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ORTHOGONAL STABILITY OF GENERALIZED CUBE ROOT FUNCTIONAL INEQUALITY IN THREE VARIABLES: A FIXED POINT APPROACH

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Abstract. This paper underlined the aspects of stability of orthogonal generalized cube root functional (GCRF) inequality

$$\begin{aligned}&\Bigg\|C\Bigg(al+bm+cn+3(al)^{\frac{2}{3}}\Big((bm)^{\frac{1}{3}}+(cn)^{\frac{1}{3}}\Big)+3(bm)^{\frac{2}{3}}\Big((al)^{\frac{1}{3}}+(cn)^{\frac{1}{3}}\Big)\\&\qquad+3(cn)^{\frac{2}{3}}\Big((bm)^{\frac{1}{3}}+(al)^{\frac{1}{3}}\Big)+6(abclmn)^{\frac{1}{3}}\Bigg)-a^{\frac{1}{3}}C(l)-b^{\frac{1}{3}}C(m)\Bigg\|\\&\qquad\leq\Big\|c^{\frac{1}{3}}C(n)\Big\|\end{aligned}$$

for all $l,\ m,\ n \in \mathbb{S}$ with $l \perp m,\ m \perp n$ and $n \perp l$ using fixed point approach where $C : \mathbb{S} \longrightarrow \mathbb{Z}$ is a mapping from an orthogonal space $(\mathbb{S},\ \perp)$ into a real Banach space, $\perp$ represents the orthogonality relation and $a,\ b,\ c$ are real numbers with $a \neq 0,\ b \neq 0,\ c \neq 0$. Using these results, we present the stability of GCRF inequality in two variables also.

1. Introduction

The crucial point from where the concept of investigating Hyers-Ulam Stability results of functional equations, differential equations, difference equations is the problem of Ulam [30]. Hyers [9] presented a partial solution to the problem of Ulam. Later, Hyers' theorem was extended and generalized in various forms by many mathematicians Aoki [28], T. Rassias [29], J. Rassias [16] and Gavruta [23]. These results instigated many mathematicians to investigate stability of various types of functional equations in different types of spaces. For detailed review of literature on this field, one can refer ([6], [22], [21], [15], [12], [3], [24], [25], [27], [8]).

The various fundamental stabilities associated with stability of reciprocal adjoint and difference functional equations were demonstrated in ([18], [19]). In recent times, there are many papers published on the stabilities and applications of some multiplicative inverse functional equations, one can refer ([1], [7], [11]).

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We make use the concept of orthogonality space in the sense of Ratz [14].

Definition 1.1. *Let $\mathbb{S}$ be a real linear space with* $\dim \mathbb{S} \geq 2$ *and $\perp$ is a binary relation on $\mathbb{S}$ with following properties:*
(1) $l \perp 0,\ 0 \perp\ l,\ \forall\ l\ \in\ \mathbb{S}$*; (totality of $\perp$ for zero)*
(2) For non-zero $l,\ m\ \in\ \mathbb{S}$, $l \perp m$, then $l,\ m$ are linearly independent; (independence)
(3) For $l,\ m\ \in\ \mathbb{S}$, $l \perp m$, then $al \perp bm\ \forall\ a,\ b\ \in \mathbb{R}$; (homogenity)
(4) If $\mathbb{T}$ is a 2-dimensional subspace of $\mathbb{S}$, $l\ \in \mathbb{T}$ and $\tau \in\ \mathbb{R}^+$, where $\mathbb{R}^+$, is the set of non-negative real numbers, then there exists $m\ \in\ \mathbb{T}$ such that $l\ \perp m$ and $l+m\ \perp\ \tau l - m$. (Thalesian property)
Then the pair $(\mathbb{S},\ \perp)$ *is called an orthogonality space and if orthogonality space associated with a normed structure then it is renamed as "orthogonality normed space".*

We can put forward some interesting and relevant examples here.
(i) The "Birkhoff-James orthogonality" on a normed space $(\mathbb{S}, \|.\|)$ defined by $l\ \perp m$ if and only if $\|l + \tau m\| \geq \|l\|$ for all $\tau \in \mathbb{R},\ l,\ m\ \in\ \mathbb{S}$.
(ii) The "James orthogonality" on a normed space $(\mathbb{S}, \|.\|)$ defined by $l\ \perp m$ if and only if $\|l + m\| = \|l - m\|$ for all $l,\ m\ \in\ \mathbb{S}$.

The authors in [20] give the orthogonal stability of a mixed type additive and quadratic functional equation.

In 2021, S. Farhadabadi et al. [31] proved the Hyers-Ulam-Aoki-Rassias stability of the orthogonal quadratic functional inequality

$$\begin{aligned}&\Big|\Big|Q(\frac{l+m+n}{2}) + Q(\frac{l-m-n}{2}) + Q(\frac{m-l-n}{2}) + Q(\frac{n-l-m}{2})\\&- Q(l) - Q(m)\Big|\Big| \leq \Big|\Big|Q(n)\Big|\Big|\end{aligned}$$

using fixed point alternatives. The generealized cube root functional equation for three variables is defined as

$$\begin{aligned}&C\Bigg(al + bm + cn + 3(al)^{\frac{2}{3}}\Big((bm)^{\frac{1}{3}} + (cn)^{\frac{1}{3}}\Big) + 3(bm)^{\frac{2}{3}}\Big((al)^{\frac{1}{3}} + (cn)^{\frac{1}{3}}\Big)\\&+ 3(cn)^{\frac{2}{3}}\Big((bm)^{\frac{1}{3}} + (al)^{\frac{1}{3}}\Big) + 6(abclmn)^{\frac{1}{3}}\Bigg) = a^{\frac{1}{3}}C(l) + b^{\frac{1}{3}}C(m) + c^{\frac{1}{3}}C(n)\end{aligned}$$

where, $C : \mathbb{S} \longrightarrow \mathbb{Z}$ is a mapping from normed linear space into real Banach space having solution is $C(l) = l^{\frac{1}{3}}$.

Fixed point theory has abundant applications in various branch of mathematics especially in stability problems. In 1996, Isac and Rassias [13] were the first to provide applications of stability theory (functional equations) as the proof of a new fixed point theorems with applications. The term "generalized metric space" was coined by Luxemburg [32].

Theorem 1.2. [17] *Let (ψ, d) be a "complete generalized metric space" and $T : \psi \longrightarrow Y$ be a strict contraction with the "Lipschitz constant" k such that*

$$d(l_0,\ A(l_0)) < +\infty \textit{ for some } l_0 \in X.$$

Then T has a unique fixed point in the set $Y := \{m \in \psi, d(l_0, m) < \infty\}$ and the sequence $\{T^n(l)\}$ converges to the fixed point l^* for every $l \in Y$. Also,

$$d(l_0, T(l_0)) \leq \omega \text{ gives } d(l^*, l_0) \leq \frac{\omega}{1-k}.$$

In this article, we investigate the stability of orthogonal generalized cube root functional (GCRF) inequality

$$\begin{aligned}\Bigg\|C\Bigg(&al+bm+cn+3(al)^{\frac{2}{3}}\Big((bm)^{\frac{1}{3}}+(cn)^{\frac{1}{3}}\Big)+3(bm)^{\frac{2}{3}}\Big((al)^{\frac{1}{3}}+(cn)^{\frac{1}{3}}\Big)\\ &+3(cn)^{\frac{2}{3}}\Big((bm)^{\frac{1}{3}}+(al)^{\frac{1}{3}}\Big)+6(abclmn)^{\frac{1}{3}}\Bigg)-a^{\frac{1}{3}}C(l)-b^{\frac{1}{3}}C(m)\Bigg\|\\ &\leq\Bigg\|c^{\frac{1}{3}}C(n)\Bigg\| \end{aligned} \tag{1.1}$$

for all $l,\ m,\ n \in \mathbb{S}$ with $l \perp m,\ m \perp n$ and $n \perp l$. and using similar arguments, we investigate the stability of orthogonal generalized cube root functional (GCRF) inequality

$$\Big\|C\Big(al+bm+3(al)^{\frac{2}{3}}(bm)^{\frac{1}{3}}+3(bm)^{\frac{2}{3}}(al)^{\frac{1}{3}}\Big)-a^{\frac{1}{3}}C(l)\Big\| \leq \Big\|b^{\frac{1}{3}}C(m)\Big\| \tag{1.2}$$

for all $l,\ m \in \mathbb{S}$ with $l \perp m$ in two variables.

2. Solution and Hyers Ulam stability of the GCRF inequality

Under this section, we can assume that $(\mathbb{S}, \perp)$ is an orthogonality space and $(\mathbb{Z}, \|.\|)$ is a real Banach space. Firstly, we go for the solution of the orthogonally GCRF inequality (1.1) by proving an orthogonal superstability proposition, and hence forth we prove its Hyers-Ulam stability in orthogonality spaces.

Definition 2.1. *A mapping $C : \mathbb{S} \longrightarrow \mathbb{Z}$ is called an (exact) orthogonally GCR mapping if*

$$C(al+bm+3.a^{\frac{2}{3}}.l^{\frac{2}{3}}.b^{\frac{1}{3}}.m^{\frac{1}{3}}+3.a^{\frac{1}{3}}.l^{\frac{1}{3}}.b^{\frac{2}{3}}.m^{\frac{2}{3}}) = a^{\frac{1}{3}}C(l)+b^{\frac{1}{3}}C(m) \tag{2.1}$$

for all $l,\ m \in \mathbb{S}$ with $l \perp m$. And it is called an approximate orthogonally GCR mapping if

$$\begin{aligned}\Bigg\|C\Bigg(&al+bm+cn+3(al)^{\frac{2}{3}}\Big((bm)^{\frac{1}{3}}+(cn)^{\frac{1}{3}}\Big)+3(bm)^{\frac{2}{3}}\Big((al)^{\frac{1}{3}}+(cn)^{\frac{1}{3}}\Big)+3(cn)^{\frac{2}{3}}\\ &\Big((bm)^{\frac{1}{3}}+(al)^{\frac{1}{3}}\Big)+6(abclmn)^{\frac{1}{3}}\Bigg)-a^{\frac{1}{3}}C(l)-b^{\frac{1}{3}}C(m)\Bigg\| \leq \Big\|c^{\frac{1}{3}}C(n)\Big\| \end{aligned} \tag{2.2}$$

for all $l,\ m,\ n \in \mathbb{S}$ with $l \perp m,\ m \perp n$ and $n \perp l$.

Proposition 2.2. *Each approximate orthogonally GCR mapping in the form (2.2) is also an (exact) orthogonally GCR mapping satisfying (2.1), where $a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}} \neq 1$.*

Proof. Assume that $C : \mathbb{S} \longrightarrow \mathbb{Z}$ is an approximate orthogonally GCR mapping satisfying (2.2). Since $0 \perp 0$, taking $l = m = n = 0$ in (2.2), we have

$$\|(1 - a^{\frac{1}{3}} - b^{\frac{1}{3}})C(0)\| \leq \|c^{\frac{1}{3}}C(0)\| = 0.$$

Hence $C(0) = 0$.
Also, $l,\ m \perp 0$ for all $l,\ m \in \mathbb{S}$, changing n by 0 in (2.2), we get

$$\left|\left|C(al+bm+3.a^{\frac{2}{3}}.l^{\frac{2}{3}}.b^{\frac{1}{3}}.m^{\frac{1}{3}}+3.a^{\frac{1}{3}}.l^{\frac{1}{3}}.b^{\frac{2}{3}}.m^{\frac{2}{3}})-a^{\frac{1}{3}}C(l)-b^{\frac{1}{3}}C(m)\right|\right| \leq \left|\left|c^{\frac{1}{3}}C(0)\right|\right| = 0$$

and hence

$$C(al + bm + 3.a^{\frac{2}{3}}.l^{\frac{2}{3}}.b^{\frac{1}{3}}.m^{\frac{1}{3}} + 3.a^{\frac{1}{3}}.l^{\frac{1}{3}}.b^{\frac{2}{3}}.m^{\frac{2}{3}}) = a^{\frac{1}{3}}C(l) + b^{\frac{1}{3}}C(m)$$

for all $l,\ m \in \mathbb{S}$ where $l \perp m$, which is the equation (2.1). Hence $C : \mathbb{S} \longrightarrow \mathbb{Z}$ is an (exact) orthogonally GCR mapping. $\square$

For the sake of proving our main results in a concise manner, let $D_{C_r} : \mathbb{S} \longrightarrow \mathbb{Z}$ be difference operators defined as follows

$$D_{C_r}(l,\ m,\ n) = C\Bigg(al + bm + cn + 3(al)^{\frac{2}{3}}\Big((bm)^{\frac{1}{3}} + (cn)^{\frac{1}{3}}\Big) + 3(bm)^{\frac{2}{3}}$$

$$\Big((al)^{\frac{1}{3}} + (cn)^{\frac{1}{3}}\Big) + 3(cn)^{\frac{2}{3}}\Big((bm)^{\frac{1}{3}} + (al)^{\frac{1}{3}}\Big) + 6(abclmn)^{\frac{1}{3}}\Bigg) - a^{\frac{1}{3}}C(l) - b^{\frac{1}{3}}C(m)$$

for all $l,\ m,\ n \in \mathbb{S}$.

Theorem 2.3. *Assume $\Omega : \mathbb{S}^3 \longrightarrow [0, \infty)$ is a function such that $\Omega(0,0,0) = 0$ and there exists an α such that $a^{\frac{-1}{3}} < \alpha < 1, a > 0$ with condition $a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}} \neq 1$,*

$$\Omega(l,\ m,\ n) \leq a^{\frac{1}{3}}\alpha\ \Omega\Big(\frac{l}{a},\ \frac{m}{a},\ \frac{n}{a}\Big) \tag{2.3}$$

for all $l,\ m,\ n\ \in \mathbb{S}$ with $l \perp m,\ m \perp n,\ n \perp l$. Let $C : \mathbb{S} \longrightarrow \mathbb{Z}$ be a mapping satisfying

$$\left|\left|D_{C_r}(l,\ m,\ n)\right|\right| \leq \left|\left|c^{\frac{1}{3}}C(n)\right|\right| + \Omega(l,\ m,\ n) \tag{2.4}$$

for all $l,\ m,\ n\ \in \mathbb{S}$, with $l \perp m,\ m \perp n$ and $n \perp l$. Then there exists a unique orthogonally GCR mapping $C_r : \mathbb{S} \longrightarrow \mathbb{Z}$ such that

$$\left|\left|C_r(l) - C(l)\right|\right| \leq \frac{\alpha}{1-\alpha}\Omega(l,\ 0,\ 0)\ \forall\ l\ \in\ \mathbb{S}. \tag{2.5}$$

Proof. Consider the set $\mathbb{H} := \{u : \mathbb{S} \longrightarrow \mathbb{Z}\}$ and introduce the generalized metric on $\mathbb{H}$:

$$d(u,\ v) = \inf\{\tau \in \mathbb{R}^+ : \|u(l) - v(l)\| \leq \tau\Omega(l,\ 0,\ 0),\ \forall\ l \in \mathbb{S}\}.$$

It is easy to show that $(H,\ d)$ is complete ([10], lemma 2.1).
Now, we consider the linear mapping $K : H \longrightarrow H$ such that

$$K(u(l)) := \frac{1}{a^{\frac{1}{3}}}u(al)$$

$\forall\ u\ \in \mathbb{H}$ and all $l \in \mathbb{S}$. Since $0 \perp 0$, letting $l = m = n = 0$ in (2.4), we have

$$\|(1 - a^{\frac{1}{3}} - b^{\frac{1}{3}})C(0)\| \leq \|c^{\frac{1}{3}}C(0)\| + \Omega(l,\ 0,\ 0).$$

Hence $C(0) = 0$.
Since $l \perp 0$ for all $l \in \mathbb{S}$, letting $m = n = 0$ in (2.4), we get

$$\|C(al) - a^{\frac{1}{3}}C(l)\| \leq \Omega(l,\ 0,\ 0) \text{ for all } l \in \mathbb{S}.$$

Since $a^{\frac{-1}{3}} < \alpha$, dividing both sides by $a^{\frac{1}{3}}$, we get

$$\Big\|\frac{C(al)}{a^{\frac{1}{3}}} - C(l)\Big\| \leq \frac{1}{a^{\frac{1}{3}}}\Omega(l,\ 0,\ 0) < \alpha\Omega(l,\ 0,\ 0)$$

for all $l \in \mathbb{S}$, which clearly gives

$$d(KC, C) \leq \alpha. \tag{2.6}$$

Let $u, v \in \mathbb{H}$ be given such that $d(u,\ v) = \omega$, then $\|u(l) - v(l)\| \leq \omega\Omega(l,\ 0,\ 0)$ for all $l \in \mathbb{S}$. Hence the definition of Ku and (2.3), yields that

$$\begin{aligned}\|Ku(l) - Kv(l)\| &= \|\frac{1}{a^{\frac{1}{3}}}u(al) - \frac{1}{a^{\frac{1}{3}}}v(al)\| \\ &\leq \frac{1}{a^{\frac{1}{3}}}\tau\ \Omega(al,\ 0,\ 0) \\ &\leq \alpha\tau\ \Omega(l,\ 0,\ 0) \text{ for all } l\ \in\ \mathbb{S}.\end{aligned}$$

Therefore,

$$d(Ku, Kv) \leq \alpha\omega = \alpha d(u,\ v) \text{ for all } u,\ v\ \in \mathbb{H}.$$

Thus K is a strictly contractive mapping with Lipschitz constant $\alpha < 1$. According to Theorem 1.2, there exists a mapping $C_r : \mathbb{S} \longrightarrow \mathbb{Z}$ satisfying the following:
(1) C_r is a fixed point of K, therefore, $KC_r = C_r$, and so

$$\frac{1}{a^{\frac{1}{3}}}C_r(al) = C_r(l)\ \forall\ l\ \in\ \mathbb{S}. \tag{2.7}$$

The mapping C_r is only one fixed point of K in the set $\{u \in \mathbb{H} : d(u,\ v) < \infty\}$. This signifying (2.7) such that there exists a non zero positive real number τ satisfying

$$\|C(l) - C_r(l)\| \leq \tau\Omega(l,\ 0,\ 0)\ \forall\ l\ \in\ \mathbb{S};$$

(2) $d(K^tC,\ C_r) \Longrightarrow 0$ as $t \longrightarrow \infty$. So, we obtain

$$\lim_{t\to\infty}\frac{1}{a^{\frac{t}{3}}}C(a^tl) = C_r(l)\ \forall\ l\ \in\ \mathbb{S}; \tag{2.8}$$

(3) $d(C,\ C_r) \leq \frac{1}{1-\alpha}d(C,\ KC)$, using (2.6), inequality becomes

$$d(C,\ C_r) \leq \frac{\alpha}{1-\alpha}.$$

Hence inequality (2.5) holds.
Now, we prove that C_r is an orthogonally GCR mapping.
Using (2.8), (2.3), (2.4) and $\alpha < 1$,

$$\begin{aligned}&\Big\|DC_r(l,\ m,\ n)\Big\| - \lim_{t\to\infty}\frac{1}{a^{\frac{t}{3}}}\Big\|DC_r(a^tl,\ a^tm,\ a^tn)\Big\| \\ &\leq \Big\|\lim_{t\to\infty}\frac{1}{a^{\frac{t}{3}}}C_r(a^tn)\Big\| + \lim_{t\to\infty}\frac{1}{a^{\frac{t}{3}}}\Omega(a^tl,\ a^tm,\ a^tn) \\ &\leq \|C_r(n)\| + \lim_{t\to\infty}\alpha^t\Omega(l,\ m,\ n) \\ &\leq \|C_r(n)\|\ \ \forall\ l,\ m,\ n\ \in \mathbb{S},\ l \perp m, m \perp n, n \perp l.\end{aligned}$$

Where,

$$DC_r(l,\ m,\ n) = C_r\Bigg(al + bm + cn + 3(al)^{\frac{2}{3}}\Big((bm)^{\frac{1}{3}} + (cn)^{\frac{1}{3}}\Big) + 3(bm)^{\frac{2}{3}}$$
$$\Big((al)^{\frac{1}{3}} + (cn)^{\frac{1}{3}}\Big) + 3(cn)^{\frac{2}{3}}\Big((bm)^{\frac{1}{3}} + (al)^{\frac{1}{3}}\Big) + 6(abclmn)^{\frac{1}{3}}\Bigg)$$
$$- a^{\frac{1}{3}}C_r(l) - b^{\frac{1}{3}}C_r(m).$$

And, now applying proposition 2.2, we get C_r is an orthogonally GCR mapping. This proves the result. $\square$

Theorem 2.4. *Assume* $\Omega : \mathbb{S}^3 \longrightarrow [0, \infty)$ *is a function such that* $\Omega(0,0,0) = 0$ *and there exists an* α *such that* $\alpha < a^{\frac{1}{3}} < 1, a > 0$ *with condition* $a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}} \neq 1$ *,*

$$\Omega(l,\ m,\ n) \leq \frac{1}{a^{\frac{1}{3}}}\alpha\ \Omega\big(al,\ am,\ an\big) \tag{2.9}$$

for all $l,\ m,\ n\ \in \mathbb{S}$ *with* $l \perp m,\ m \perp n,\ n \perp l$*. Let* $C : \mathbb{S} \longrightarrow \mathbb{Z}$ *ba a mapping satisfying*

$$\Big|\Big|D_{C_r}(l,\ m,\ n)\Big|\Big| \leq \Big|\Big|c^{\frac{1}{3}}C(n)\Big|\Big| + \Omega(l,\ m,\ n) \tag{2.10}$$

for all $l,\ m,\ n\ \in \mathbb{S}$*, with* $l \perp m,\ m \perp n$ *and* $n \perp l$*. Then there exists a unique orthogonally GCR mapping* $C_r : \mathbb{S} \longrightarrow \mathbb{Z}$ *such that*

$$\Big|\Big|C_r(l) - C(l)\Big|\Big| \leq \frac{1}{1-\alpha}\Omega(l,\ 0,\ 0) \text{ for all } l\ \in\ \mathbb{S}. \tag{2.11}$$

Proof. Let $(\mathbb{H},\ d)$ be "generalized metric space" as defined in the proof of previous Theorem 2.3. Now, we consider the linear mapping $K : H \longrightarrow H$ such that

$$K(u(l)) := a^{\frac{1}{3}}u\Big(\frac{l}{a}\Big) \text{ for all } u \in \mathbb{H} \text{ and all } l \in \mathbb{S}.$$

Similar to the proof of Theorem 2.3, using condition $\alpha < a^{\frac{1}{3}}$ and (2.10), we obtain

$$\|C(al) - a^{\frac{1}{3}}C(l)\| \leq \Omega(l,\ 0,\ 0) \text{ for all } l\ \in \mathbb{S}$$

$$\begin{aligned}\|C(l) - a^{\frac{1}{3}}C(\frac{l}{a}) &\leq \Omega(\frac{l}{a},\ 0,\ 0)\\ &\leq \frac{\alpha}{a^{\frac{1}{3}}}\Omega(l,\ 0,\ 0)\\ &\leq \Omega(l,\ 0,\ 0) \text{ for all } l\ \in \mathbb{S}.\end{aligned}$$

Hence, we obtain

$$d(KC, C) \leq 1. \tag{2.12}$$

We can also show that K is a strictly contractive mapping with Lipschitz constant $\alpha < 1$. According to Theorem 1.2 again, there exists a mapping $C_r : \mathbb{S} \longrightarrow \mathbb{Z}$ satisfying

$$d(C,\ C_r) \leq \frac{1}{1-\alpha}d(C,\ KC),$$

using (2.12), inequality becomes

$$d(C,\ C_r) \leq \frac{1}{1-\alpha}.$$

Hence inequality (2.11) holds.
The remaining proof follows the same steps as in previous Theorem 2.3. $\square$

Corollary 2.5. *Let $\mathbb{S}$ be a normed orthogonally space. Let ω be a non-negative real number and $\tau \neq \frac{1}{3}$ be a positive real number. Let $C : \mathbb{S} \longrightarrow \mathbb{Z}$ be a mapping satisfying*

$$\left|\left|D_{C_r}(l,\ m,\ n)\right|\right| \leq \left|\left|c^{\frac{1}{3}}C(n)\right|\right| + \omega\Big(\|l\|^\tau + \|m\|^\tau + \|n\|^\tau\Big)$$

for all $l,\ m,\ n\ \in \mathbb{S}$, with $l \perp m,\ m \perp n,\ n \perp l$ with condition $a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}} \neq 1$, where $a > 0$. Then there exists a unique orthogonally GCR mapping $C_r : \mathbb{S} \longrightarrow \mathbb{Z}$ such that
for $0 < \tau < \frac{1}{3}$ with condition $a^{\frac{-1}{3}} < \alpha < 1$, we obtain

$$\left|\left|C_r(l) - C(l)\right|\right| \leq \frac{a^\tau}{a^{\frac{1}{3}} - a^\tau}\omega\|l\|^\tau \tag{2.13}$$

and for $\tau > \frac{1}{3}$ with condition $\alpha < a^{\frac{1}{3}} < 1$, we obtain

$$\left|\left|C_r(l) - C(l)\right|\right| \leq \frac{a^\tau}{a^\tau - a^{\frac{1}{3}}}\omega\|l\|^\tau \tag{2.14}$$

for all $l\ \in\ \mathbb{S}$.

Proof. Let $\Omega(l,\ m,\ n)$ be described by $\Omega(l,\ m,\ n) = \omega\Big(\|l\|^\tau + \|m\|^\tau + \|n\|^\tau\Big)$ for all $l,\ m,\ n\ \in \mathbb{S}$.
Firstly, assume that $0 < \tau < \frac{1}{3}$, Take $\alpha =\ a^{\tau - \frac{1}{3}}$. Since $\tau < \frac{1}{3}$, therefore $\alpha < 1$ such that

$$\begin{aligned}
\Omega(l,\ m,\ n) &= \omega\Big(\|l\|^\tau + \|m\|^\tau + \|n\|^\tau\Big) \\
&= a^{\frac{1}{3}}\alpha a^{-\tau}\omega\Big(\|l\|^\tau + \|m\|^\tau + \|n\|^\tau\Big) \\
&= a^{\frac{1}{3}}\alpha\omega\left(\left|\left|\frac{l}{a}\right|\right|^\tau + \left|\left|\frac{m}{a}\right|\right|^\tau + \left|\left|\frac{n}{a}\right|\right|^\tau\right) \\
&= a^{\frac{1}{3}}\alpha\Omega\left(\frac{l}{a},\ \frac{m}{a},\ \frac{n}{a}\right)
\end{aligned}$$

for all $l,\ m,\ n\ \in\ \mathbb{S}$. The recent terms allows to use Theorem 2.3, so by applying Theorem 2.3, it follows from (2.5) that

$$\left|\left|C_r(l) - C(l)\right|\right| \leq \frac{a^\tau}{a^{\frac{1}{3}} - a^\tau}\omega\|l\|^\tau \text{ for all } l\ \in\ \mathbb{S}.$$

Now, using same arguments we obtain result for $\tau > \frac{1}{3}$ by putting $\alpha = a^{\frac{1}{3} - \tau}$. Hence, we get the reuired result. $\square$

Corollary 2.6. *Assume $\mathbb{S}$ is a normed orthogonally space. Let ω be a non-negative real number and $\tau = p + q + r \neq \frac{1}{3}$ be a positive real number. Let $C : \mathbb{S} \longrightarrow \mathbb{Z}$ be a mapping satisfying*

$$\begin{aligned}
\left|\left|D_{C_r}(l,\ m,\ n)\right|\right| \leq \left|\left|c^{\frac{1}{3}}C(n)\right|\right| + \omega\Big(&\|l\|^p\|m\|^q\|n\|^r + \|l\|^{p+q+r} + \|m\|^{p+q+r} \\
&+ \|n\|^{p+q+r}\Big)
\end{aligned}$$

for all $l,\ m,\ n\ \in \mathbb{S}$, *with* $l \perp m,\ m \perp n,\ n \perp l$ *with condition* $a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}} \neq 1$, *where* $a > 0$. *Then there exists a unique orthogonally GCR mapping* $C_r : \mathbb{S} \longrightarrow \mathbb{Z}$ *such that*
for $0 < \tau = p + q + r < \frac{1}{3}$ *with condition* $a^{\frac{-1}{3}} < \alpha < 1$, *we obtain*

$$\Big|\Big|C_r(l) - C(l)\Big|\Big| \leq \frac{a^{\tau}}{a^{\frac{1}{3}} - a^{\tau}} \omega \|l\|^{\tau}$$

and for $\tau = p + q + r > \frac{1}{3}$ *with condition* $\alpha < a^{\frac{1}{3}} < 1$, *we obtain*

$$\Big|\Big|C_r(l) - C(l)\Big|\Big| \leq \frac{a^{\tau}}{a^{\tau} - a^{\frac{1}{3}}} \omega \|l\|^{\tau} \text{ for all } l\ \in\ \mathbb{S}.$$

Proof. Let $\Omega(l,\ m,\ n)$ be described by

$$\Omega(l,\ m,\ n) = \omega\Big(\|l\|^p\|m\|^q\|n\|^r + \|l\|^{p+q+r} + \|m\|^{p+q+r} + \|n\|^{p+q+r}\Big)$$

for all $l,\ m,\ n\ \in \mathbb{S}$.
For $0 < \tau < \frac{1}{3}$, assume $\alpha =\ a^{\tau - \frac{1}{3}}$ and using same arguments as in the previous corollary we get the required result.
For $\tau > \frac{1}{3}$, assume $\alpha = a^{\frac{1}{3} - \tau}$ we get the desired result. □

Orthogonal stability of GCRF inequality in two variables. Throughout this subsection, let us assume that $(\mathbb{S}, \perp)$ is an orthogonality space and $(\mathbb{Z}, \|.\|)$ is a real Banach space. In the following theorems and corollaries, we present the stability results of GCRF inequality (1.2)in two variables. The arguments of proving stability results of inequality (1.2) are akin to the proofs of Section 2. For the sake of completness, we furnish below the statement of theorems and corollaries only pertinent to various fundamental stabilities.
For the convenience, let $D_{C_2} : \mathbb{S} \longrightarrow \mathbb{Z}$ be difference operators defined as follow

$$D_{C_2}(l,\ m,\ n) = C\Big(al + bm + 3(al)^{\frac{2}{3}}(bm)^{\frac{1}{3}} + 3(bm)^{\frac{2}{3}}(al)^{\frac{1}{3}}\Big) - a^{\frac{1}{3}}C(l)$$

for all $l, m \in \mathbb{S}$.

Theorem 2.7. *Assume* $\Omega : \mathbb{S}^2 \longrightarrow [0, \infty)$ *is a function such that* $\Omega(0,0) = 0$ *and there exists an* α *such that* $a^{\frac{-1}{3}} < \alpha < 1, a > 0$ *with condition* $a^{\frac{1}{3}} + b^{\frac{1}{3}} \neq 1$,

$$\Omega(l,\ m) \leq a^{\frac{1}{3}}\alpha\ \Omega\Big(\frac{l}{a},\ \frac{m}{a}\Big)$$

for all $l,\ m\ \in \mathbb{S}$ *with* $l \perp m$. *Let* $C : \mathbb{S} \longrightarrow \mathbb{Z}$ *ba a mapping satisfying*

$$\Big|\Big|D_{C_2}(l,\ m)\Big|\Big| \leq \Big|\Big|b^{\frac{1}{3}}C(m)\Big|\Big| + \Omega(l,\ m)$$

for all $l,\ m\ \in \mathbb{S}$, *with* $l \perp m$. *Then there exists a unique orthogonally GCR mapping* $C_2 : \mathbb{S} \longrightarrow \mathbb{Z}$ *such that*

$$\Big|\Big|C_2(l) - C(l)\Big|\Big| \leq \frac{\alpha}{1 - \alpha}\Omega(l,\ 0) \text{ for all } l\ \in\ \mathbb{S}.$$

Theorem 2.8. *Assume* $\Omega : \mathbb{S}^2 \longrightarrow [0, \infty)$ *is a function such that* $\Omega(0,0) = 0$ *and there exists an* α *such that* $\alpha < a^{\frac{1}{3}} < 1, a > 0$ *with condition* $a^{\frac{1}{3}} + b^{\frac{1}{3}} \neq 1$,

$$\Omega(l,\ m) \leq \frac{1}{a^{\frac{1}{3}}}\alpha\ \Omega\big(al,\ am\big)$$

for all $l,\ m\ \in \mathbb{S}$ with $l \perp m$. Let $C : \mathbb{S} \longrightarrow \mathbb{Z}$ ba a mapping satisfying

$$\Big|\Big|D_{C_2}(l,\ m)\Big|\Big| \leq \Big|\Big|b^{\frac{1}{3}}C(m)\Big|\Big| + \Omega(l,\ m)$$

for all $l,\ m\ \in \mathbb{S}$, with $l \perp m$. Then there exists a unique orthogonally GCR mapping $C_2 : \mathbb{S} \longrightarrow \mathbb{Z}$ such that

$$\Big|\Big|C_2(l) - C(l)\Big|\Big| \leq \frac{1}{1-\alpha}\Omega(l,\ 0) \text{ for all } l\ \in\ \mathbb{S}.$$

Corollary 2.9. *Assume $\mathbb{S}$ is a normed orthogonally space. Let ω be a non-negative real number and $\tau \neq \frac{1}{3}$ be a positive real number. Let $C : \mathbb{S} \longrightarrow \mathbb{Z}$ be a mapping satisfying*

$$\Big|\Big|D_{C_2}(l,\ m)\Big|\Big| \leq \Big|\Big|b^{\frac{1}{3}}C(m)\Big|\Big| + \omega\Big(\|l\|^{\tau} + \|m\|^{\tau}\Big)$$

for all $l,\ m \in \mathbb{S}$, with condition $l \perp m$ and $a^{\frac{1}{3}} + b^{\frac{1}{3}} \neq 1$, where $a > 0$. Then there exists a unique orthogonally GCR mapping $C_2 : \mathbb{S} \longrightarrow \mathbb{Z}$ such that
for $0 < \tau < \frac{1}{3}$ with condition $a^{\frac{-1}{3}} < \alpha < 1$, we obtain

$$\Big|\Big|C_2(l) - C(l)\Big|\Big| \leq \frac{a^{\tau}}{a^{\frac{1}{3}} - a^{\tau}}\omega\|l\|^{\tau}$$

and for $\tau > \frac{1}{3}$ with condition $\alpha < a^{\frac{1}{3}} < 1$, we obtain

$$\Big|\Big|C_2(l) - C(l)\Big|\Big| \leq \frac{a^{\tau}}{a^{\tau} - a^{\frac{1}{3}}}\omega\|l\|^{\tau}$$

for all $l\ \in\ \mathbb{S}$.

Corollary 2.10. *Assume $\mathbb{S}$ is a normed orthogonally space. Let ω be a non-negative real number and $\tau = p + q \neq \frac{1}{3}$ be a positive real number. Let $C : \mathbb{S} \longrightarrow \mathbb{Z}$ be a mapping satisfying*

$$\Big|\Big|D_{C_2}(l,\ m)\Big|\Big| \leq \Big|\Big|b^{\frac{1}{3}}C(m)\Big|\Big| + \omega\Big(\|l\|^{p}\|m\|^{q} + \|l\|^{p+q} + \|m\|^{p+q}\Big)$$

for all $l,\ m\ \in \mathbb{S}$, with condition $l \perp m$ and $a^{\frac{1}{3}} + b^{\frac{1}{3}} \neq 1$, where $a > 0$. Then there exists a unique orthogonally GCR mapping $C_2 : \mathbb{S} \longrightarrow \mathbb{Z}$ such that
for $0 < \tau = p + q < \frac{1}{3}$ with condition $a^{\frac{-1}{3}} < \alpha < 1$, we obtain

$$\Big|\Big|C_2(l) - C(l)\Big|\Big| \leq \frac{a^{\tau}}{a^{\frac{1}{3}} - a^{\tau}}\omega\|l\|^{\tau}$$

and for $\tau = p + q > \frac{1}{3}$ with condition $\alpha < a^{\frac{1}{3}} < 1$, we obtain

$$\Big|\Big|C_2(l) - C(l)\Big|\Big| \leq \frac{a^{\tau}}{a^{\tau} - a^{\frac{1}{3}}}\omega\|l\|^{\tau}$$

for all $l\ \in\ \mathbb{S}$.

Conclusion. We wind up this paper with a conclusion that we have proved stability results of GCRF inequality associating a general control function, sum of powers of norms and mixed product-sum of powers of norms appropriate to the results established by Gavruta [23] and Rassias [16] in orthogonality space using fixed point approach. After that we have given the stability results of orthogonal GCRF inequality in two variables also which is obtained by following the similar steps as described in section 2.

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