 2.8, Theorem 2.9). The product of two members of m-quasi-$(n, \mathbf{A})$-paranormal operators is also studied(Theorem 2.10, Theorem 2.11). Section 3, is devoted to describe some properties of tensor product of some members related to m-quasi-$(n, \mathbf{A})$-paranormal operators. We show that the class of m-quasi-$(n, \mathbf{A})$-paranormal operators is closed under tensor product (Theorem 3.4).

2. Properties of m-quasi-$(n, \mathbf{A})$-paranormal operators

In this section, we define the class of m-quasi-$(n, \mathbf{A})$-paranormal operators in semi-Hilbertian spaces and we investigate some properties of such operators.

Firstly, we start with the definition of this class.

Definition 2.1. *Let m and n be positive integers, an operator $\mathbf{N} \in \mathcal{B}_{\mathbf{A}}\big[\mathcal{Z}\big]$ is called an m-quasi-$(n, \mathbf{A})$-paranormal if*

$$\left\|\mathbf{N}^{m+n+1}\varphi\right\|_{A} \|\mathbf{N}^{m}\varphi\|_{A}^{n} \geq \left\|\mathbf{N}^{m+1}\varphi\right\|_{\mathbf{A}}^{n+1}$$

for all $\varphi \in \mathcal{Z}$.

Let $\mathcal{Q}[m] \cap \mathcal{P}_{\mathbf{A}}[n]$ be the sets of all m-quasi-$(n, \mathbf{A})$-paranormal operators.

Remark. (1) *if $m = 0$ we get the class of $(n, \mathbf{A})$-paranormal operators introduced in* [15].

(2) *If $m = 1$, $\mathbf{N}$ is a quasi-(n, A)-paranormal operator.*

(3) *If $\mathbf{A} = \mathbf{I}$, then every m-quasi-$(n, \mathbf{A})$-paranormal is m-quasi-n-paranormal operators (*[26]*).*

(5) *The following inclusions hold:*

$$\mathcal{P}_{\mathbf{A}}[1] \subseteq \mathcal{P}_{\mathbf{A}}[n] \subseteq \mathcal{Q}[m] \cap \mathcal{P}_{\mathbf{A}}[n] \subset \mathcal{Q}[m+1] \cap \mathcal{P}_{A}[n].$$

From the above inclusion we can see that $\mathcal{P}_{\mathbf{A}}[n]$ form a subclass of $\mathcal{Q}[m] \cap \mathcal{P}_{\mathbf{A}}[n]$ for all positive integers m and n. The following example shows that the converse is not true in general.

Example 2.1. *Let $\mathcal{Z} = \mathbb{C}^3, \mathbf{N} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. A direct calculation shows that $\mathbf{A} \geq 0$ and $\mathbf{Ran}\big(\mathbf{N}^*\mathbf{A}\big) \subset \mathbf{Ran}(\mathbf{A})$ Thus $\mathbf{N} \in \mathcal{B}_A\big[\mathcal{Z}\big]$. Moreover $\mathbf{N}$ satisfies*

$$\big\|\mathbf{N}^{m+n+1}\varphi\big\|_A \|\mathbf{N}^m\varphi\|_A^n \geq \big\|\mathbf{N}^{m+1}\varphi\big\|_{\mathbf{A}}^{n+1}$$

for all $\varphi \in \mathcal{Z}$, $m \geq 2, n \geq 2$. But

$$\big\|\mathbf{N}^{n+1}\varphi\big\|_A \|\varphi\|_A^n \geq \|\mathbf{N}\varphi\|_{\mathbf{A}}^{n+1}$$

not satisfied for $n \geq 2$ and $\varphi_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Hence $\mathbf{N}$ is in $\mathcal{Q}[m] \cap \mathcal{P}_{\mathbf{A}}[n]$ for $m \geq 2$ and $n \geq 2$ but $\mathbf{N}$ is not in $\mathcal{P}_{\mathbf{A}}[n]$ for $n \geq 2$.

Lemma 2.1. *(*[10]*) Let a and b two positive number, then $a^\alpha b^\mu \leq \alpha a + \mu b$ holds for $\alpha, \mu > 0$ such that $\alpha + \mu = 1$.*

In [15] it has been shown that $\mathbf{N} \in \mathcal{B}_{\mathbf{A}}\big[\mathcal{Z}\big]$ is an $(n, \mathbf{A})$-paranormal if and only if

$$\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^\sharp\mathbf{N} + n\beta^{n+1}\mathbf{P} \geq_{\mathbf{A}} 0, \quad \forall\ \beta > 0. \tag{2.1}$$

Similarly, we have the following characterization for the members of the class of m-quasi-$(n, \mathbf{A})$-paranormal operators. It is similar to [13, Theorem 2.1] for Hilbert space operators.

Theorem 2.2. *Let $\mathbf{N} \in \mathcal{B}_A\big[\mathcal{Z}\big]$. Then $\mathbf{N}$ is an m-quasi-$(n, \mathbf{A})$-paranormal if and only if*

$$\big(\mathbf{N}^\#\big)^m \Big(\mathbf{N}^{\#^{n+1}}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^\#\mathbf{N} + n\beta^{n+1}\mathbf{P}\Big)\mathbf{N}^m \geq_{\mathbf{A}} 0, \tag{2.2}$$

for all $\beta > 0$. Equivalently, $\mathbf{N}$ is an m-quasi-$(n, \mathbf{A})$-paranormal if and only if

$$\frac{1}{n+1}\Big(\beta^{-n}\mathbf{N}^{\sharp(m+n+1)}\mathbf{N}^{m+n+1} + n\beta\mathbf{N}^{\sharp m}\mathbf{P}\mathbf{N}^m\Big) \geq_{\mathbf{A}} \big(\mathbf{N}^\sharp\big)^{m+1}\mathbf{N}^{m+1}, \qquad \forall\ \beta > 0.$$

Proof. First we show the direct implication. Assume that

$$\left\|\mathbf{N}^{m+n+1}\varphi\right\|_A \|\mathbf{N}^m\varphi\|_A^n \geq \left\|\mathbf{N}^{m+1}\varphi\right\|_{\mathbf{A}}^{n+1}$$

for all $\varphi \in \mathcal{H}$ or equivalently

$$\left\|\mathbf{N}^{m+n+1}\varphi\right\|_A^{\frac{1}{n+1}} \|\mathbf{N}^m\varphi\|_A^{\frac{n}{n+1}} \geq \left\|\mathbf{N}^{m+1}\varphi\right\|_{\mathbf{A}}$$

for all $\varphi \in \mathcal{H}$. Then by taking into account Lemma 2.1 , we my write

$$\begin{aligned}
& \frac{1}{n+1}\left\langle \beta^{-n}\left(\mathbf{N}^\sharp\right)^{m+n+1}\mathbf{N}^{m+n+1}\varphi \mid \varphi\right\rangle_A + \frac{n}{n+1}\left\langle \beta\mathbf{N}^{\sharp m}\mathbf{P}\mathbf{N}^m\varphi \mid \varphi\right\rangle_A \\
\geq\ & \left\langle \beta^{-n}\left(\mathbf{N}^\sharp\right)^{m+n+1}\mathbf{N}^{m+n+1}\varphi \mid \varphi\right\rangle_A^{\frac{1}{n+1}} \left\langle \beta\mathbf{N}^{\sharp m}\mathbf{P}\mathbf{N}^m\varphi \mid \varphi\right\rangle_A^{\frac{n}{n+1}} \\
\geq\ & \|\mathbf{N}^{m+n+1}\varphi\|_A^{\frac{2}{n+1}} \|\mathbf{N}^m\varphi\|_A^{\frac{2n}{n+1}} \\
\geq\ & \|\mathbf{N}^{m+1}\varphi\|_A^2.
\end{aligned}$$

This implies that

$$\frac{1}{n+1}\left\langle \beta^{-n}\left(\mathbf{N}^\sharp\right)^{m+n+1}\mathbf{N}^{m+n+1}\varphi \mid \varphi\right\rangle_A + \frac{n}{n+1}\left\langle \beta\mathbf{N}^{\sharp m}\mathbf{P}\mathbf{N}^m\varphi \mid \varphi\right\rangle_A - \left\langle \mathbf{N}^{\sharp(m+1)}\mathbf{N}^{(m+1)}\varphi \mid \varphi\right\rangle \geq_A 0,$$

the above inequality forces

$$\left\langle \left(\mathbf{N}^\#\right)^m \left(\mathbf{N}^{\#^{n+1}}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^\#\mathbf{N} + n\beta^{n+1}\mathbf{P}\right)\mathbf{N}^m\varphi \mid \varphi\right\rangle_A \geq 0.$$

This leads to,

$$\left(\mathbf{N}^\#\right)^m \left(\mathbf{N}^{\#^{n+1}}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^\#\mathbf{N} + n\beta^{n+1}\mathbf{P}\right)\mathbf{N}^m \geq_{\mathbf{A}} 0,$$

for all $\beta > 0$.

For the other direction, assume that (2.2) holds. If $\varphi_0 \in \mathcal{Z}$ such that $\|\mathbf{N}^{m+n+1}\varphi_0\|_{\mathbf{A}} = 0$ or equivalently, $\mathbf{N}^{m+n+1}\varphi_0 \in \mathbf{Null}(\mathbf{A})$ we have by equation (2.2) that

$$-(n+1)\|\mathbf{N}^{m+1}\varphi_0\|_{\mathbf{A}}^2 + n\beta\|\mathbf{N}^m\varphi_0\|_{\mathbf{A}}^2 \geq 0\Big).$$

If $\beta \longrightarrow 0$ we obtain $\|\mathbf{N}^{m+1}\varphi_0\|_{\mathbf{A}} = 0$. Therefore,

$$\left\|\mathbf{N}^{m+n+1}\varphi_0\right\|_A \|\mathbf{N}^m\varphi_0\|_A^n \geq \left\|\mathbf{N}^{m+1}\varphi_0\right\|_{\mathbf{A}}^{n+1}.$$

Suppose that $\|\mathbf{N}^{m+n+1}\varphi\|_{\mathbf{A}} \neq 0$ for all $\varphi \in \mathcal{Z}$. From (2.2) we have for all $\beta > 0$

$$\frac{1}{n+1}\left(\beta^{-n}\|\mathbf{N}^{m+n+1}\varphi\|_{\mathbf{A}}^2 + \beta n\|\mathbf{N}^m\varphi\|_{\mathbf{A}}^2\right) \geq \|\mathbf{N}^{m+1}\varphi\|_{\mathbf{A}}^2 \qquad \forall\, \varphi \in \mathcal{Z}.$$

Choosing $\beta = \left(\dfrac{\|\mathbf{N}^{m+n+1}\varphi\|_{\mathbf{A}}}{\|\mathbf{N}^m\varphi\|_{\mathbf{A}}}\right)^{\frac{2}{n+1}}$ we get

$$\frac{1}{n+1}\left(\frac{\|\mathbf{N}^{m+n+1}\varphi\|_{\mathbf{A}}^2}{\|\mathbf{N}^m\varphi\|_{\mathbf{A}}^2}\right)^{\frac{-n}{n+1}}\|\mathbf{N}^{m+n+1}\varphi\|_{\mathbf{A}}^2 + \frac{n}{n+1}\left(\frac{\|\mathbf{N}^{m+n+1}\varphi\|_{\mathbf{A}}}{\|\mathbf{N}^m\varphi\|_{\mathbf{A}}}\right)^{\frac{2}{n+1}}\|\mathbf{N}^m\varphi\|_{\mathbf{A}}^2 \geq \|\mathbf{N}^{m+1}\varphi\|_{\mathbf{A}}^2 \quad \varphi \in \mathcal{Z}.$$

This leads to

$$\frac{1}{n+1}\|\mathbf{N}^m\varphi\|_{\mathbf{A}}^{\frac{2n}{n+1}}\|\mathbf{N}^{m+n+1}\varphi\|_{\mathbf{A}}^{\frac{2}{n+1}} + \frac{n}{n+1}\|\mathbf{N}^m\varphi\|_A^{\frac{2n}{n+1}}\|\mathbf{N}^{m+n+1}\varphi\|_A^{\frac{2}{n+1}} \geq \|\mathbf{N}^{m+1}\varphi\|_A^2 \qquad \forall\, \varphi \in \mathcal{Z}.$$

This yields,

$$\|\mathbf{N}^{m+n+1}\varphi\|_{\mathbf{A}}^{\frac{2}{n+1}}\|\mathbf{N}^m\varphi\|_{\mathbf{A}}^{\frac{2n}{n+1}} \geq \|\mathbf{N}^{m+1}\varphi\|_{\mathbf{A}}^2, \qquad \varphi \in \mathcal{Z}.$$

Therefore,

$$\|\mathbf{N}^{m+n+1}\varphi\|_{\mathbf{A}}\|\mathbf{N}^{m}\varphi\|_{\mathbf{A}}^{n} \geq \|\mathbf{N}^{m+1}\varphi\|_{\mathbf{A}}^{n+1}, \qquad \varphi \in \mathcal{Z}.$$

Hence $\mathbf{N}$ is an m-quasi $(n, \mathbf{A})$-paranormal. □

Remark. *It should be noted that (2.2) is equivalent to*

$$\|\mathbf{N}^{m+n+1}\varphi\|_{\mathbf{A}}^{2} - (n+1)\beta^{n}\|\mathbf{N}^{m+1}\varphi\|_{\mathbf{A}}^{2} + n\beta^{n+1}\|\mathbf{N}^{m}\varphi\|_{\mathbf{A}}^{2} \geq 0, \tag{2.3}$$

for all $\varphi \in \mathcal{Z}$ and $\beta > 0$.

Proposition 2.3. *If $\mathbf{N} \in \mathcal{Q}[m] \cap \mathcal{P}_{\mathbf{A}}[n]$, then $\lambda\mathbf{N} \in \mathcal{Q}[m] \cap \mathcal{P}_{\mathbf{A}}[n]$ for all $\lambda \in \mathbb{C}$.*

Proof. For $\lambda \neq 0$, we have for all $\beta > 0$

$$\begin{aligned}
& \big(\lambda\mathbf{N}\big)^{\sharp(m+n+1)}\big(\lambda\mathbf{N}\big)^{m+n+1} - (n+1)\beta^{n}\big(\lambda\mathbf{N}\big)^{\sharp m+1}\big(\lambda\mathbf{N}\big)^{m+1} + n\beta^{n+1}\big(\lambda\mathbf{N}\big)^{\sharp m}\mathbf{P}\big(\lambda\mathbf{N}\big)^{m} \\
= \;& |\lambda|^{2(m+n+1)}\mathbf{N}^{\sharp(m+n+1)}\mathbf{N}^{m+n+1} - (n+1)|\lambda|^{2(m+1)}\beta^{n}\mathbf{N}^{\sharp(m+1}\mathbf{N}^{m+1} + n\beta^{n+1}|\lambda|^{2m}\mathbf{N}^{\sharp m}\mathbf{P}\mathbf{N}^{m} \\
= \;& |\lambda|^{2(m+n+1)}\left(\mathbf{N}^{\sharp(m+n+1)}\mathbf{N}^{m+n+1} - (n+1)\left(\frac{\beta}{|\lambda|^{2}}\right)^{n}\mathbf{N}^{\sharp(m+1)}\mathbf{N}^{m+1} + n\left(\frac{\beta}{|\lambda|^{2}}\right)^{n+1}\mathbf{N}^{\sharp m}\mathbf{P}\mathbf{N}^{m}\right) \\
\geq_{A} \;& 0 \quad \left(\text{since } \mathbf{N} \in \mathcal{Q}[m] \cap \mathcal{P}_{\mathbf{A}}[n]\right).
\end{aligned}$$

Henceforth, $\lambda\mathbf{N} \in \mathcal{Q}[m] \cap \mathcal{P}_{\mathbf{A}}[n]$ by Theorem 2.2. □

Proposition 2.4. *Let $\mathbf{N} \in \mathcal{B}_{A}\big[\mathcal{Z}\big]$ be an m-quasi-$(n, \mathbf{A})$-paranormal. If $\overline{\mathbf{Ran}(\mathbf{N}^{m})} = \mathcal{Z}$, then $\mathbf{N}$ is an $(n, \mathbf{A})$-paranormal.*

Proof. Since $\mathbf{N} \in \mathcal{Q}[m] \cap \mathcal{P}_{\mathbf{A}}[n]$ it follows by Theorem 2.2

$$\mathbf{N}^{\#m}\left(\mathbf{N}^{\#(n+1)}\mathbf{N}^{n+1} - (n+1)\beta^{n}\mathbf{N}^{\#}\mathbf{N} + n\beta^{n+1}\mathbf{P}\right)\mathbf{N}^{m} \geq_{A} 0,$$

for all $\varphi \in \mathcal{Z}$ and for all $\beta > 0$. It results that

$$\left\langle\left(\mathbf{N}^{\#(n+1)}\mathbf{N}^{n+1} - (n+1)\beta^{n}\mathbf{N}^{\#}\mathbf{N} + n\beta^{n+1}\mathbf{P}\right)\mathbf{N}^{m}\varphi \mid \mathbf{N}^{m}\varphi\right\rangle_{A} \geq 0,$$

for all $\varphi \in \mathcal{Z}$ and for all $\beta > 0$. The last inequality is equivalent to

$$\mathbf{N}^{\#(n+1)}\mathbf{N}^{n+1} - (n+1)\beta^{n}\mathbf{N}^{\#}\mathbf{N} + n\beta^{n+1}\mathbf{P} \geq_{A} 0 \;\text{ on } \overline{\mathbf{ran}(\mathbf{N}^{m})} = \mathcal{Z}.$$

This implies that $\mathbf{N}$ is an $(n, \mathbf{A})$-paranormal by [22, Theorem 2.4]. □

Proposition 2.5. *Let $\mathbf{N} \in \mathcal{Q}[m] \cap \mathcal{P}_{\mathbf{A}}[n]$ is such that $\mathbf{Ran}(\mathbf{N}^{m}) = \mathbf{Ran}(\mathbf{N}^{j})$ for some integer $j \in \{1, \cdots, m-1\}$, then $\mathbf{N} \in \mathcal{Q}[j] \cap \mathcal{P}_{\mathbf{A}}[n]$.*

Proof. Since $\mathbf{N} \in \mathcal{Q}[m] \cap \mathcal{P}_{\mathbf{A}}[n]$, we have in view of Theorem 2.2 that

$$\big(\mathbf{N}^{\sharp}\big)^{m}\left(\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1} - (n+1)\beta^{n}\mathbf{N}^{\sharp}\mathbf{N} + n\beta^{n+1}\mathbf{P}\right)\mathbf{N}^{m} \geq_{\mathbf{A}} 0, \tag{2.4}$$

for all $\beta > 0$. Therefore

$$\left\langle\left(\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1} - (n+1)\beta^{n}\mathbf{N}^{\sharp}\mathbf{N} + n\beta^{n+1}\mathbf{P}\right)\mathbf{N}^{m}\varphi \mid \mathbf{N}^{m}\varphi\right\rangle_{\mathbf{A}} \geq 0 \quad \forall\, \varphi \in \mathcal{Z}.$$

From the range condition $\mathbf{Ran}(\mathbf{N}^{j}) = \mathbf{Ran}(\mathbf{N}^{m})$ it is enough to see that

$$\left\langle\left(\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1} - (n+1)\beta^{n}\mathbf{N}^{\sharp}\mathbf{N} + n\beta^{n+1}\mathbf{P}\right)\mathbf{N}^{j}\psi \mid \mathbf{N}^{j}\psi\right\rangle_{\mathbf{A}} \geq 0 \quad \forall\, \psi \in \mathcal{Z}.$$

This yields to

$$\left\langle N^{\sharp j}\left(\mathbf{N}^{\#^{n+1}}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^{\#}\mathbf{N} + n\beta^{n+1}\mathbf{P}\right)\mathbf{N}^j\psi \mid \psi\right\rangle_{\mathbf{A}} \geq 0 \quad \forall\, \psi \in \mathcal{Z}.$$

So we have,

$$\mathbf{N}^{\sharp j}\left(\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^{\sharp}\mathbf{N} + n\beta^{n+1}\mathbf{P}\right)\mathbf{N}^j \geq_A 0.$$

This shows that $\mathbf{N}$ is a j-quasi-$(n, \mathbf{A})$-paranormal. □

Lemma 2.6. [5, Lemma 3.1] *Let* $(\mathbf{N}_k)_{1\leq k\leq 4}$ *where* $\mathbf{N}_k \in \mathcal{B}_{\mathbf{A}}\big[\mathcal{Z}\big]$ *for all* $k = 1, 2, 3, 4$*. Then* $\mathbf{N} = \begin{pmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_4 \end{pmatrix} \in \mathcal{B}_{A_0}(\mathcal{H}\oplus\mathcal{H})$ *where* $\mathbf{A}_0 = \begin{pmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A} \end{pmatrix}$*. Furthermore,* $\mathbf{N}^{\sharp_{\mathbf{A}_0}} = \begin{pmatrix} \mathbf{N}_1^{\sharp} & \mathbf{N}_3^{\sharp} \\ \mathbf{N}_2^{\sharp} & \mathbf{N}_4^{\sharp} \end{pmatrix}$.

Theorem 2.7. *Let* $\mathbf{N}_1, \mathbf{N}_2 \in \mathcal{B}\big[\mathcal{Z}\big]$ *and let* $\mathbf{N}$ *be the operator on* $\mathcal{B}_{A_0}\big[\mathcal{H}\oplus\mathcal{H}\big]$ *defined as*

$$\mathbf{N} = \begin{pmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ 0 & 0 \end{pmatrix}.$$

If $\mathbf{N}_1$ *is an* $(m-1)$*-quasi-*$(n, \mathbf{A})$*-paranormal, then* $\mathbf{N}$ *is an* m*-quasi-*$(n, \mathbf{A}_0)$*-paranormal for* $m \geq 2$*..*

Proof. From Lemma 2.6, we have $\mathbf{N}^{\sharp_{\mathbf{A}_0}} = \begin{pmatrix} \mathbf{N}_1^{\sharp} & 0 \\ \mathbf{N}_2^{\sharp} & 0 \end{pmatrix}$ and with simple calculation we show that

$$\mathbf{N}^{\sharp m}\left(\mathbf{N}^{\sharp(n+1)}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^{\sharp}\mathbf{N} + n\beta^{n+1}\mathbf{P}\right)\mathbf{N}^m$$

$$= \begin{pmatrix} \mathbf{N}_1^{\sharp m}\Psi_n\big(\mathbf{N}_1, \mathbf{N}_1^{\sharp}\big)\mathbf{N}_1^m & \mathbf{N}_1^{\sharp m}\Psi_n\big(\mathbf{N}_1, \mathbf{N}_1^{\sharp}\big)\mathbf{N}_1^{m-1}\mathbf{N}_2 \\ \mathbf{N}_2^{\sharp}\mathbf{N}_1^{\sharp(m-1)}\Psi_n\big(\mathbf{N}_1, \mathbf{N}_2^{\sharp}\big)N_1^m & \mathbf{N}_2^{\sharp}\mathbf{N}_1^{\sharp(m-1)}\Psi_n\big(\mathbf{N}_1, \mathbf{N}_1^{\sharp}\big)\mathbf{N}^{m-1}\mathbf{N}_2 \end{pmatrix},$$

where

$$\Psi_n\big(\mathbf{N}_1, \mathbf{N}_1^{\sharp}\big) = \mathbf{N}_1^{\#(n+1)}\mathbf{N}_1^{n+1} - (n+1)\beta^n\mathbf{N}_1^{\#}\mathbf{N}_1 + n\beta^{n+1}\mathbf{P}$$

for all $\lambda > 0$.

Let $\varphi = \psi_1 \oplus \psi_2 \in \mathcal{Z}\oplus\mathcal{Z}$ and taking into account that $\mathbf{N}_1$ is an $(m-1)$-quasi-$(n, \mathbf{A})$-paranormal, we have

$$\begin{aligned}
&\left\langle \mathbf{N}^{\#m}\left(\mathbf{N}^{\#(n+1)}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^{\#}\mathbf{N} + n\beta^{n+1}\mathbf{P}\right)\mathbf{N}^m\psi \mid \psi\right\rangle_{A_0} \\
= \;& \left\langle \mathbf{N}_1^{\#m}\Psi_n\big(\mathbf{N}_1, \mathbf{N}_1^{\sharp}\big)\mathbf{N}_1^m\psi_1 \mid \psi_1\right\rangle_A + \left\langle \mathbf{N}_1^{\#m}\Psi_n\big(\mathbf{N}_1, \mathbf{N}_1^{\sharp}\big)\mathbf{N}_1^{m-1}\mathbf{N}_2\psi_2 \mid \psi_1\right\rangle_A \\
& + \left\langle \mathbf{N}_2^{\#}\mathbf{N}_1^{\sharp(m-1)}\Psi_n\big(\mathbf{N}_1, \mathbf{N}_2^{\sharp}\big)N_1^m\psi_1 \mid \psi_2\right\rangle_A + \left\langle \mathbf{N}_2^{\#}\mathbf{N}_1^{\sharp(m-1)}\Psi_n\big(\mathbf{N}_1, \mathbf{N}_1^{\sharp}\big)\mathbf{N}_1^{m-1}\mathbf{N}_2\psi_2 \mid \psi_2\right\rangle_A \\
= \;& \left\langle \mathbf{N}_1^{\sharp(m-1)}\Psi_n\big(\mathbf{N}_1, \mathbf{N}_1^{\sharp}\big)\mathbf{N}_1^{m-1}\left(\mathbf{N}_1\psi_1 + \mathbf{N}_2\psi_2\right), \left(\mathbf{N}_1\psi_1 + \mathbf{N}_2\psi_2\right)\right\rangle_A \geq 0.
\end{aligned}$$

□

The following theorem presents the sufficient conditions for which the product of a member of $\mathcal{Q}[m] \cap \mathcal{P}_{\mathbf{A}}[n]$ with an $\mathbf{A}$-isometry remains in $\mathcal{Q}[m] \cap \mathcal{P}_{\mathbf{A}}[n]$.

Theorem 2.8. *Let $\mathbf{N}, \mathbf{T} \in \mathcal{B}_{\mathbf{A}}[\mathcal{Z}]$ be such that $\mathbf{N} \in \mathcal{Q}[m] \cap \mathcal{P}_{\mathbf{A}}[n]$ and $\mathbf{T}$ be an $\mathbf{A}$-isometry. Assume that*

$$\begin{cases} \mathbf{TN} = \mathbf{NT}, \quad \mathbf{TN}^\sharp = \mathbf{N}^\sharp\mathbf{T} \\ \mathbf{T}(\mathbf{Null}(\mathbf{A})^\perp) \subseteq \mathbf{Null}(\mathbf{A})^\perp \\ \mathbf{N}(\mathbf{Null}(\mathbf{A})^\perp) \subseteq \mathbf{Null}(\mathbf{A})^\perp \end{cases}$$

Then $\mathbf{TN} \in \mathcal{Q}[k] \cap \mathcal{P}_{\mathbf{A}}[n]$.

Proof. Since $\mathbf{T}$ is an $\mathbf{A}$-isometry, we have $\mathbf{T}^\sharp\mathbf{T} = \mathbf{P}$. However, by the conditions $\mathbf{T}(\mathbf{Null}(A)^\perp) \subseteq \mathbf{Null}(A)^\perp$ and $\mathbf{N}(\mathbf{Null}(A)^\perp) \subseteq \mathbf{Null}(A)^\perp$ it follows that $\mathbf{Null}(A)$ is a reducing subspace for both $\mathbf{T}$ and $\mathbf{N}$. This yield to

$$\mathbf{PT} = \mathbf{TP},\ \mathbf{PT}^\sharp = \mathbf{T}^\sharp\mathbf{P}$$

and

$$\mathbf{PN} = \mathbf{NP},\ \mathbf{PN}^\sharp = \mathbf{N}^\sharp\mathbf{P}.$$

We deduce that $\mathbf{T}^{\sharp(n+1)}\mathbf{T}^{n+1} = \mathbf{P}$.

In order to prove that $\mathbf{TN} \in \mathcal{Q}[m] \cap \mathcal{P}_{\mathbf{A}}[n]$, we will apply Theorem 2.2. To this gaol we have for all $\beta > 0$,

$$\begin{aligned}
& \big(\mathbf{NT}\big)^{\sharp m}\Big(\big(\mathbf{NT}\big)^{\sharp(n+1)}\big(\mathbf{NT}\big)^{n+1} - (n+1)\beta^n\big(\mathbf{NT}\big)^{\sharp}\big(\mathbf{NT}\big) + n\beta^{n+1}\mathbf{P}\Big)\big(\mathbf{NT}\big)^m \\
= \;& \mathbf{T}^{\sharp m}\mathbf{N}^{\sharp m}\Big(\mathbf{T}^{\sharp(n+1)}\mathbf{T}^{n+1}\mathbf{N}^{\sharp(n+1)}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{T}^\sharp\mathbf{TNN}^\sharp + n\beta^{n+1}\mathbf{P}\Big)\mathbf{N}^m\mathbf{T}^m \\
= \;& \mathbf{T}^{\sharp m}\mathbf{N}^{\sharp m}\mathbf{P}\Big(\mathbf{N}^{\sharp(n+1)}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^\sharp\mathbf{N} + n\beta^{n+1}\mathbf{P}\Big)\mathbf{N}^m\mathbf{T}^m \\
= \;& \\
= \;& \big(\mathbf{PT}\big)^{\sharp m}\underbrace{\Big[\mathbf{N}^{\sharp m}\Big(\mathbf{N}^{\sharp(n+1)}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^\sharp\mathbf{N} + n\beta^{n+1}\mathbf{P}\Big)\mathbf{N}^m\Big]}_{\geq_{\mathbf{A}} 0}\Big)\big(\mathbf{PT}\big)^m \\
\geq_{\mathbf{A}} \;& 0,
\end{aligned}$$

where the last inequality follows from the assumption that $\mathbf{N} \in \mathcal{Q}[m] \cap \mathcal{P}_{\mathbf{A}}[n]$ and [24, Lemma 2.1]$\big)$. □

Theorem 2.9. *Let $\mathbf{N} \in \mathcal{B}\big[\mathcal{Z}\big]$ be in $\mathcal{Q}[m] \cap \mathcal{P}_{\mathbf{A}}[n]$. Assume that $\mathbf{N}(\mathbf{Null}(\mathbf{A})^\perp) \subseteq \mathbf{Null}(\mathbf{A})^\perp$ and $\mathbf{U} \in \mathcal{B}_{\mathbf{A}}[\mathcal{Z}]$ be an $\mathbf{A}$-unitarily operator, then $\mathbf{UNU}^\sharp \in \mathcal{Q}[m] \cap \mathcal{P}_{\mathbf{A}}[n]$.*

Proof. Since $\mathbf{U}$ is an $\mathbf{A}$-unitary operator,we get $\mathbf{UU}^\sharp = \big(\mathbf{U}^\sharp\big)^\sharp\mathbf{U}^\sharp = \mathbf{P}$ or equivalently

$$\|\mathbf{U}\varphi\|_{\mathbf{A}} = \|\mathbf{U}^\sharp\varphi\|_{\mathbf{A}} = \|\varphi\|_{\mathbf{A}} \ \ \forall\ \ \varphi \in \mathcal{Z}.$$

By the assumption that $\mathbf{N}(\mathbf{Null}(\mathbf{A})^\perp) \subseteq \mathbf{Null}(\mathbf{A})^\perp$ it follows that $\mathbf{Null}(\mathbf{A})$ is a reducing subspace for $\mathbf{N}$, from which we can write $\mathbf{NP} = \mathbf{PN}$ and
$\mathbf{PA} = \mathbf{AP} = \mathbf{A}$.

So we have that

$$\begin{aligned}(\mathbf{U}\mathbf{N}\mathbf{U}^\sharp)^j &= \underbrace{(\mathbf{U}\mathbf{N}\mathbf{U}^\sharp)(\mathbf{U}\mathbf{N}\mathbf{U}^\sharp).\cdots(\mathbf{U}\mathbf{N}\mathbf{U}^\sharp)}_{j-\text{times}}\\ &= (\mathbf{U}\mathbf{N}\mathbf{P}\mathbf{N}\mathbf{U}^\sharp)\cdots(\mathbf{U}\mathbf{N}\mathbf{U}^\sharp)\\ &= \mathbf{U}\mathbf{P}\mathbf{N}^2\mathbf{U}^\sharp\cdots(\mathbf{U}\mathbf{N}\mathbf{U}^\sharp)\\ &= \vdots\\ &= \mathbf{U}\mathbf{P}\mathbf{N}^j\mathbf{U}^\sharp. \end{aligned}\tag{2.5}$$

By the assumptions on $\mathbf{N}$, $\mathbf{U}$ and (2.5), we infer that

$$\begin{aligned}\left\|(\mathbf{U}\mathbf{N}\mathbf{U}^\sharp)^{m+1}\varphi\right\|_{\mathbf{A}}^{n+1} &= \left\|\mathbf{U}\mathbf{P}\mathbf{N}^{m+1}\mathbf{U}^\sharp\varphi\right\|_{\mathbf{A}}^{n+1}\\ &\leq \left\|\mathbf{N}^{m+n+1}\mathbf{U}^\sharp\varphi\right\|_{\mathbf{A}}\left\|\mathbf{N}^{m}\mathbf{U}^\sharp\varphi\right\|_{\mathbf{A}}^{n}\\ &= \left\|\mathbf{P}\mathbf{N}^{n+1}\mathbf{U}^\sharp\varphi\right\|_{\mathbf{A}}\left\|\mathbf{P}\mathbf{N}^{m}\mathbf{U}^\sharp\varphi\right\|_{\mathbf{A}}^{n} \quad \left(\text{since } \mathbf{N}\left(\overline{\mathbf{ran}(\mathbf{A})}\right)\subseteq\overline{\mathbf{ran}(\mathbf{A})}\right)\\ &= \left\|\mathbf{U}\mathbf{P}\mathbf{N}^{m+n+1}\mathbf{U}^\sharp\varphi\right\|_{\mathbf{A}}\left\|\mathbf{U}\mathbf{P}\mathbf{N}^{m}\mathbf{U}^\sharp\varphi\right\|_{\mathbf{A}}^{n}\\ &= \left\|(\mathbf{U}\mathbf{N}\mathbf{U}^\sharp)^{m+n+1}\varphi\right\|_{\mathbf{A}}\left\|(\mathbf{U}\mathbf{N}\mathbf{U}^\sharp)^{m}\varphi\right\|_{\mathbf{A}}^{n}.\end{aligned}$$

So we get

$$\left\|(\mathbf{U}\mathbf{N}\mathbf{U}^\sharp)^{n+1}\varphi\right\|_{\mathbf{A}}\left\|(\mathbf{U}\mathbf{N}\mathbf{U}^\sharp)^{m}\varphi\right\|_{\mathbf{A}}^{n} \geq \left\|(\mathbf{U}\mathbf{N}\mathbf{U}^\sharp)^{m+1}\varphi\right\|_{\mathbf{A}}^{n+1}, \quad \forall\ \varphi\in\mathcal{Z},$$

which immediately gives that $\mathbf{U}\mathbf{N}\mathbf{U}^\sharp \in \mathcal{Q}[m]\cap\mathcal{P}_{\mathbf{A}}[n]$. □

Theorem 2.10. *Let $\mathbf{N},\mathbf{T}\in\mathcal{Q}[m]\cap\mathcal{P}_{\mathbf{A}}[n]$ such that $\mathbf{N}\mathbf{T}=\mathbf{T}\mathbf{N}$. If $\mathbf{N}$ and $\mathbf{T}$ satisfy*

$$\|\mathbf{N}^{m+n+1}\mathbf{T}^m\varphi\|_{\mathbf{A}}\|\mathbf{T}^{m+n+1}\mathbf{N}^m\varphi\|_{\mathbf{A}} \leq \|(\mathbf{N}\mathbf{T})^{m+n+1}\varphi\|_{\mathbf{A}}\|(\mathbf{N}\mathbf{T})^{m}\varphi\|_{\mathbf{A}}, \tag{2.6}$$

$\forall\,\varphi\in\mathcal{Z}$. Then $\mathbf{N}.\mathbf{T}\in\mathcal{Q}[m]\cap\mathcal{P}_{\mathbf{A}}[n]$.

Proof.

$$\begin{aligned} &\|(\mathbf{N}\mathbf{T})^{m+1}\varphi\|_{\mathbf{A}}^{n+1} = \|\mathbf{N}^{m+1}\mathbf{T}^{m+1}\varphi\|_{\mathbf{A}}^{n+1}\\ \leq\ & \|\mathbf{N}^{m+n+1}\mathbf{T}^{m+1}\|_{\mathbf{A}}\|\mathbf{N}^{m}\mathbf{T}^{m+1}\varphi\|_{\mathbf{A}}^{n}\\ =\ & \|\mathbf{T}^{m+1}\mathbf{N}^{m+n+1}\varphi\|_{A}\|\mathbf{T}^{m+1}\mathbf{N}^{m}\varphi\|_{\mathbf{A}}^{n}\\ \leq\ & \left(\|\mathbf{T}^{m+n+1}\mathbf{N}^{m+n+1}\varphi\|_{A}\|\mathbf{T}^{m}\mathbf{N}^{m+n+1}\varphi\|_{\mathbf{A}}^{n}\right)^{\frac{1}{n+1}}\left(\|\mathbf{T}^{m+n+1}\mathbf{N}^{m}\varphi\|_{\mathbf{A}}\|\mathbf{T}^{m}\mathbf{N}^{m}\varphi\|_{\mathbf{A}}^{n}\right)^{\frac{n}{n+1}}\\ =\ & \|(\mathbf{N}\mathbf{T})^{m+n+1}\|_{\mathbf{A}}^{\frac{1}{n+1}}\|(\mathbf{N}\mathbf{T})^{m}\varphi\|_{A}^{\frac{n^2}{n+1}}\left(\|\mathbf{N}^{m+n+1}\mathbf{T}^{m}\varphi\|_{\mathbf{A}}\|\mathbf{T}^{m+n+1}\mathbf{N}^{m}\varphi\|_{\mathbf{A}}\right)^{\frac{n}{n+1}}\\ \leq\ & \|(\mathbf{N}\mathbf{T})^{m+n+1}\|_{\mathbf{A}}^{\frac{1}{n+1}}\|(\mathbf{N}\mathbf{T})^{m}\varphi\|_{A}^{\frac{n^2}{n+1}}\left(\|(\mathbf{N}\mathbf{T})^{m+n+1}\varphi\|_{\mathbf{A}}\|(\mathbf{T}\mathbf{N})^{m}\varphi\|_{\mathbf{A}}\right)^{\frac{n}{n+1}}\\ =\ & \|(\mathbf{N}\mathbf{T})^{m+n+1}\varphi\|_{\mathbf{A}}\|(\mathbf{T}\mathbf{N})^{m}\varphi\|_{\mathbf{A}}^{n}. \end{aligned}$$

This leads to

$$\|(\mathbf{N}\mathbf{T})^{m+n+1}\varphi\|_{\mathbf{A}}\|(\mathbf{T}\mathbf{N})^{m}\varphi\|_{\mathbf{A}}^{n} \geq \|(\mathbf{N}\mathbf{T})^{m+1}\varphi\|_{\mathbf{A}}^{n+1} \qquad \forall\,\varphi\in\mathcal{Z}.$$

□

Theorem 2.11. *Let* $\mathbf{T} \in \mathcal{B}_{\mathbf{I}}\big[\mathcal{Z}\big]$ *be an invertible operator and* $\mathbf{N}$ *be an operator such that* $\big[\mathbf{N}, \mathbf{T}^\sharp\mathbf{T}\big] = 0$*. Then,* $\mathbf{N}$ *is in* $\mathcal{Q}[m] \cap \mathcal{P}_{\mathbf{I}}[n]$ *if and only if* $\mathbf{T}\mathbf{N}\mathbf{T}^{-1}$ *is in* $\mathcal{Q}[m] \cap \mathcal{P}_{\mathbf{I}}[n]$.

Proof. Assume that $\mathbf{N}$ is in $\mathcal{Q}[m] \cap \mathcal{P}_{\mathbf{I}}[n]$, it follows from Theorem 2.2 that

$$\big(\mathbf{N}^\sharp\big)^m \left(\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^\sharp\mathbf{N} + n\beta^{n+1}\mathbf{I}\right)\mathbf{N}^m \geq_{\mathbf{I}} 0.$$

From this we have that

$$\mathbf{T}\big(\mathbf{N}^\sharp\big)^m \left(\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^\sharp\mathbf{N} + n\beta^{n+1}\mathbf{I}\right)\mathbf{N}^m\mathbf{T}^\sharp \geq_{\mathbf{I}} 0.$$

Since $\big[\mathbf{N}, \mathbf{T}^\sharp\mathbf{T}\big] = 0$ we have $\big[\mathbf{N}^\sharp, \mathbf{T}^\sharp\mathbf{T}\big] = 0$ and we my write

$$\begin{aligned}
& \mathbf{T}\big(\mathbf{N}^\sharp\big)^m \left(\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^\sharp\mathbf{N} + n\beta^{n+1}\mathbf{I}\right)\mathbf{N}^m\mathbf{T}^\sharp\big(\mathbf{T}\mathbf{T}^\sharp\big) \\
= \; & \mathbf{T}\big(\mathbf{N}^\sharp\big)^m \left(\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^\sharp\mathbf{N} + n\beta^{n+1}\mathbf{I}\right)\mathbf{N}^m\big(\mathbf{T}^\sharp\mathbf{T}\big)\mathbf{T}^\sharp \\
= \; & \mathbf{T}\big(\mathbf{T}^\sharp\mathbf{T}\big)\big(\mathbf{N}^\sharp\big)^m \left(\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^\sharp\mathbf{N} + n\beta^{n+1}\mathbf{I}\right)\mathbf{N}^m\mathbf{T}^\sharp \\
= \; & \big(\mathbf{T}\mathbf{T}^\sharp\big)\mathbf{T}\big(\mathbf{N}^\sharp\big)^m \left(\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^\sharp\mathbf{N} + n\beta^{n+1}\mathbf{I}\right)\mathbf{N}^m\mathbf{T}^\sharp.
\end{aligned}$$

This implies that

$$\left[\mathbf{T}\mathbf{T}^\sharp, \mathbf{T}\big(\mathbf{N}^\sharp\big)^m \left(\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^*\mathbf{N} + n\beta^{n+1}\mathbf{I}\right)\mathbf{N}^m\mathbf{T}^\sharp\right] = 0$$

and hence

$$\left[\big(\mathbf{T}\mathbf{T}^\sharp\big)^{-1}, \mathbf{T}\big(\mathbf{N}^\sharp\big)^m \left(\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^\sharp\mathbf{N} + n\beta^{n+1}\mathbf{I}\right)\mathbf{N}^m\mathbf{T}^\sharp\right] = 0.$$

By observing that $\big(\mathbf{T}\mathbf{T}^\sharp\big)^{-1} \geq_{\mathbf{I}} 0$ and

$$\mathbf{T}\big(\mathbf{N}^\sharp\big)^m \left(\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^\sharp\mathbf{N} + n\beta^{n+1}\mathbf{I}\right)\mathbf{N}^m\mathbf{T}^\sharp \geq 0$$

it follows that

$$\mathbf{T}\big(\mathbf{N}^\sharp\big)^m \left(\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^\sharp\mathbf{N} + n\beta^{n+1}\mathbf{I}\right)\mathbf{N}^m\mathbf{T}^\sharp\big(\mathbf{T}\mathbf{T}^\sharp\big)^{-1} \geq 0.$$

According to the condition $\big[\mathbf{N}, \mathbf{T}^\sharp\mathbf{T}\big] = 0$ we my write

$$\big(\mathbf{T}\mathbf{N}\mathbf{T}^{-1}\big)^{\sharp k} = \mathbf{T}^{\sharp-1}\mathbf{N}^{\sharp k}\mathbf{T}^\sharp \;\text{ and } \big(\mathbf{T}\mathbf{N}\mathbf{T}^{-1}\big)^k = \mathbf{T}^{-1}\mathbf{N}^k\mathbf{T}$$

and

$$\big(\mathbf{T}\mathbf{N}\mathbf{T}^{-1}\big)^\sharp\big(\mathbf{T}\mathbf{N}\mathbf{T}^{-1}\big) = \mathbf{T}\mathbf{N}^\sharp\mathbf{N}\mathbf{T}^{-1} \quad \text{and } \big(\mathbf{T}\mathbf{N}\mathbf{T}^{-1}\big)^{\sharp(n+1)}\big(\mathbf{T}\mathbf{N}\mathbf{T}^{-1}\big)^{n+1} = \mathbf{T}\mathbf{N}^{\sharp(n+1)}\mathbf{N}^{n+1}\mathbf{T}^{-1}.$$

Now we are ready to show that $\mathbf{S} = \mathbf{T}\mathbf{N}\mathbf{T}^{-1}$ is in $\mathcal{Q}[m] \cap \mathcal{P}_{\mathbf{I}}[n]$. Indeed,

$$\begin{aligned}
& \mathbf{S}^{\sharp m}\left(\mathbf{S}^{\sharp(n+1)}\mathbf{S}^{n+1} - (n+1)\beta^n\mathbf{S}^\sharp\mathbf{S} + n\beta^{n+1}\mathbf{I}\right)\mathbf{S}^m \\
= \; & \mathbf{T}^{\sharp-1}\mathbf{N}^{\sharp m}\mathbf{T}^\sharp\left(\mathbf{T}\mathbf{N}^{\sharp(n+1)}\mathbf{N}^{n+1}\mathbf{T}^{-1} - (n+1)\beta^n\mathbf{T}\mathbf{N}^\sharp\mathbf{N}\mathbf{T}^{-1} + n\beta^{n+1}\mathbf{I}\right)\mathbf{T}\mathbf{N}^m\mathbf{T}^{-1} \\
= \; & \mathbf{T}\mathbf{N}^{\sharp m}\left(\mathbf{N}^{\sharp(n+1)}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^\sharp\mathbf{N} + n\beta^{n+1}\mathbf{I}\right)\mathbf{N}^m\mathbf{T}^{-1}.
\end{aligned}$$

In order to show that the last expression is positive, we take in our consideration that

$$\mathbf{T}\big(\mathbf{N}^\sharp\big)^m \left(\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^\sharp\mathbf{N} + n\beta^{n+1}\mathbf{I}\right)\mathbf{N}^m\mathbf{T}^\sharp\big(\mathbf{T}\mathbf{T}^\sharp\big)^{-1} \geq 0.$$

This leads to

$$\mathbf{T}\big(\mathbf{N}^{\sharp}\big)^{m}\Big(\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1}-(n+1)\beta^{n}\mathbf{N}^{\sharp}\mathbf{N}+n\beta^{n+1}\mathbf{I}\Big)\,\mathbf{N}^{m}\mathbf{T}^{\sharp}\mathbf{T}^{\sharp-1}\mathbf{T}^{-1}\geq 0$$

and therefore

$$\mathbf{T}\big(\mathbf{N}^{\sharp}\big)^{m}\Big(\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1}-(n+1)\beta^{n}\mathbf{N}^{\sharp}\mathbf{N}+n\beta^{n+1}\mathbf{I}\Big)\,\mathbf{N}^{m}\mathbf{T}^{-1}\geq 0.$$

This does means that $\mathbf{T}\mathbf{N}\mathbf{T}^{-1}$ is in $\mathcal{Q}[m]\cap\mathcal{P}_{\mathbf{I}}[n]$.

Conversely, assume that $\mathbf{S}=\mathbf{T}\mathbf{N}\mathbf{T}^{-1}$ is in $\mathcal{Q}[m]\cap\mathcal{P}_{\mathbf{I}}[n]$. Similarly, we have that

$$\begin{aligned}
&\mathbf{S}^{\sharp m}\Big(\mathbf{S}^{\sharp(n+1)}\mathbf{S}^{n+1}-(n+1)\beta^{n}\mathbf{S}^{\sharp}\mathbf{S}+n\beta^{n+1}\mathbf{I}\Big)\mathbf{S}^{m}\geq_{\mathbf{I}} 0\\
\Longrightarrow\quad &\mathbf{T}\big(\mathbf{N}^{\sharp}\big)^{m}\Big(\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1}-(n+1)\beta^{n}\mathbf{N}^{\sharp}\mathbf{N}+n\beta^{n+1}\mathbf{I}\Big)\,\mathbf{T}^{\sharp}\mathbf{N}^{m}\mathbf{T}^{-1}\geq_{\mathbf{I}} 0\\
\Longrightarrow\quad &\mathbf{T}^{\sharp}\mathbf{T}\big(\mathbf{N}^{\sharp}\big)^{m}\Big(\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1}-(n+1)\beta^{n}\mathbf{N}^{\sharp}\mathbf{N}+n\beta^{n+1}\mathbf{I}\Big)\,\mathbf{N}^{m}\mathbf{T}^{-1}\mathbf{T}\geq_{\mathbf{I}} 0\\
\Longrightarrow\quad &\mathbf{T}^{\sharp}\mathbf{T}\big(\mathbf{N}^{\sharp}\big)^{m}\Big(\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1}-(n+1)\beta^{n}\mathbf{N}^{\sharp}\mathbf{N}+n\beta^{n+1}\mathbf{I}\Big)\,\mathbf{N}^{m}\geq_{\mathbf{I}} 0.
\end{aligned}$$

Since $\big[\mathbf{N},\mathbf{T}^{\sharp}\mathbf{T}\big]=0$, $\big[\mathbf{T}^{\sharp}\mathbf{T},\mathbf{R}\big]=0$ and $\Big[\big(\mathbf{T}^{\sharp}\mathbf{T}\big)^{-1},\mathbf{R}\Big]=0$ where

$$\mathbf{R}=\big(\mathbf{T}^{\sharp}\mathbf{T}\big)\Big(\big(\mathbf{N}^{\sharp}\big)^{m}\Big(\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1}-(n+1)\beta^{n}\mathbf{N}^{\sharp}\mathbf{N}+n\beta^{n+1}\mathbf{I}\Big)\,\mathbf{N}^{m}\Big).$$

Moreover $\big(\mathbf{T}^{\sharp}\mathbf{T}\big)$,$\big(\mathbf{T}^{\sharp}\mathbf{T}\big)^{-1}$ and $\mathbf{R}$ are $\mathbf{I}$-positive we deduce that

$$\big(\mathbf{T}^{\sharp}\mathbf{T}\big)^{-1}\mathbf{R}\geq_{\mathbf{I}} 0.$$

This yields that

$$\big(\mathbf{N}^{\sharp}\big)^{m}\Big(\mathbf{N}^{\sharp^{n+1}}\mathbf{N}^{n+1}-(n+1)\beta^{n}\mathbf{N}^{\sharp}\mathbf{N}+n\beta^{n+1}\mathbf{I}\Big)\,\mathbf{N}^{m}\geq_{\mathbf{I}} 0.$$

This does means $\mathbf{N}$ is in $\mathcal{Q}[m]\cap\mathcal{P}_{\mathbf{I}}[n]$. □

3. Tensor product of m-quasi-$(n, A\otimes\mathbf{B})$-paranormal operators

In this section, we prove under suitable conditions that the tensor product of an m-quasi-$(n,\mathbf{A})$-paranormal and an A-isometry is an m-quasi-$(n,\mathbf{A}\otimes\mathbf{A})$-paranormal operator (Proposition 3.2). However, the tensor product of an m-quasi-$(n,\mathbf{A})$-paranormal and an m-quasi-$(n,\mathbf{B})$-paranormal is an m-quasi $(n,\mathbf{A}\otimes\mathbf{B}$-paranormal (Theorem 3.4).

Let $\mathcal{Z}\overline{\otimes}\mathcal{Z}$ denote the completion, endowed with a reasonable uniform cross norm, of the algebraic tensor product of $\mathcal{Z}$ with itself. An inner product on $\mathcal{Z}\overline{\otimes}\mathcal{Z}$ is defines as

$$\langle\varphi_1\otimes\varphi_2\mid\psi_1\otimes\psi_2\rangle:=\langle\varphi_1\mid\psi_1\rangle\langle\varphi_2\mid\psi_2\rangle\ \text{ where }\ \varphi_k,\psi_k\in\mathcal{Z},\ \text{for } k=1,2$$

Let $\mathbf{A},\mathbf{B}\in\mathcal{B}\big[\mathcal{Z}\big]$ are positive operators. The operator $\mathbf{A}\otimes\mathbf{B}\in\mathcal{B}[\mathcal{Z}\overline{\otimes}\mathcal{Z}]$ is positive and defines a positive semi-definite sesquilinear form

$$\langle .\mid .\rangle_{\mathbf{A}\otimes\mathbf{B}}:\mathcal{Z}\otimes\mathcal{Z}\times\mathcal{Z}\otimes\mathcal{Z}\longrightarrow\mathbb{C}$$

given by

$$\langle\varphi_1\otimes\varphi_2\mid\psi_1\otimes\psi_2\rangle_{\mathbf{A}\otimes\mathbf{B}}=\langle\mathbf{A}\varphi_1\mid\psi_1\rangle\,\langle\mathbf{B}\varphi_2\mid\psi_2\rangle.$$

This semi-inner product induces a semi-norm $\|.\|_{\mathbf{A}\otimes\mathbf{B}}$ defined by

$$\begin{aligned}\|\varphi\otimes\psi\|^2_{\mathbf{A}\otimes\mathbf{B}} &= \langle\varphi\otimes\psi \mid \varphi\otimes\psi\rangle_{\mathbf{A}\otimes\mathbf{B}}\\ &= \langle\mathbf{A}\varphi \mid \varphi\rangle\langle\mathbf{B}\psi \mid \psi\rangle\\ &= \|\varphi\|^2_{\mathbf{A}}\|\psi\|^2_{\mathbf{B}}.\end{aligned}$$

It should be noted that $\|\varphi\otimes\psi\|_{\mathbf{A}\otimes\mathbf{B}} = 0$ if and only if $\varphi\in\mathbf{Null}(\mathbf{A})$ or $\psi\in\mathbf{Null}(\mathbf{B})$. For $\mathbf{N}\in\mathcal{B}\big[\mathcal{Z}\big]$ and $\mathbf{T}\in\mathcal{B}\big[\mathcal{Z}\big]$, $\mathbf{N}\otimes\mathbf{T}\in\mathcal{B}\big[\mathcal{Z}\overline{\otimes}Z\big]$ denotes the tensor product of $\mathbf{N}$ and $\mathbf{T}$ given by $\big(\mathbf{N}\otimes\mathbf{T}\big)\big(\varphi\otimes\psi\big) = \mathbf{N}\varphi\otimes\mathbf{T}\psi$ for $\varphi,\psi\in\mathcal{Z}$.

We begin this section by the following lemma.

Lemma 3.1. *Let $\mathbf{N}\in\mathcal{B}_{\mathbf{A}}\big[\mathcal{Z}\big]$ be in $\mathcal{Q}[m]\cap\mathcal{P}_{\mathbf{A}}[n]$, then $\mathbf{N}\otimes\mathbf{I}$ and $\mathbf{I}\otimes\mathbf{N}$ are in $\mathcal{Q}[m]\cap\mathcal{P}_{\mathbf{A}\otimes\mathbf{A}}[n]$.*

Proof. For all $\beta>0$, we have

$$\begin{aligned}&\big(\mathbf{N}\otimes\mathbf{I}\big)^{\sharp m}\Bigg(\big(\mathbf{N}\otimes\mathbf{I}\big)^{\sharp(n+1)}\big(\mathbb{N}\otimes\mathbf{I}\big)^{n+1} - (n+1)\beta^n\big(\mathbf{N}\otimes\mathbf{I}\big)^{\sharp}\big(\mathbf{N}\otimes\mathbf{I}\big) + n\beta^{n+1}\mathbf{P}\Bigg)\big(\mathbf{N}\otimes\mathbf{I}\big)^m\\ =\;& \mathbf{N}^{\sharp m}\Bigg(\mathbf{N}^{\sharp(n+1)}\mathbf{N}^{n+1} - (n+1)\beta^n\mathbf{N}^{\sharp}\mathbf{N} + n\beta^{n+1}\mathbf{P}\Bigg)\mathbf{N}^m\otimes\mathbf{P}\\ \geq_{\mathbf{A}\otimes\mathbf{A}}\;& 0.\end{aligned}$$

□

Proposition 3.2. *Let $\mathbf{N},\mathbf{T}\in\mathcal{B}_{\mathbf{A}}\big[\mathcal{Z}\big]$ such that $\mathbf{null}(\mathbf{A})^{\perp}$ is invariant for both $\mathbf{N}$ and $\mathbf{T}$. If $\mathbf{N}$ is in $\mathcal{Q}[m]\cap\mathcal{P}_{\mathbf{A}}[n]$ and $\mathbf{T}$ is an $\mathbf{A}$-isometry, then $\mathbf{N}\otimes\mathbf{T}\in\mathcal{B}_{\mathbf{A}\otimes\mathbf{A}}(\mathcal{Z}\,\overline{\otimes}\mathcal{Z})$ is in $\mathcal{Q}[m]\cap\mathcal{P}_{\mathbf{A}\otimes\mathbf{A}}[n]$.*

Proof. We like to notice that $\mathbf{N}\otimes\mathbf{T} = \big(\mathbf{N}\otimes\mathbf{I}\big)\big(\mathbf{I}\otimes\mathbf{T}\big) = \big(\mathbf{I}\otimes\mathbf{T}\big)\big(\mathbf{N}\otimes\mathbf{I}\big)$.On the other hand we have $\mathbf{Null}(\mathbf{A})^{\perp}$ is invariant for $\mathbf{N}$, we obtain $\mathbf{NP}=\mathbf{PN}$ and hence

$$\big(\mathbf{N}\otimes\mathbf{I}\big)\big(\mathbf{I}\otimes\mathbf{T}\big)^{\#} = \big(\mathbf{I}\otimes\mathbf{T}\big)^{\#}\big(\mathbf{N}\otimes\mathbf{I}\big).$$

Since $\mathbf{N}$ is in $\mathcal{Q}[m]\cap\mathcal{P}_{\mathbf{A}}[n]$ and $\mathbf{T}$ is an A-isometry, it follows that $\mathbf{N}\otimes\mathbf{I}\in\mathcal{Q}[m]\cap\mathcal{P}_{\mathbf{A}\otimes\mathbf{A}}[n]$. Moreover

$$\big(\mathbf{N}\otimes\mathbf{I}\big)\big(\mathbf{Null}\big(\mathbf{A}\otimes\mathbf{A}\big)^{\perp}\big)\subset\mathbf{Null}\big(\mathbf{A}\otimes\mathbf{A}\big)^{\perp}.$$

In fact, let $\varphi_1\otimes\varphi_2\in\mathbf{Null}\big(\mathbf{A}\otimes\mathbf{A}\big)^{\perp}$ and $\psi_1\otimes\psi_2\in\mathbf{Null}\big(\mathbf{A}\otimes\mathbf{A}\big)$, we have

$$\begin{aligned}\big\langle\big(\mathbf{N}\otimes\mathbf{I}\big)\big(\varphi_1\otimes\varphi_2\big)\,,\,\psi_1\otimes\psi_2\big\rangle &= \big\langle\mathbf{N}\varphi_1\otimes\varphi_2 \mid \big(\psi_1\otimes\psi_2\big)\big\rangle\\ &= \langle\mathbf{N}\varphi_1 \mid \psi_1\rangle\langle\varphi_2 \mid \psi_2\rangle\\ &= \langle\varphi_1 \mid \mathbf{N}^*\psi_1\rangle\langle\varphi_2 \mid \psi_2\rangle\\ &= \langle\varphi_1\otimes\varphi_2 \mid \mathbf{N}^*\psi_1\otimes\psi_2\rangle.\end{aligned}$$

According to the fact that $\psi_1\otimes\psi_2\in\mathbf{Null}\big(\mathbf{A}\otimes\mathbf{A}\big)$ we get $\psi_1\in\mathbf{Null}(\mathbf{A})$ or $\psi_2\in\mathbf{Null}(\mathbf{A})$. This above consideration shows that

$$\mathbf{N}^*\psi_1\in\mathbf{Null}(\mathbf{A})\ \text{ or }\psi_2\in\mathbf{Null}(\mathbf{A})\ \ \Bigg(\text{because }\mathbf{Null}(\mathbf{A})\ \text{ reduces }\mathbf{N}\Bigg),$$

which implies that

$$\big\langle\big(\mathbf{N}\otimes\mathbf{I}\big)\big(\varphi_1\otimes\varphi_2\big) \mid \psi_1\otimes\psi_2\big\rangle = 0.$$

Repeating this argument, we show that

$$\big(\mathbf{I}\otimes\mathbf{T}\big)\big(\mathbf{Null}\big(\mathbf{A}\otimes\mathbf{A}\big)^{\perp}\big)\subset\mathbf{Null}\big(\mathbf{A}\otimes\mathbf{A}\big)^{\perp}.$$

By applying Theorem 2.8 to $\mathbf{N}\otimes\mathbf{I}$ and $\mathbf{I}\otimes\mathbf{T}$ we can assert that $\mathbf{N}\otimes\mathbf{T}\in\mathcal{Q}[m]\cap\mathcal{P}_{\mathbf{A}\otimes\mathbf{A}}[n]$. The proposition is proved.

□

Corollary 3.3. *Let* $\mathbf{N},\mathbf{T}\in\mathcal{B}_{\mathbf{A}}\big[\mathcal{Z}\big]$ *such that* $\mathbf{Null}(\mathbf{A})^{\perp}$ *is invariant for both* $\mathbf{N}$ *and* $\mathbf{T}$. *If* $\mathbf{N}$ *is in* $\mathcal{Q}[m]\cap\mathcal{P}_{\mathbf{A}}[n]$ *and* $\mathbf{T}$ *is an* $\mathbf{A}$*-isometry, then* $\mathbf{N}\otimes\mathbf{T}^{q}\in\mathcal{Q}[n]\cap\mathcal{P}_{\mathbf{A}\otimes\mathbf{A}}[n]$ *for all positive integer* q.

Proof. Since $\mathbf{T}$ is an $\mathbf{A}$-isometry and $\mathbf{T}\big(\mathbf{Null}(\mathbf{A})^{\perp}\big)\subset\mathbf{Null}(\mathbf{A})^{\perp}$ it follows that $\mathbf{T}^{q}$ is an $\mathbf{A}$-isometry for all positive q. The desired result follows using Proposition 3.2. □

Theorem 3.4. *Let* $\mathbf{N}\in\mathcal{B}_{\mathbf{A}}\big[\mathcal{Z}\big]$ *and* $\mathbf{T}\in\mathcal{B}_{\mathbf{B}}\big[\mathcal{Z}\big]$. *If* $\mathbf{N}$ *is in* $\mathcal{Q}[m]\cap\mathcal{P}_{\mathbf{A}}[n]$ *and* $\mathbf{T}$ *is in* $\mathcal{Q}[m]\cap\mathcal{P}_{\mathbf{B}}[n]$ *, then* $\mathbf{N}\otimes\mathbf{T}$ *is in* $\mathcal{Q}[m]\cap\mathcal{P}_{\mathbf{A}\otimes\mathbf{B}}[n]$.

Proof. From assumptions $\mathbf{N}\in\mathcal{B}_{\mathbf{A}}(\mathcal{Z})$ and $\mathbf{T}\in\mathcal{B}_{\mathbf{B}^{\frac{1}{2}}}(\mathcal{Z})$ we obtain

$$\|\mathbf{N}^{m+1}\varphi\|_{\mathbf{A}}^{m+1}\leq\|\mathbf{N}^{m+n+1}\varphi\|_{\mathbf{A}}^{2}\|\mathbf{N}^{m}\varphi\|_{\varphi}^{n},\quad\forall\,\varphi\in\mathcal{Z}$$

and

$$\|\mathbf{T}^{m+1}\psi\|_{\mathbf{B}}^{n+1}\leq\|\mathbf{T}^{m+n+1}\psi\|_{\mathbf{B}}\|\mathbf{T}^{m}\psi\|_{\mathbf{B}}^{n},\quad\forall\,\psi\in\mathcal{Z}.$$

So we have that

$$\|\mathbf{N}^{m+1}\varphi\|_{\mathbf{A}}^{n+1}\|\mathbf{T}^{m+1}\psi\|_{\mathbf{B}}^{n+1}\leq\|\mathbf{N}^{m+n+1}\varphi\|_{\mathbf{A}}\|\mathbf{T}^{m+n+1}\psi\|_{\mathbf{B}}\|\mathbf{N}^{m}\varphi\|_{\mathbf{A}}^{n}\|\mathbf{T}^{m}\psi\|_{\mathbf{B}}^{n},$$

$\forall\,\varphi,\psi\in\mathcal{Z}$. This shows that

$$\|\mathbf{N}^{m+1}\otimes\mathbf{T}^{m+1}(\varphi\otimes\psi)\|_{\mathbf{A}\otimes\mathbf{B}}^{n+1}\leq\|\mathbf{N}^{m+n+1}\otimes\mathbf{T}^{m+n+1}(\varphi\otimes\psi)\|_{\mathbf{A}\otimes\mathbf{B}}\|\mathbf{N}^{m}\otimes\mathbf{T}^{m}(\varphi\otimes\psi)\|_{\mathbf{A}\otimes\mathbf{B}}^{n},$$

$\forall\,\varphi,\psi\in\mathcal{Z}$ or equivalently,

$$\|\big(\mathbf{N}\otimes\mathbf{T}\big)^{m+1}(\varphi\otimes\psi)\|_{\mathbf{A}\otimes\mathbf{B}}^{n+1}\leq\|\big(\mathbf{N}\otimes\mathbf{T}\big)^{m+n+1}(\varphi\otimes\psi)\|_{\mathbf{A}\otimes\mathbf{B}}\|\big(\mathbf{N}\otimes\mathbf{T}\big)^{m}(\varphi\otimes\psi)\|_{\mathbf{A}\otimes\mathbf{B}}^{n},$$

$\forall\,\varphi,\psi\in\mathcal{Z}$. Therefore we have $\mathbf{N}\otimes\mathbf{T}$ is in $\mathcal{Q}[m]\cap\mathcal{P}_{\mathbf{A}\otimes\mathbf{B}}[n]$. □

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