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CLASS OF OPERATORS RELATED TO (α, β)-CLASS $(\mathcal{Q})$ OPERATORS

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Abstract. In this paper, we introduce new class of operators related to the class (α, β)-Class $(\mathcal{Q})$ operators which is named m-quasi-(α, β)-Class $(\mathcal{Q})$ operators. A bounded linear operator $\mathbf{R}$ on a complex Hilbert space $\mathcal{Y}$ is said to be m-quasi (α, β)-Class $(\mathcal{Q})$ operator if

$$\alpha^2(\mathbf{R}^*)^{m+2}\mathbf{R}^{m+2} \leq (\mathbf{R}^*)^m(\mathbf{R}^*\mathbf{R})^2\mathbf{R}^m \leq \beta^2(\mathbf{R}^*)^{m+2}\mathbf{R}^{m+2},$$

where $0 \leq \alpha \leq 1 \leq \beta$ and m is nonnegative integer. We investigate some basic properties that this class enjoys. Product and tensor product results were also investigated.

1. Introduction

Let $\mathcal{Y}$ be a complex separable Hilbert space. If $\mathbf{R} \in \mathcal{B}[\mathcal{Y}]$, we denote by $\ker(\mathbf{R})$ its kernel, $\mathbf{Ran}(\mathbf{R})$ its range and $\mathbf{R}^*$ for its adjoint. Moreover For $\mathbf{R} \in \mathcal{B}[\mathcal{Y}]$, we write $\sigma_s(\mathbf{R})$, $\sigma(\mathbf{R})$ and $\sigma_{ap}(\mathbf{R})$ for the surjective spectrum, the spectrum and the approximate spectrum of $\mathbf{R}$, respectively. An operator $\mathbf{R} \in \mathcal{B}[\mathcal{Y}]$ is said to be

(1) normal if $R^*R = RR^*$ $\left(\Leftrightarrow \|Rw\| = \|R^*w\| \ \forall\ w \in \mathcal{Y}\right)$[1, 2],

(2) hyponormal if $R^*R \geq RR^*$ $\left(\Leftrightarrow \|Rw\| \geq \|R^*w\| \ \forall\ w \in \mathcal{Y}\right)$[1, 2],

(3) (α, β)-normal operator $(0 \leq \alpha \leq 1 \leq \beta)$ ([2],[3], [11]) if

$$\alpha^2\mathbf{R}^*\mathbf{R} \leq \mathbf{R}\mathbf{R}^* \leq \beta^2\mathbf{R}^*\mathbf{R}, \quad \left(\alpha\|Rw\| \leq \|R^*w\| \leq \beta\|Rw\| \ \forall\ w \in \mathcal{Y}\right),$$

(4) m-quasi-(α, β)-normal operator $(0 \leq \alpha \leq 1 \leq \beta)$ ([12]) if

$$\alpha^2\big(\mathbf{R}^*\big)^{m+1}\mathbf{R}^{m+1} \leq \big(\mathbf{R}^*\big)^m\mathbf{R}\mathbf{R}^*\big(\mathbf{R}\big)^m \leq \beta^2\big(\mathbf{R}^*\big)^{m+1}\mathbf{R}^{m+1}.$$

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In the development of operator inequality, many operator classes which include normal operators were defined and many authors studied these new classes.We mention here the classes for which our work represents an extension. An operator $\mathbf{R}\in\mathcal{B}\big[\mathcal{Y}\big]$ is said to be

(1) Class $(\mathcal{Q})$ operator if $\mathbf{R}^{*2}\mathbf{R}^2=\big(\mathbf{R}^*\mathbf{R}\big)^2 \quad \Big(\Leftrightarrow \|\mathbf{R}^2w\|=\|\mathbf{R}^*\mathbf{R}w\| \quad \forall\, w\in\mathcal{Y}\Big)$ ([6]),

(2) Almost Class $(\mathcal{Q})$ if $\big(\mathbf{R}^*\mathbf{R}\big)^2\leq\big(\mathbf{R}^*\big)^2\mathbf{R}^2 \quad \Big(\Leftrightarrow \|\mathbf{R}^*\mathbf{R}w\|\leq\|\mathbf{R}^2w\| \quad \forall\, w\in\mathcal{Y}\Big)$ ([15]),

(3) (α,β)-Class $(\mathcal{Q})$ operators $(0\leq\alpha\leq 1\leq\beta)$ ([14]) if
$\alpha^2\mathbf{R}^{*2}\mathbf{R}^2\leq\big(\mathbf{R}^*\mathbf{R}\big)^2\leq\beta^2\mathbf{R}^{*2}\mathbf{R}^2 \quad \Big(\alpha\|\mathbf{R}^2w\|\leq\|\mathbf{R}^*\mathbf{R}w\|\leq\beta\|\mathbf{R}^2w\| \quad \forall\, w\in\mathcal{Y}\Big).$

There are many classes of operators that have been studied by many authors in recent years, so we direct the readers to [4, 5, 9, 13].

Referring to definitions of class $(\mathcal{Q})$ operators and (α,β)-Class $(\mathcal{Q})$ operators, we wanted to present a new class of operators termed as m-quasi-(α,β)-Class $(\mathcal{Q})$ operators parallel to (α,β)-normal operators ([2, 11]) and m-quasi-(α,β)-normal operators ([10, 12]). We study some properties of some members of this class of operators.

2. m-quasi-(α,β)-Class $(\mathcal{Q})$ operators

In this section, we are interested to introduce a new concept of operators known as m-quasi-(α,β)-Class $(\mathcal{Q})$ operators. We investigate various structural properties of this class of operators and study some relations about it.

Definition 2.1. *An operator $\mathbf{R}\in\mathcal{B}\big[\mathcal{Y}\big]$ is said to be an m-quasi-(α,β)-Class $(\mathcal{Q})$ operator for $0\leq\alpha\leq 1$ and $1\leq\beta$ if $\mathbf{R}$ satisfies*

$$\alpha^2(\mathbf{R}^*)^{m+2}\mathbf{R}^{m+2}\leq(\mathbf{R}^*)^m(\mathbf{R}^*\mathbf{R})^2\mathbf{R}^m\leq\beta^2(\mathbf{R}^*)^{m+2}\mathbf{R}^{m+2},$$

for some nonnegative integer m.

Remark. (1) *If $m=0$, then $\alpha^2(\mathbf{R}^*)^2\mathbf{R}^2\leq\big(\mathbf{R}^*\mathbf{R}\big)^2\leq\beta^2(\mathbf{R}^*)^2\mathbf{R}^2$.*

(2) *If $\mathbf{R}$ is (α,β)-Class $(\mathcal{Q})$ operator, then $\mathbf{R}$ is an m-quasi-(α,β)-Class $(\mathcal{Q})$ operator.*

Theorem 2.1. *Let $\mathbf{R}\in\mathcal{B}\big[\mathcal{Y}\big]$, then $\mathbf{R}$ is an m-quasi-(α,β)-class $(\mathcal{Q})$ operator, if and only if*

$$\alpha\|\mathbf{R}^{m+2}w\|\leq\|\mathbf{R}^*\mathbf{R}^{m+1}w\|\leq\beta\|\mathbf{R}^{m+2}w\|,$$

for all $w\in\mathcal{Y}$.

Proof. The proof is an immediate consequence of Definition 2.1. □

Remark. *Clearly every an m-quasi (α,β)-Class $(\mathcal{Q})$ operator is an $(m+1)$-quasi-(α,β)-Class $(\mathcal{Q})$ operator. We want to find an example of an operator $\mathbf{R}$ which is a m-quasi (α,β)-Class $(\mathcal{Q})$ operators but not a $(m-1)$-quasi-(α,β)-Class $(\mathcal{Q})$ operator.*

Example 2.1. *Consider the operator* $\mathbf{R} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *acting on* $\mathcal{Y} = \mathbb{C}^4$. *Direct computation shows that* $\mathbf{R}$ *satisfies*

$$\alpha\|\mathbf{R}^4 w\| \leq \|\mathbf{R}^*\mathbf{R}^3 w\| \leq \beta\|\mathbf{R}^4 w\|,$$

but not satisfies

$$\alpha\|\mathbf{R}^3 w\| \leq \|\mathbf{R}^*\mathbf{R}^2 w\| \leq \beta\|\mathbf{R}^3 w\|.$$

This means that $\mathbf{R}$ *is a* 2*-quasi* (α, β)*-Class* $(\mathcal{Q})$ *operators but* $\mathbf{R}$ *is not a* 1*-quasi-*(α, β)*-Class* $(\mathcal{Q})$ *operator.*

Now we are ready to give a sufficient condition for an m-quasi-(α, β)-Class $(\mathcal{Q})$ operator to be a quasi-(α, β)-Class $(\mathcal{Q})$ operator.

Theorem 2.2. *Let* $\mathbf{R}$ *be an* m*-quasi-*(α, β)*-Class* $(\mathcal{Q})$ *operator for* $m \geq 2$ *and satisfies* $\mathbf{Ran}(\mathbf{R}^m) = \mathbf{Ran}(\mathbf{R}^j)$ *for some integer* $j \in \{1, 2, \cdots, m-1\}$. *Then* $\mathbf{R}$ *is an* j*-quasi-*(α, β)*-Class* $(\mathcal{Q})$ *operator.*

Proof. The proof is an immediate consequence of Theorem 2.1. □

Proposition 2.3. *Every* m*-quasi-*(α, β)*-normal operator is an* m*-quasi-*(α, β)*-Class* $(\mathcal{Q})$ *operator.*

Proof. Let $\mathbf{R}$ be an m-quasi-(α, β)-normal operator, then we have

$$\alpha\|\mathbf{R}^{m+1} w\| \leq \|\mathbf{R}^*\mathbf{R}^m w\| \leq \beta\|\mathbf{R}^{m+1} w\| \quad \forall w \in \mathcal{Y},$$

which implies that

$$\alpha\|\mathbf{R}^{m+2} w\| \leq \|\mathbf{R}^*\mathbf{R}^{m+1} w\| \leq \beta\|\mathbf{R}^{m+2} w\| \quad \forall\, w \in \mathcal{Y}.$$

Therefore $\mathbf{R}$ is m-quasi-(α, β)-Class $(\mathcal{Q})$ operator. □

Theorem 2.4. *Let* $\mathbf{R}$ *be an* m*-quasi-*(α, β)*-Class* $(\mathcal{Q})$ *operator. If* $\overline{\mathbf{Ran}(\mathbf{R}^m)} = \mathcal{Y}$, *then* $\mathbf{R}$ *is* (α, β)*-Class* $(\mathcal{Q})$ *operator.*

Proof. According to $\overline{\mathbf{Ran}(\mathbf{R}^m)} = \mathcal{Y}$ we have for $w \in \mathcal{Y}$ there exists a sequence (w_n) in $\mathcal{Y}$ such that $\mathbf{R}^m(w_n) \to w$ as $n \to \infty$.
Since $\mathbf{R}$ is an m-quasi-(α, β)-Class $(\mathcal{Q})$ operator, we have

$$\alpha\|\mathbf{R}^{m+2} w\| \leq \|\mathbf{R}^*\mathbf{R}^{m+1} w\| \leq \beta\|\mathbf{R}^{m+2} w\|$$

for all $w \in \mathcal{Y}$. In particular,

$$\alpha\|\mathbf{R}^{m+2} w_n\| \leq \|\mathbf{R}^*\mathbf{R}^{m+1} w_n\| \leq \beta\|\mathbf{R}^{m+2} w_n\|$$

for $w_n \in \mathcal{Y}$. It follows that

$$\alpha\|\mathbf{R}^2 w\| \leq \|\mathbf{R}^*\mathbf{R} w\| \leq \beta\|\mathbf{R}^2 w\|.$$

for all $w \in \mathcal{Y}$. Therefore $\mathbf{R}$ is (α, β)-Class $(\mathcal{Q})$ operator. □

The following theorem gives a matrix representation of m-quasi-(α, β)-Class $(\mathcal{Q})$ operator.

Theorem 2.5. *Let $\mathbf{R}\in\mathcal{B}[\mathcal{Y}]$ such that $\mathbf{R}^m$ does not have a dense range, then the following statements are equivalent.*

(1) $\mathbf{R}$ *is an m-quasi-(α,β)-class-$(\mathcal{Q})$-operator.*

(2) $\mathbf{R}=\begin{pmatrix}\mathbf{R}_1 & \mathbf{R}_2\\ 0 & \mathbf{R}_3\end{pmatrix}$ *on* $\mathcal{Y}=\overline{\mathbf{Ran}(\mathbf{R}^m)}\oplus ker(\mathbf{R}^{*m})$, *where*

$$\alpha^2\mathbf{R}_1^{*2}\mathbf{R}_1^2\le\big(\mathbf{R}_1^*\mathbf{R}_1\big)^2+\mathbf{R}_1^*\mathbf{R}_2\mathbf{R}_2^*\mathbf{R}_1\le\beta^2\mathbf{R}_1^{*2}\mathbf{R}_1^2,$$

and $\mathbf{R}_3^m=0$. *Furthermore* $\sigma(\mathbf{R})=\sigma(\mathbf{R}_1)\cup\{0\}$.

Proof. (1) $\Rightarrow$ (2). Consider the matrix representation of $\mathbf{R}$ with respect to the decomposition $\mathcal{Y}=\overline{\mathbf{Ran}(\mathbf{R}^m)}\oplus ker(\mathbf{R}^{*m})$: $\mathbf{R}=\begin{pmatrix}\mathbf{R}_1 & \mathbf{R}_2\\ 0 & \mathbf{R}_3\end{pmatrix}$. Let $\mathbf{P}$ be the projection onto $\overline{Ran(\mathbf{R}^m)}$. Then $\begin{pmatrix}\mathbf{R}_1 & 0\\ 0 & 0\end{pmatrix}=\mathbf{RP}=\mathbf{PRP}$. Since $\mathbf{R}$ is an m-quasi-(α,β)-Class $(\mathcal{Q})$, we have

$$\alpha^2(\mathbf{R}^*)^{m+2}\mathbf{R}^{m+2}\le(\mathbf{R}^*)^m(\mathbf{R}^*\mathbf{R})^2\mathbf{R}^m\le\beta^2(\mathbf{R}^*)^{m+2}\mathbf{R}^{m+2}$$

and it follows that

$$\alpha^2\mathbf{P}\Big(\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{P}\le\mathbf{P}\Big(\big(\mathbf{R}^*\mathbf{R}\big)^2\Big)\mathbf{P}\le\beta^2\mathbf{P}\Big(\mathbf{R}^{*2}\mathbf{R}^2\Big)\mathbf{P}$$

which implies that

$$\alpha^2\mathbf{R}_1^{*2}\mathbf{R}_1^2\le\big(\mathbf{R}_1^*\mathbf{R}_1\big)^2+\mathbf{R}_1^*\mathbf{R}_2\mathbf{R}_2^*\mathbf{R}_1\le\beta^2\mathbf{R}_1^{*2}\mathbf{R}_1^2.$$

Observe that for $w=w_1+w_2\in\mathcal{Y}=\overline{\mathbf{Ran}(\mathbf{R}^m)}\oplus ker(\mathbf{R}^{*m})$ we have by a simple computation that

$$\begin{aligned}\langle\mathbf{R}_3^m w_2,\ ,\ w_2\rangle &= \langle\mathbf{R}^m(I-P)w,(I-P)w\rangle\\ &= \langle(I-P)w,\mathbf{R}^{*m}(I-P)w\rangle=0.\end{aligned}$$

Hence, $\mathbf{R}_3^m=0$.

Since $\sigma(\mathbf{R})\cup\mathcal{S}=\sigma(\mathbf{R}_1)\cup\sigma(\mathbf{R}_3)$, where $\mathcal{S}$ is the union of the holes in $\sigma(\mathbf{R})$ which happen to be subset of $\sigma(\mathbf{R}_1)\cap\sigma(\mathbf{R}_3)$ by Corollary 7 of [7], and $\sigma(\mathbf{R}_1)\cap\sigma(\mathbf{R}_3)$ has no interior point and $\mathbf{R}_3$ is nilpotent, we have $\sigma(\mathbf{R})=\sigma(\mathbf{R}_1)\cup\{0\}$.

(2) $\Rightarrow$ (1) Assume that $\mathbf{R}=\begin{pmatrix}\mathbf{R}_1 & \mathbf{R}_2\\ \\ 0 & \mathbf{R}_3\end{pmatrix}$ onto $\mathcal{Y}=\overline{\mathbf{Ran}(\mathbf{R}^m)}\oplus ker(\mathbf{R}^{*m})$ with

$$\alpha^2\mathbf{R}_1^{*2}\mathbf{R}_1^2\le\big(\mathbf{R}_1^*\mathbf{R}_1\big)^2+\mathbf{R}_1^*\mathbf{R}_2\mathbf{R}_2^*\mathbf{R}_1\le\beta^2\mathbf{R}_1^{*2}\mathbf{R}_1^2$$

and $\mathbf{R}_3^m=0$.

As $\mathbf{R}^m=\begin{pmatrix}\mathbf{R}_1^m & \sum_{j=0}^{m-1}\mathbf{R}_1^j\mathbf{R}_2\mathbf{R}_3^{k-1-j}\\ \\ 0 & 0\end{pmatrix}$ and $\mathbf{R}^*\mathbf{R}=\begin{pmatrix}\mathbf{R}_1^*\mathbf{R}_1 & \mathbf{R}_1^*\mathbf{R}_2\\ \\ \mathbf{R}_2^*\mathbf{R}_1 & \mathbf{R}_2^*\mathbf{R}_2+\mathbf{R}_3^*\mathbf{R}_3\end{pmatrix}$.

Further

$$\begin{aligned}\mathbf{R}^m\mathbf{R}^{*m} &= \begin{pmatrix} \mathbf{R}_1^m\mathbf{R}_1^{*m} + \left(\sum_{j=0}^{m-1}\mathbf{R}_1^j\mathbf{R}_2\mathbf{R}_3^{k-1-j}\right)\left(\sum_{j=0}^{m-1}\mathbf{R}_1^j\mathbf{R}_2\mathbf{R}_3^{k-1-j}\right)^* & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{D}_m & 0 \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

where $\mathbf{D}_m = \mathbf{R}_1^m\mathbf{R}_1^{*m} + \left(\sum_{j=0}^{m-1}\mathbf{R}_1^j\mathbf{R}_2\mathbf{R}_3^{k-1-j}\right)\left(\sum_{j=0}^{m-1}\mathbf{R}_1^j\mathbf{R}_2\mathbf{R}_3^{k-1-j}\right)^* = \mathbf{D}_m^*$.

We get

$$\begin{aligned}&\alpha^2\mathbf{R}^m\mathbf{R}^{*m}\big(\mathbf{R}^{*2}\mathbf{R}^2\big)\mathbf{R}^m\mathbf{R}^{*m} \\ = &\begin{pmatrix} \alpha^2\mathbf{D}_m\big(\mathbf{R}_1^{*2}\mathbf{R}_1^2\big)\mathbf{D}_m & 0 \\ 0 & 0 \end{pmatrix} \\ \leq &\begin{pmatrix} \mathbf{D}_m\Big(\big(\mathbf{R}_1^*\mathbf{R}_1\big)^2 + \mathbf{R}_1^*\mathbf{R}_2\mathbf{R}_2^*\mathbf{R}_1\Big)\mathbf{D}_m & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{R}^m\mathbf{R}^{*m}\big(\mathbf{R}^*\mathbf{R}\big)^2\mathbf{R}^m\mathbf{R}^{*m} \\ \leq &\begin{pmatrix} \beta^2\mathbf{D}_m\mathbf{R}_1^{*2}\mathbf{R}_1^2\mathbf{D}_m & 0 \\ 0 & 0 \end{pmatrix} = \beta^2\mathbf{R}^m\mathbf{R}^{*m}\big(\mathbf{R}^{*2}\mathbf{R}^2\mathbf{R}^m\mathbf{R}^{*m}).\end{aligned}$$

Which implies that

$$\alpha^2\mathbf{R}^{*m}\big(\mathbf{R}^{*2}\mathbf{R}^2\big)\mathbf{R}^m \leq \mathbf{R}^{*m}\big(\mathbf{R}^*\mathbf{R}\big)^2\mathbf{R}^m \leq \beta^2\mathbf{R}^{*m}\big(\mathbf{R}^{*2}\mathbf{R}^2\big)\mathbf{R}^m,$$

on $\mathcal{Y} = \mathbf{Ran}(\mathbf{R}^{*m}) \oplus ker(\mathbf{R}^m)$. Therefore, $\mathbf{R}$ is an m-quasi-(α,β)-class $(\mathcal{Q})$ operator. □

Theorem 2.6. *Let* $\mathbf{R} = \begin{pmatrix} \mathbf{R}_1 & \mathbf{R}_2 \\ 0 & \mathbf{R}_3 \end{pmatrix} \in \mathcal{B}\big[\mathcal{Y} \oplus \mathcal{Y}\big]$. *If* $\mathbf{R}_1$ *is surjective* (α,β)-*Class* $(\mathcal{Q})$ *operator and* $\mathbf{R}_3^m = 0$, *then* $\mathbf{R}$ *is similar to an* m*-quasi-* (α,β)*-Class* $(\mathcal{Q})$ *operator.*

Proof. Sine $\mathbf{R}_1$ is surjective and $\mathbf{R}_3^m = 0$, we have $\sigma_s(\mathbf{R}_1) \cap \sigma_{ap}(\mathbf{R}_3) = \emptyset$. From the statement (c) in [8, Theorem 3.5.1], there exists some operator $\mathbf{N} \in \mathcal{B}\big[\mathcal{Y}\big]$ for which $\mathbf{R}_1\mathbf{N} - \mathbf{N}\mathbf{R}_3 = \mathbf{R}_2$.

$$\begin{pmatrix} \mathbf{I} & \mathbf{N} \\ 0 & \mathbf{I} \end{pmatrix}\begin{pmatrix} \mathbf{R}_1 & \mathbf{R}_2 \\ 0 & \mathbf{R}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{R}_1 & 0 \\ 0 & \mathbf{R}_3 \end{pmatrix}\begin{pmatrix} \mathbf{I} & \mathbf{N} \\ 0 & \mathbf{I} \end{pmatrix}.$$

Hence $\mathbf{R}$ is similar to $\mathbf{A} = \begin{pmatrix} \mathbf{R}_1 & 0 \\ 0 & \mathbf{R}_3 \end{pmatrix}$.

In fact, since $\mathbf{R}_1$ is (α,β)-Class $(\mathcal{Q})$ operator and $\mathbf{R}_3^m=0$, we obtain

$$\begin{aligned}\big(\mathbf{A}^*\big)^{m+2}\mathbf{A}^{m+2} &= \begin{pmatrix}\big(\mathbf{R}_1^*\big)^{m+2}\mathbf{R}_1^{m+2} & 0\\ & \\ 0 & 0\end{pmatrix}\\ &\leq \begin{pmatrix}\big(\mathbf{R}_1^*\big)^{m}\big(\mathbf{R}_1^*\mathbf{R}_1\big)^2\mathbf{R}_1^{m} & 0\\ 0 & 0\end{pmatrix}=\big(\mathbf{A}^*\big)^m\big(\mathbf{A}^*\mathbf{A}\big)^2\mathbf{A}^m\\ &\leq \begin{pmatrix}\beta^2\big(\mathbf{R}_1^*\big)^{m+2}\mathbf{R}_1^{m+2} & 0\\ 0 & 0\end{pmatrix}=\beta^2\big(\mathbf{A}^*\big)^{m+2}\mathbf{A}^{m+2}.\end{aligned}$$

Therefore $\mathbf{R}$ is similar to an m-quasi-(α,β)-Class $(\mathcal{Q})$ operator. □

Theorem 2.7. *Let $\mathbf{R},\mathbf{N}\in\mathcal{B}\big[\mathcal{Y}\big]$ are doubly commuting operators. If $\mathbf{N}$ is an m-quasi-(α',β')-class $(\mathcal{Q})$, and $\mathbf{R}$ is an m-quasi-(α,β)-Class $(\mathcal{Q})$ and then $\mathbf{RN}$ is an m-quasi-$(\alpha\alpha',\beta\beta')$-Class $(\mathcal{Q})$ operator.*

Proof. Under the assumptions that $\mathbf{R}$ is an m-quasi-(α,β)-class $(\mathcal{Q})$ and $\mathbf{N}$ is an m-quasi-(α',β')-class $(\mathcal{Q})$ operator such that $\big[\mathbf{R},\mathbf{N}\big]=\big[\mathbf{R},\mathbf{N}^*\big]=0$ we have

$$\alpha\alpha'\|(\mathbf{RN})^{m+2}w\| \;=\; \alpha\alpha'\|\mathbf{R}^{m+2}\mathbf{N}^{m+2}w|\leq\alpha'\|\mathbf{R}^*\mathbf{R}^{m+1}\mathbf{N}^{m+2}w\|\leq\|\mathbf{N}^*\mathbf{R}^*\mathbf{R}^{m+1}\mathbf{N}^{m+1}w\|$$

and

$$\|\mathbf{N}^*\mathbf{N}^{m+1}\mathbf{R}^*\mathbf{R}^{m+1}w\|\leq\beta'\|\mathbf{N}\mathbf{N}^{m+1}\mathbf{R}^*\mathbf{R}^{m+1}w\|=\beta'\|\mathbf{R}^*\mathbf{R}^{m+1}\mathbf{N}\mathbf{N}^{m+1}w\|\leq\beta\beta'|\|\mathbf{R}^{m+2}\mathbf{N}^{m+2}w\|.$$

Consequently,

$$\alpha\alpha'\|(\mathbf{RN})^{m+2}w\|\leq\|(\mathbf{RN})^*\big(\mathbf{RN}\big)^{m+1}w\|\leq\beta\beta'\|\big(\mathbf{RN}\big)^{m+2}w\|.$$

□

Theorem 2.8. *Let $\mathbf{R}\in\mathcal{B}\big[\mathcal{Y}\big]$ such that $\mathbf{Ran}(\mathbf{R}^{m+1})=\mathbf{Ran}(\mathbf{R}^{*m+1})$. If $\mathbf{R}$ is an m-quasi-(α,β)-class $(\mathcal{Q})$ for $0<\alpha\leq 1$ and $1\leq\beta$, then $\mathbf{R}^*$ is an m-quasi-$(\frac{1}{\beta},\frac{1}{\alpha})$-Class $(\mathcal{Q})$ operator.*

Proof. According to that $\mathbf{R}$ is an m-quasi-(α,β)-Class $(\mathcal{Q})$, we have that

$$\alpha\|\mathbf{R}^{m+2}w\|\leq\|\mathbf{R}^*\mathbf{R}^{m+1}w\|\leq\beta\|\mathbf{R}^{m+2}w\|,\quad\forall\, w\in\mathcal{Y}.$$

This means that

$$\alpha\|\mathbf{R}\big(\mathbf{R}^*\big)^{m+1}v\|\leq\|\mathbf{R}^*\big(\mathbf{R}^*\big)^{m+1}v\|\leq\beta\|\mathbf{R}\big(\mathbf{R}^*\big)^{m+1}v\|,\quad\forall\, v\in\mathcal{Y}.$$

Combining these inequalities,

$$\frac{1}{\beta}\|\big(\mathbf{R}^*\big)^{m+2}v\|\leq\|\mathbf{R}\big(\mathbf{R}^*\big)^{m+1}v\|\leq\frac{1}{\alpha}\|\big(\mathbf{R}^*\big)^{m+2}v\|.$$

This shows that $\mathbf{R}^*$ is an m-quasi-$(\frac{1}{\beta},\frac{1}{\alpha})$-Class $(\mathcal{Q})$ operator. □

Corollary 2.9. *Let $(\alpha,\beta)\in\mathbb{R}^2$ such that $0<\alpha\leq 1\leq\beta$ and let $\mathbf{R}\in\mathcal{B}\big[\mathcal{Y}\big]$ such that $\mathbf{Ran}(\mathbf{R}^{m+1})=\mathbf{Ran}(\mathbf{R}^{*m+1})$. If $\alpha\beta=1$ then $\mathbf{R}$ is an m-quasi-(α,β)-Class $(\mathcal{Q})$ operator if and only if $\mathbf{R}^*$ is an m-quasi-(α,β)-Class $(\mathcal{Q})$ operator.*

Theorem 2.10. *Let $\mathbf{R}, \mathbf{N} \in \mathcal{B}[\mathcal{Y}]$ are m-quasi-(α, β)-Class $(\mathcal{Q})$ operator, then the following assertions hold.*

(1) $\mathbf{R} \oplus \mathbf{N}$ *is an m-quasi-(α, β)-Class $(\mathcal{Q})$ operator.*

(2) $\mathbf{R} \otimes \mathbf{N}$ *is an m-quasi-(α^2, β^2)-Class $(\mathcal{Q})$ operator.*

Proof. The outline of the proof is analogous to the one given in [12, Proposition 2], so we can omitted it. □

Theorem 2.11. *The class of m-quasi-(α, β)-Class $(\mathcal{Q})$ operators $(0 \leq \alpha \leq 1 \leq \beta)\}$ is arcwise connected for $m \in \mathbb{N}$.*

Proof. Let $\mathbf{R}$ be m-quasi-(α, β)-Class $(\mathcal{Q})$ operator and $\lambda \in \mathbb{C}, \lambda \neq 0$. Direct calculation shows that

$$\beta \|\big(\lambda \mathbf{R}\big)^{m+2} w\| \geq \|\big(\lambda \mathbf{R}\big)^{*}\big(\lambda \mathbf{R}\big)^{m} w\| \geq \alpha \|\big(\lambda \mathbf{R}\big)^{m+2} w\| \quad \forall\, w \in \mathcal{Y}.$$

□

Proposition 2.12. *Let $\mathbf{V} \in \mathcal{B}[\mathcal{Y}]$ be an isometry and let $\mathbf{R} \in \mathcal{B}[\mathcal{Y}]$ be an m-quasi-(α, β)-Class $(\mathcal{Q})$ operator for $(0 \leq \alpha \leq 1$ and $1 \leq \beta$. Then $\mathbf{VRV}^*$ is an m-quasi-(α, β)-Class $(\mathcal{Q})$ operator.*

Proof. In view of assumptions that $\mathbf{R}$ is an m-quasi-(α, β)-Class $(\mathcal{Q})$ operator for $(0 \leq \alpha \leq 1 \leq \beta)$ and $\mathbf{V}$ is an isometry. Direct calculation shows that

$$\beta^2 \big(\big(\mathbf{VRV}^*\big)^*\big)^{m+2} \big(\mathbf{VRV}^*\big)^{m+2} \geq \big(\big(\mathbf{VRV}^*\big)^*\big)^{m} \left(\big(\mathbf{VRV}\big)^* \left(\big(\mathbf{VRV}^*\big)\right)^2\right) \big(\mathbf{VRV}^*\big)^{m}$$

and

$$\big(\big(\mathbf{VRV}^*\big)^*\big)^{m} \left(\big(\mathbf{VRV}\big)^* \left(\big(\mathbf{VRV}^*\big)\right)^2\right) \big(\mathbf{VRV}^*\big)^{m} \geq \alpha^2 \big(\big(\mathbf{VRV}^*\big)^*\big)^{m+2} \big(\mathbf{VRV}^*\big)^{m+2}.$$

Therefore, $\mathbf{VRV}^*$ is an m-quasi-(α, β)-Class $(\mathcal{Q})$ operator. □

Proposition 2.13. *Let $\mathbf{R}, \mathbf{N} \in \mathcal{B}[\mathcal{Y}]$ are commuting operator and such that $\mathbf{R}$ is an m-quasi-(α, β)-Class $(\mathcal{Q})$ operator. The following statements are true.*

(1) *If $\mathbf{N}$ is unitary and $\mathbf{R}^*\mathbf{N} = \mathbf{N}\mathbf{R}^*$, then $\mathbf{RN}$ is an m-quasi- (α, β)-Class $(\mathcal{Q})$ operator.*

(2) *If $\mathbf{N}$ is selfadjoint and $\mathbf{R}^*\mathbf{N} = \mathbf{N}\mathbf{R}^*$ then $\mathbf{RN}$ is an m-quasi-(α, β)-Class $(\mathcal{Q})$ operator.*

Proof. (1) In view of the fact that $\mathbf{N}$ is unitary we have $\mathbf{N}^*\mathbf{N} = \mathbf{N}\mathbf{N}^* = \mathbf{I}$.

Now direct calculations give

$$\beta^2 \left(\big((\mathbf{RN})^*\big)^{m+2} (\mathbf{RN})^{m+2} \right) =$$

$$\beta^2 \left(\big(\mathbf{R}^*\big)^{m+2} \big(\mathbf{N}^*\big)^{m+2} \mathbf{N}^{m+2} \mathbf{R}^{m+2} \right) = \beta^2 \left(\big(\mathbf{R}^*\big)^{m+2} \mathbf{R}^{m+2} \right)$$

$$\geq \underbrace{\left(\big(\mathbf{R}^*\big)^{m} \big(\mathbf{R}^*\mathbf{R}\big)^2 \mathbf{R}^{m} \right)}_{(1)} \geq \alpha^2 \underbrace{\left(\mathbf{R}^*\right)^{m+2} \mathbf{R}^{m+2} \Big)}_{(2)}$$

$$\geq \big(\mathbf{R}^*\big)^m\big(\mathbf{R}^*\mathbf{R}\big)^2\mathbf{R}^m \geq \alpha^2\big(\mathbf{R}^*\big)^{m+2}\big(\mathbf{N}^*\big)^{m+2}\mathbf{N}^{m+2}\mathbf{R}^{m+2}$$
$$= \mathbf{R}^{*m}\mathbf{N}^{*m}\mathbf{N}^m\big(\mathbf{R}^*\mathbf{N}^*\mathbf{N}\mathbf{R}\big)^2\mathbf{R}^m \geq \alpha^2\big(\big(\mathbf{R}\mathbf{N}\big)^*\big)^{m+2}\big(\mathbf{R}\mathbf{N}\big)^{m+2}$$
$$= \big(\big(\mathbf{R}\mathbf{N}\big)^*\big)^m\big(\mathbf{R}\mathbf{N}\big)^*\big(\mathbf{R}\mathbf{N}\big)\big)^2\big(\mathbf{R}\mathbf{N}\big)^m \geq \alpha^2\big(\big(\mathbf{R}\mathbf{N}\big)^*\big)^{m+2}\big(\mathbf{R}\mathbf{N}\big)^{m+2}.$$

Therefore, $\mathbf{RN}$ is an m-quasi-(α,β)-Class $(\mathcal{Q})$ operator.

(2) In similar way the proof of the statement (2) follows. □

Theorem 2.14. *If* $\mathbf{R} \in \mathcal{B}\big[\mathcal{Y}\big]$ *is an* m*-quasi-*(α,β)*-Class* $(\mathcal{Q})$ *operator, then* $\ker\big(\mathbf{R}^{m+1}\big) = \ker\big(\mathbf{R}^{m+2}\big)$.

Proof. Since $\mathbf{R}$ is m-quasi-(α,β)-Class $(\mathcal{Q})$ operator

$$\alpha\|\mathbf{R}^{m+2}w\| \leq \|\mathbf{R}^*\mathbf{R}^{m+1}w\| \leq \beta\|\mathbf{R}^{m+2}w\| \quad \forall\, w \in \mathcal{Y}.$$

Let $w \in \ker(\mathbf{R}^{m+2})$, we get $\mathbf{R}^*\mathbf{R}^{m+1}w = 0$ and therefore $\big(\mathbf{R}^*\big)^{m+1}\mathbf{R}^{m+1}w = 0$, with implies that $w \in \ker\big(\big(\mathbf{R}^*\big)^{m+1}\mathbf{R}^{m+1}\big) = \ker\big(\mathbf{R}^{m+1}\big)$. Consequently, $\ker\big(\mathbf{R}^{m+1}\big) = \ker\big(\mathbf{R}^{m+2}\big)$. □

Corollary 2.15. *If* $\mathbf{R}$ *is an* m*-quasi-*(α,β)*-Class* $(\mathcal{Q})$ *operator, then* $\mathbf{R}$ *has SVEP.*

Proof. According to Theorem 2.14 we have $\ker\big(\mathbf{R}^{m+1}\big) = \ker\big(\mathbf{R}^{m+2}\big)$. Hence $\mathbf{R}$ has finite ascent and therefore $\mathbf{R}$ has SVEP by [1, Theorem 3.8]. □

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