Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 2 Issue 3(2010), Pages 53-64.

NEW CLASSES OF P-VALENT HARMONIC FUNCTIONS

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Abstract. In the present paper we have studied new subclasses of p-valent harmonic functions in the unit disc and obtain the basic properties such as coefficient bound, distortion properties, extreme points and also we apply integral operator for the same.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain C if both u and v are real harmonic in D. In any simply connected domain $D \subset C$ we can write $f = h + \overline{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $\left|h^{'}(z)\right| > \left|g^{'}(z)\right|$, Clunie and Sheil - Small [7] (see also, [2], [8] and [13]).

Denote by H the family of functions $f = h + \overline{g}$, which are harmonic univalent and sense-preserving in the open unit disc $U = \{z : |z| < 1\}$ with normalization $f(0) = h(0) = f_z(0) - 1 = 0$.

Recently, Ahuja and Jahangiri [1] defined the class $H_p(n)(p, n \in N = \{1, 2, ...\})$ consisting of all p-valent harmonic functions $f = h + \overline{g}$ that are sense-preserving in U, and h, g are of the form:

$$h(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k,\ g(z) = \sum_{k=n+p-1}^{\infty} b_k z^k,\ |b_{n+p-1}| < 1. \qquad (1.1)$$

For $f = h + \overline{g}$ given by (1.1), the modified multiplier transformation of f is defined as:

$$D_p^{m,\ell} f(z) = D_p^{m,\ell} h(z) + (-1)^m \overline{D_p^{m,\ell} g(z)}; \qquad (1.2)$$

where

$$D_p^{m,\ell} h(z) = z^p + \sum_{k=n+p}^{\infty} (\frac{k+\ell}{p+\ell})^m a_k z^k$$

2000 *Mathematics Subject Classification.* 30C55.

Key words and phrases. Harmonic function, p-valent, multiplier transformation.

Submitted December 3, 2009. Published July 29, 2010.

and

$$D_p^{m,\ell} g(z) = \sum_{k=n+p-1}^{\infty} (\frac{k+\ell}{p+\ell})^m b_k z^k$$

(see [5], [6], [10] and [14]). We note that $D_p^{m,0} f(z) = D_p^m f(z)$, where $D_p^m f(z)$ is the p-valent Salagean operator (see [3] and [9]).

Also, the subclasses denoted by $H_p^m(n)$ consist of harmonic functions $f_m = h + \overline{g_m}$, so that h and g_m are of the form:

$$h(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \quad g_m(z) = (-1)^m \sum_{k=n+p-1}^{\infty} b_k z^k,$$

for $a_k, b_k \geq 0, |b_{n+p-1}| < 1$.

For $0 \leq \alpha < p, m \in N_0 = N \cup \{0\}, \ell \geq 0, \lambda \geq 0, p \in N$ and $z = re^{i\theta} \in U$, a function f in $H_p(n)$ is said to be in the class $H_p^m(n, \ell; \lambda, \alpha)$ if

$$\operatorname{Re}\left\{(1-\lambda)p^m \frac{D_p^{m,\ell} f(z)}{\dfrac{\partial^m}{\partial\theta^m} z^p} + \lambda p^{m+1} \frac{D_p^{m+1,\ell} f(z)}{\dfrac{\partial^{m+1}}{\partial\theta^{m+1}} z^p}\right\} > \frac{\alpha}{p^{m+1}},$$

where $D_p^{m,\ell} f$ is defined by (1.2).

We define the subclass $\overline{H}_p^m(n, \ell; \lambda, \alpha) = H_p^m(n, \ell; \lambda, \alpha) \cap H_p^m(n)$.

We note that : (i) $\overline{H}_p^m(n, 0; \lambda, \alpha) = \overline{H}_p^m(n; \lambda, \alpha)$ (Yalcin et al. [15]);

(ii) $\overline{H}_p^0(n, 0; \lambda, \alpha) = \overline{H}_p(n; \lambda, \alpha)$ (Ahuja and Jahangiri [1]);

(iii) $\overline{H}_p^m(n, \ell; 0, \alpha) = \overline{H}_p^m P(n, \ell; \alpha)$

$$= \left\{ f \in H_p^m(n) : \operatorname{Re}\left(p^m \frac{D_p^{m,\ell} f(z)}{\frac{\partial^m}{\partial\theta^m} z^p} \right) > \frac{\alpha}{p^{m+1}}, z \in U \right\};$$

(iv) $\overline{H}_p^m(n, \ell; 1, \alpha) = \overline{H}_p^m Q(n, \ell; \alpha)$

$$= \left\{ f \in H_p^m(n) : \operatorname{Re}\left(p^{m+1} \frac{D_p^{m+1,\ell} f(z)}{\frac{\partial^m}{\partial\theta^m} z^p} \right) > \frac{\alpha}{p^{m+1}}, z \in U \right\}.$$

In this paper, we obtain sufficient coefficient bounds for functions in $H_p^m(n, \ell; \lambda, \alpha)$. These sufficient coefficient conditions are shown to be also necessary for functions in $\overline{H}_p^m(n, \ell; \lambda, \alpha)$. A representation theorem, inclusion properties, and distortion bounds for the class $\overline{H}_p^m(n, \ell; \lambda, \alpha)$ are also obtained.

2. Coefficient bounds

Theorem 1. *Let $f = h + \overline{g}$ given by (1.1). Then $f \in H_p^m(n, \ell; \lambda, \alpha)$ if*

$$\sum_{k=n+p}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{\lambda k + (1-\lambda)p}{p+\ell}\right| |a_k| +$$

$$\sum_{k=n+p-1}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{\lambda(k+\ell) - (1-\lambda)(p+\ell)}{p+\ell}\right| |b_k| \le p^{m+1} - \alpha. \tag{2.1}$$

Proof. Using the fact that $\mathrm{Re}\zeta \ge 0$ if and only if $|1+\zeta| \ge |1-\zeta|$ in U, it sufficies to show that

$$\left|p^{m+1} - \alpha + p^{m+1} w\right| \ge \left|p^{m+1} + \alpha - p^{m+1} w\right|, \tag{2.2}$$

where

$$w(z) = (1-\lambda)p^m \frac{D_p^{m,\ell} f(z)}{\frac{\partial^m}{\partial\theta^m} z^p} + \lambda p^{m+1} \frac{D_p^{m+1,\ell} f(z)}{\frac{\partial^{m+1}}{\partial\theta^{m+1}} z^p}.$$

Substituting for h and g in w, we obtain

$$\left|p^{m+1} - \alpha + p^{m+1} w\right| \ge 2p^{m+1} - \alpha$$
$$- \sum_{k=n+p}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{p+\ell+\lambda(k-p)}{p+\ell}\right| |a_k| \, |z|^{k-p}$$
$$- \sum_{k=n+p-1}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{p+\ell-\lambda(k+p+2\ell)}{p+\ell}\right| |b_k| \, |z|^{k-p}$$

and

$$\left|p^{m+1} + \alpha - p^{m+1} w\right| \le \alpha + \sum_{k=n+p}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{p+\lambda(k-p)}{p+\ell}\right| |a_k| \, |z|^{k-p}$$
$$+ \sum_{k=n+p-1}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{p+\ell-\lambda(k+p+2\ell)}{p+\ell}\right| |b_k| \, |z|^{k-p}$$

these two inequalities in conjunction with the required condition (2.1) yields

$$\left|p^{m+1} - \alpha + p^{m+1} w\right| - \left|p^{m+1} + \alpha - p^{m+1} w\right|$$
$$\ge 2\left[p^{m+1} - \alpha - \sum_{k=n+p}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{p+\lambda(k-p)}{p+\ell}\right| |a_k| - \right.$$
$$\left. \sum_{k=n+p-1}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{p+\ell-\lambda(k+p+2\ell)}{p+\ell}\right| |b_k| \right] \ge 0.$$

The coefficient bound (2.1) gave in Theorem 1 is sharp for the function

$$f(z) = z^p + \sum_{k=n+p}^{\infty} (\frac{p+\ell}{k+\ell})^m \frac{(p+\ell)x_k}{p^{m+1}\left|\lambda k + (1-\lambda)p\right|} z^k$$
$$+ \sum_{k=n+p-1}^{\infty} (\frac{p+\ell}{k+\ell})^m \frac{(p+\ell)\overline{y_k}}{p^{m+1}\left|\lambda(k+\ell) - (1-\lambda)(p+\ell)\right|} \overline{z}^k,$$

where $\sum\limits_{k=n+p}^{\infty}|x_k|+\sum\limits_{k=n+p-1}^{\infty}|y_k|=p^{m+1}-\alpha$.

Theorem 2. *Let $f_m=h+\overline{g_m}$ be given by (1.2). Then $f_m\in\overline{H}_p^m(n,\ell;\lambda,\alpha)$ if and only if*

$$\sum_{k=n+p}^{\infty}p^{m+1}(\frac{k+\ell}{p+\ell})^m\left|\frac{\lambda k+(1-\lambda)p}{p+\ell}\right|a_k+$$
$$\sum_{k=n+p-1}^{\infty}p^{m+1}(\frac{k+\ell}{p+\ell})^m\left|\frac{\lambda(k+\ell)-(1-\lambda)(p+\ell)}{p+\ell}\right|b_k\le p^{m-1}-\alpha. \tag{2.3}$$

Proof. In view of Theorem 1, we only need to prove the only if part of the theorem, since $f_m\in\overline{H}_p^m(n,\ell;\lambda,\alpha)\subset H_p^m(n,\ell;\lambda,\alpha)$. If $f_m\in\overline{H}_p^m(n,\ell;\lambda,\alpha)$ then, for $z=re^{i\theta}\in U$, we get

$$\operatorname{Re}\left\{(1-\lambda)p^m\frac{D_p^{m,\ell}f_m(z)}{\frac{\partial^m}{\partial\theta^m}z^p}+\lambda p^{m+1}\frac{D_p^{m+1,\ell}f_m(z)}{\frac{\partial^{m+1}}{\partial\theta^{m+1}}z^p}\right\}$$
$$=\operatorname{Re}\left\{(1-\lambda)p^m\left(\frac{D_p^{m,\ell}h(z)+(-1)^m\overline{D_p^{m,\ell}g_m(z)}}{\frac{\partial^m}{\partial\theta^m}z^p}\right)\right.$$
$$\left.+\lambda p^{m+1}\left(\frac{D_p^{m+1,\ell}h(z)-(-1)^m\overline{D_p^{m+1,\ell}g_m(z)}}{\frac{\partial^{m+1}}{\partial\theta^{m+1}}z^p}\right)\right\}$$
$$=\operatorname{Re}\left\{(1-\lambda)\left(\frac{D_p^{m,\ell}h(z)+(-1)^m\overline{D_p^{m,\ell}g_m(z)}}{i^mz^p}\right)\right.$$
$$\left.+\lambda\left(\frac{D_p^{m+1,\ell}h(z)-(-1)^m\overline{D_p^{m+1,\ell}g_m(z)}}{i^{m+1}z^p}\right)\right\}$$
$$\ge 1-\sum_{k=n+p}^{\infty}(\frac{k+\ell}{p+\ell})^m\left|\frac{\lambda k+(1-\lambda)p}{p+\ell}\right|a_kr^{k-p}$$
$$-\sum_{k=n+p-1}^{\infty}(\frac{k+\ell}{p+\ell})^m\left|\frac{\lambda(k+\ell)-(1-\lambda)(p+\ell)}{p+\ell}\right|b_kr^{k-p}\ge\frac{\alpha}{p^{m+1}}.$$

This inequality must holds for all $z\in U$. In particular, choosing the values of z on the positive real axis, letting $r\to 1$, it yields the required condition.

Putting $\lambda=0$ in Theorem 2, we obtain the following corollary.

Corollary 1. *Let $f_m=h+g_m$ be given by (1.2). Then $f_m\in\overline{H}_p^mP(n,\ell;\alpha)$ if and only if*

$$\sum_{k=n+p}^{\infty}p^{m+1}(\frac{k+\ell}{p+\ell})^m(\frac{p}{p+\ell})a_k+\sum_{k=n+p-1}^{\infty}p^{m+1}(\frac{k+\ell}{p+\ell})^mb_k\le p^{m+1}-\alpha$$

Putting $\lambda=1$ in Theorem 2, we obtain the following corollary.

Corollary 2. *Let $f_m = h + g_m$ be given by (1.2). Then $f_m \in \overline{H}_p^m Q(n,\ell;\alpha)$ if and only if*

$$\sum_{k=n+p}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m(\frac{k}{p+\ell})a_k + \sum_{k=n+p-1}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^{m+1}b_k \le p^{m+1}-\alpha.$$

3. . Extreme points and distortion theorem

Our next theorem is on the extreme points of convex hulls of $\overline{H}_p^m(n,\ell;\lambda,\alpha)$ denoted by clco $\overline{H}_p^m(n,\ell;\lambda,\alpha)$.

Theorem 3. *Let f_m be given by (1.2). Then $f_m \in \overline{H}_p^m(n,\ell;\lambda,\alpha)$ if and only if f_m can be expressed as*

$$f_m(z) = X_p h_p(z) + \sum_{k=n+p}^{\infty} X_k h_k(z) + \sum_{k=n+p-1}^{\infty} Y_k g_{k_m}(z),$$

where

$$h_p(z) = z^p, h_k(z) = z^p - \frac{p^{m+1}-\alpha}{\left|\frac{\lambda k+(1-\lambda)p}{p+\ell}\right| p^{m+1}(\frac{k+\ell}{p+\ell})^m} z^k$$

$$(k = n+p, n+p+1, ...), g_{k_m}(z) = z^p + (-1)^m \frac{p^{m+1}-\alpha}{\left|\frac{\lambda(k-\lambda)-(1-\lambda)(p+\ell)}{p+\ell}\right| p^{m+1}(\frac{k+\ell}{p+\ell})^m} \overline{z}^k$$

$$(k = n+p-1, n+p, ...), X_p \ge 0, Y_{n+p-1} \ge 0, X_p + \sum_{k=n+p}^{\infty} X_k$$

$$+\sum_{k=n+p-1}^{\infty} Y_k = 1 \quad \text{and} \quad X_k \ge 0, Y_k \ge 0 \quad \text{for} \quad k = n+p, n+p+1, ... \ .$$

Proof. For functions f_m of the form (2.2), we have

$$f_m(z) = X_p h_p(z) + \sum_{k=n+p}^{\infty} X_k h_k(z) + \sum_{k=n+p-1}^{\infty} Y_k g_{k_m}(z)$$

$$= z^p - \sum_{k=n+p}^{\infty} \frac{p^{m+1}-\alpha}{p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{\lambda k+(1-\lambda)p}{p+\ell}\right|} X_k z^k$$

$$+(-1)^m \sum_{k=n+p-1}^{\infty} \frac{p^{m+1}-\alpha}{p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{\lambda(k+\ell)-(1-\lambda)(p+\ell)}{p+\ell}\right|} Y_k \overline{z}^k.$$

Consequently, $f_m \in \overline{H}_p^m(n,\ell;\lambda,\alpha)$, since by (2.2), we have

$$\sum_{k=n+p}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{\lambda k+(1-\lambda)p}{p+\ell}\right| a_k +$$

$$\sum_{k=n+p-1}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{\lambda(k+\ell)-(1-\lambda)(p+\ell)}{p+\ell}\right| b_k$$

$$= \sum_{k=n+p}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{\lambda k+(1-\lambda)p}{p+\ell}\right| \left|\frac{p^{m+1}-\alpha}{p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{\lambda k+(1-\lambda)p}{p+\ell}\right|}\right| |X_k|$$

$$+\sum_{k=n+p-1}^{\infty} p^{m+1}\left(\frac{k+\ell}{p+\ell}\right)^m \left|\frac{\lambda(k+\ell)-(1-\lambda)(p+\ell)}{p+\ell}\right| \left|\frac{p^{m+1}-\alpha}{p^{m+1}\left(\frac{k+\ell}{p+\ell}\right)^m \left|\frac{\lambda(k+\ell)-(1-\lambda)(p+\ell)}{p+\ell}\right|}\right| |Y_k|$$

$$= (p^{m+1}-\alpha)\left(\sum_{k=n+p}^{\infty}|X_k| + \sum_{k=n+p-1}^{\infty}|Y_k|\right) = (p^{m+1}-\alpha)(1-X_p)$$

$$\leq p^{m+1}-\alpha$$

and so $f_m \in \overline{H}_p^m(n,\ell;\lambda,\alpha)$.

Conversely, suppose $f_m \in \overline{H}_p^m(n,\ell;\lambda,\alpha)$. Letting $X_p = 1 - \sum\limits_{k=n+p}^{\infty} X_k - \sum\limits_{k=n+p-1}^{\infty} Y_k$, where

$$X_k = \frac{p^{m+1}\left(\frac{k+\ell}{p+\ell}\right)^m \left|\frac{\lambda k+(1-\lambda)p}{p+\ell}\right|}{p^{m+1}-\alpha} a_k$$

and

$$Y_k = \frac{p^{m+1}\left(\frac{k+\ell}{p+\ell}\right)^m \left|\frac{\lambda(k+\ell)-(1-\lambda)(p+\ell)}{p+\ell}\right|}{p^{m+1}-\alpha} b_k,$$

we obtain the required representation, since

$$f_m(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k + (-1)^m \sum_{k=n+p-1}^{\infty} b_k \overline{z}^k$$

$$= z^p - \sum_{k=n+p}^{\infty} \frac{(p^{m+1}-\alpha)X_k}{p^{m+1}\left(\frac{k+\ell}{p+\ell}\right)^m \left|\frac{\lambda k+(1-\lambda)p}{p+\ell}\right|} z^k$$

$$+(-1)^m \sum_{k=n+p-1}^{\infty} \frac{(p^{m+1}-\alpha)Y_k}{p^{m+1}\left(\frac{k+\ell}{p+\ell}\right)^m \left|\frac{\lambda(k+\ell)-(1-\lambda)(p+\ell)}{p+\ell}\right|} \overline{z}^k$$

$$= z^p - \sum_{k=n+p}^{\infty} (z^p - h_k(z))X_k - \sum_{k=n+p-1}^{\infty} (z^p - g_{k_m}(z))Y_k(z)$$

$$= (1 - \sum_{k=n+p}^{\infty} X_k - \sum_{k=n+p-1}^{\infty} Y_k(z))z^p + \sum_{k=n+p}^{\infty} h_k(z)X_k + \sum_{k=n+p-1}^{\infty} g_{k_m}(z)Y_k$$

$$= X_p h_p(z) + \sum_{k=n+p}^{\infty} X_k h_k(z) + \sum_{k=n+p-1}^{\infty} Y_k g_{k_m}.$$

This completes the proof of Theorem 3.

The inclusion relations between the classes $\overline{H}_p^m P(n,\ell;\alpha), \overline{H}_p^m Q(n,\ell;\alpha)$ and $\overline{H}_p^m(n,\ell;\lambda,\alpha)$ for different values of λ are not so obvious. In the following theorem we discuss the inclusion relation between the above mentiond classes.

Theorem 4. *For $n \in N$ and $0 \leq \alpha < p$, we have*

(i) $\overline{H}_p^m Q(n,\ell;\alpha) \subset \overline{H}_p^m P(n,\ell;\alpha)$,

(ii) $\overline{H}_p^m Q(n,\ell;\alpha) \subset \overline{H}_p^m(n,\ell;\lambda,\alpha), 0 < \lambda \leq 1$,

(iii) $\overline{H}_p^m(n,\ell;\lambda,\alpha) \subset \overline{H}_p^m Q(n,\ell;\alpha), \lambda \geq 1.$

Proof. (i) In view of Corollaries 1 and 2, since

$$\sum_{k=n+p}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m \frac{p}{p+\ell} a_k + \sum_{k=n+p-1}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m b_k$$

$$\leq \sum_{k=n+p}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m \frac{k}{p+\ell} a_k + \sum_{k=n+p-1}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^{m+1} b_k$$

$$\leq p^{m+1} - \alpha,$$

the result follows.

(ii) For $0 \leq \lambda < 1$, we have

$$\sum_{k=n+p}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{\lambda k + (1-\lambda)p}{p+\ell}\right| a_k +$$

$$\sum_{k=n+p-1}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{\lambda(k+\ell) - (1-\lambda)(p+\ell)}{p+\ell}\right| b_k$$

$$= \sum_{k=n+p}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{\lambda(k-p)+p}{p+\ell}\right| a_k +$$

$$\sum_{k=n+p-1}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{\lambda(k+p)+\ell(2\lambda-1)-p}{p+\ell}\right| b_k$$

$$\leq \sum_{k=n+p}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m (\frac{k}{p+\ell}) a_k + \sum_{k=n+p-1}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^{m+1} b_k \leq p^{m+1} - \alpha$$

by Corollary 2. Thus, (ii) is obtained from Theorem 2.

(iii) If $\lambda \geq 1$, then,

$$\sum_{k=n+p}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m (\frac{k}{p+\ell}) a_k + \sum_{k=n+p-1}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^{m+1} b_k +$$

$$\leq \sum_{k=n+p}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{\lambda(k-p)+p}{p+\ell}\right| a_k +$$

$$\sum_{k=n+p-1}^{\infty} p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{\lambda(k+\ell)+\ell(2\lambda-1)-p}{p+\ell}\right| b_k \leq p^{m+1} - \alpha.$$

Thus, (iii) is obtained from Corollary 2.

Finally, we give a distortion theorem for functions in $\overline{H}_p^m(n,\ell;\lambda,\alpha)$, which leads to a covering result for this family.

Theorem 5. *Let the functions $f_m(z)$ defined by* (1.2) *be in the class* $\overline{H}_p^m(n,\ell;\lambda,\alpha)(\lambda \geq 1)$*. Then for $|z| = r < 1$, we have*

$$|f_m(z)| \leq (1 + b_{n+p-1} r^{n-1}) r^p +$$

$$\left\{\frac{p^{m+1}-\alpha}{p^{m+1}(\frac{n+p+\ell}{p+\ell})^m(\frac{\lambda n+p}{p+\ell})}-\right.$$

$$\left.\frac{(\frac{n+p-1+\ell}{p+\ell})^m\left[\lambda(n+2p-1)-p+\ell(2\lambda-1)\right]}{(\frac{n+p+\ell}{p+\ell})^m(\lambda n+p)}b_{n+p-1}\right\}r^{n+p}$$

and

$$|f_m(z)|\geq(1-b_{n+p-1}r^{n-1})r^p-$$

$$\left\{\frac{p^{m+1}-\alpha}{p^{m+1}(\frac{n+p+\ell}{p+\ell})^m(\frac{\lambda n+p}{p+\ell})}-\right.$$

$$\left.\frac{(\frac{n+p-1+\ell}{p+\ell})^m\left[\lambda(n+2p-1)-p+\ell(2\lambda-1)\right]}{(\frac{n+p+\ell}{p+\ell})^m(\lambda n+p)}b_{n+p-1}\right\}r^{n+p}.$$

Proof. We prove the left hand side inequality for $|f_m|$. The proof for the right hand side inequality can be done by using similar arguments

Let $f_m\in\overline{H}_p^m(n,\ell;\lambda,\alpha)$, then from Theorem 2, we have

$$|f_m(z)|=\left|z^p+(-1)^m b_{n+p-1}\overline{z}^{n+p-1}+\sum_{k=n+p}^{\infty}(a_kz^k+(-1)^m b_k\overline{z}^k)\right|$$

$$\geq r^p-b_{n+p-1}r^{n+p-1}-$$

$$\frac{p^{m+1}-\alpha}{p^{m+1}(\frac{n+p+\ell}{p+\ell})^m(\frac{\lambda n+p}{p+\ell})}\sum_{k=n+p}^{\infty}\left\{\frac{(\frac{\lambda n+p}{p+\ell})}{p^{m+1}-\alpha}a_k+\right.$$

$$\left.\frac{(\frac{\lambda n+p}{p+\ell})}{p^{m+1}-\alpha}b_k\right\}p^{m+1}(\frac{n+p+\ell}{p+\ell})^m r^k$$

$$\geq r^p-b_{n+p-1}r^{n+p-1}-$$

$$\frac{p^{m+1}-\alpha}{p^{m+1}(\frac{n+p+\ell}{p+\ell})^m(\frac{\lambda n+p}{p+\ell})}\sum_{k=n+p}^{\infty}\left\{\frac{(\frac{\lambda(k-p)+p}{p+\ell})}{p^{m+1}-\alpha}a_k+\right.$$

$$\left.\frac{\frac{\lambda(k+p)-p+\ell(2\lambda-1)}{p+\ell}}{p^{m+1}-\alpha}b_k\right\}p^{m+1}(\frac{k+\ell}{p+\ell})^m r^k$$

$$\geq(1-b_{n+p-1}r^{n-1})r^p-$$

$$\frac{p^{m+1}-\alpha}{p^{m+1}(\frac{n+p+\ell}{p+\ell})^m(\frac{\lambda n+p}{p+\ell})}\left\{1-p^{m+1}\frac{(\frac{n+p+\ell-1}{p+\ell})^m\left[\frac{\lambda(n+2p-1)-p+p(2\lambda-1)}{p+\ell}\right]}{p^{m+1}-\alpha}b_{n+p-1}\right\}r^{n+p}$$

$$\geq (1-b_{n+p-1}r^{n-1})r^p-$$

$$\left\{\frac{p^{m+1}-\alpha}{p^{m+1}(\frac{n+p+\ell}{p+\ell})^m(\frac{\lambda n+p}{p+\ell})}-\right.$$

$$\left.\frac{(\frac{n+p+\ell-1}{p+\ell})^m\left[\lambda(n+2p-1)-p+p(2\lambda-1)\right]}{(\frac{n+p+\ell}{p+\ell})^m(\lambda n+p)}b_{n+p-1}\right\}r^{n+p}.$$

This completes the proof of Theorem 5

The following covering result follows from the left side inequality in Theorem 5.

Corollary 3. *Let* $f_m \in \overline{H}_p^m(n,\ell;\lambda,\alpha)$, *then the set*

$$\left\{w:|w|<\right.$$

$$\left.\frac{p^{m+1}(\frac{n+p+\ell}{p+\ell})^m(\frac{\lambda n+p}{p+\ell})-p^{m-1}+\alpha-\left\{p^{m+1}(\frac{n+p+\ell}{p+\ell})^m(\frac{\lambda n+p}{p+\ell})+p^{m+1}(\frac{n+p+\ell-1}{p+\ell})^m\left[\frac{\lambda(n+2p-1)-p+p(2\lambda-1)}{p+\ell}\right]\right\}b_{n+p-1}}{p^{m+1}(\frac{n+p+\ell}{p+\ell})^m(\frac{\lambda n+p}{p+\ell})}\right\}$$

is included in $f_m(U)$.

Putting $\lambda=0$ in Theorem 5, we obtain the following corollary.

Corollary 4. *Let the functions* $f_m(z)$ *defined by* (1.2) *be in the class* $\overline{H}_p^m P(n,\ell;\alpha)$, *then for* $|z|=r<1$,

$$|f_m(z)|\leq (1+b_{n+p-1}r^{n-1})r^p+$$

$$\left\{\frac{p^{m+1}-\alpha}{p^{m+1}(\frac{n+p+\ell}{p+\ell})^m(\frac{p}{p+\ell})}+\frac{(\frac{n+p-1+\ell}{p+\ell})^m}{(\frac{n+p+\ell}{p+\ell})^m(\frac{p}{p+\ell})}b_{n+p-1}\right\}r^{n+p}$$

and

$$|f_m(z)|\geq (1-b_{n+p-1}r^{n-1})r^p-$$

$$\left\{\frac{p^{m+1}-\alpha}{p^{m+1}(\frac{n+p+\ell}{p+\ell})^m(\frac{p}{p+\ell})}+\frac{(\frac{n+p-1+\ell}{p+\ell})^m}{(\frac{n+p+\ell}{p+\ell})^m(\frac{p}{p+\ell})}b_{n+p-1}\right\}r^{n+p}.$$

Putting $\lambda=1$ in Theorem 5, we obtain the following corollary.

Corollary 5. *Let the functions* $f_m(z)$ *defined by* (1.2) *be in the class* $\overline{H}_p^m Q(n,\ell;\alpha)$. *Then for* $|z|=r<1$, *we have*

$$|f_m(z)|\leq (1+b_{n+p-1}r^{n-1})r^p+$$

$$\left\{\frac{p^{m+1}-\alpha}{p^{m+1}(\frac{n+p+\ell}{p+\ell})^m(\frac{n+p}{p+\ell})}-\frac{(\frac{n+p-1+\ell}{p+\ell})^{m+1}}{(\frac{n+p+\ell}{p+\ell})^m(\frac{n+p}{p+\ell})}\right\}b_{n+p-1}$$

and

$$|f_m(z)| \geq (1 - b_{n+p-1} r^{n-1}) r^p -$$

$$\left\{ \frac{p^{m+1} - \alpha}{p^{m+1} (\frac{n+p+\ell}{p+\ell})^m (\frac{n+p}{p+\ell})} - \frac{(\frac{n+p-1+\ell}{p+\ell})^{m+1}}{(\frac{n+p+\ell}{p+\ell})^m (\frac{n+p}{p+\ell})} b_{n+p-1} \right\} r^{n+p}.$$

Now we will examine the closure properties of the class $\overline{H}_p^m(n, \ell; \lambda, \alpha)$ under the generalized Bernardi-Libera-Livingston integral operator (see [4], [11] and [12]) $L_{c,p}(f)$ which is defined by

$$L_{c,p}(f)(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt \qquad (c > -p). \tag{3.1}$$

Theorem 6. *Let* $f \in \overline{H}_p^m(n, \ell; \lambda, \alpha)$. *Then* $L_{c,p}(f)(z)$ *belongs to the class* $\overline{H}_p^m(n, \ell; \lambda, \alpha)$.

Proof. From the representation of $L_{c,p}(f)(z)$, it follows that

$$L_{c,p}(f)(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} \left\{ h(t) + \overline{g(t)} \right\} dt$$

$$= \frac{c+p}{z^c} \left\{ \int_0^z t^{c-1} (t^p - \sum_{k=n+p}^{\infty} |a_k| \, t^k) dt + \overline{\int_0^z t^{c-1} (\sum_{k=n+p-1}^{\infty} |b_k| \, t^k) dt} \right\}$$

$$= z^p - \sum_{k=n+p}^{\infty} A_k z^k + \sum_{k=n+p-1}^{\infty} B_k \overline{z}^k,$$

where

$$A_k = (\frac{c+p}{c+k}) a_k \qquad \text{and} \qquad B_k = (\frac{c+p}{c+k}) b_k.$$

Therefore

$$\sum_{k=n+p}^{\infty} p^{m+1} (\frac{k+\ell}{p+\ell})^m \left| \frac{\lambda k + (1-\lambda) p}{p+\ell} \right| (\frac{c+p}{c+k}) a_k$$

$$+ \sum_{k=n+p-1}^{\infty} p^{m+1} (\frac{k+\ell}{p+\ell})^m \left| \frac{\lambda (k+\ell) - (1-\lambda)(p+\ell)}{p+\ell} \right| (\frac{c+p}{c+k}) b_k$$

$$\leq \sum_{k=n+p-1}^{\infty} p^{m+1} (\frac{k+\ell}{p+\ell})^m \left| \frac{\lambda k + (1-\lambda) p}{p+\ell} \right| a_k$$

$$+ \sum_{k=n+p-1}^{\infty} p^{m+1} (\frac{k+\ell}{p+\ell})^m \left| \frac{\lambda (k+\ell) - (1-\lambda)(p+\ell)}{p+\ell} \right| b_k \leq p^{m+1} - \alpha.$$

Since $f \in \overline{H}_p^m(n, \ell; \lambda, \alpha)$, by Theorem 2, we have $L_{c,p}(f)(z) \in \overline{H}_p^m(n, \ell; \lambda, \alpha)$.

For harmonic functions of the form:

$$f_m(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k + (-1)^m \sum_{k=n+p-1}^{\infty} b_k \overline{z}^k (a_k \geq 0, b \geq 0) \tag{3.2}$$

and

$$F_m(z) = z^p - \sum_{k=n+p}^{\infty} A_k z^k + (-1)^m \sum_{k=n+p-1}^{\infty} B_k \overline{z}^k (A_k \geq 0, B \geq 0), \tag{3.3}$$

we define the convolution of two harmonic functions f_m and F_m as

$$\begin{aligned}(f_m * F_m)(z) &= f_m(z) * F_m(z) \\ &= z^p - \sum_{k=n+p}^{\infty} a_k A_k z^k + (-1)^m \sum_{k=n+p-1}^{\infty} b_k B_k \overline{z}^k\end{aligned}$$

Using this definition, we show that the class $\overline{H}_p^m(n, \ell; \lambda, \alpha)$ is closed under convolution.

Theorem 7. *For* $0 \leq \beta \leq \alpha < p, m \in N_0, p \in N, \ell \geq 0$ and $\lambda \geq 0$, let $f_m \in \overline{H}_p^m(n, \ell; \lambda, \alpha)$ and $F_m \in \overline{H}_p^m(n, \ell; \lambda, \beta)$. Then $f_m * F_m \in \overline{H}_p^m(n, \ell; \lambda, \alpha) \subset \overline{H}_p^m(n, \ell; \lambda, \beta)$.

Proof. Let the functions $f_m(z)$ defined by (1.2) be in the class $\overline{H}_p^m(n, \ell; \lambda, \alpha)$ and let the functions $F_m(z)$ defined by (3.3) be in the class $\overline{H}_p^m(n, \ell; \lambda, \beta)$. Then the convolution $f_m * F_m$ is given by (3.4). We wish to show that the coefficients of $f_m * F_m$ satisfy the required condition given in Theorem 2. For $F_m \in \overline{H}_p^m(n, \ell; \lambda, \beta)$ we note that $0 \leq A_k \leq 1$ and $0 \leq B_k \leq 1$. Now, for the convolution $f_m * F_m$ we obtain

$$\begin{aligned}
&\sum_{k=n+p}^{\infty} \frac{p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{\lambda k + (1-\lambda)p}{p+\ell}\right|}{p^{m+1}-\beta} a_k A_k + \\
&\sum_{k=n+p-1}^{\infty} \frac{p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{\lambda(k+\ell) - (1-\lambda)(p+\ell)}{p+\ell}\right|}{p^{m+1}-\beta} b_k B_k \\
&\leq \sum_{k=n+p}^{\infty} \frac{p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{\lambda k + (1-\lambda)p}{p+\ell}\right|}{p^{m+1}-\beta} a_k + \\
&\sum_{k=n+p-1}^{\infty} \frac{p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{\lambda(k+\ell) - (1-\lambda)(p+\ell)}{p+\ell}\right|}{p^{m+1}-\beta} b_k \\
&\leq \sum_{k=n+p}^{\infty} \frac{p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{\lambda k + (1-\lambda)p}{p+\ell}\right|}{p^{m+1}-\alpha} a_k \\
&+ \sum_{k=n+p-1}^{\infty} \frac{p^{m+1}(\frac{k+\ell}{p+\ell})^m \left|\frac{\lambda(k+\ell) - (1-\lambda)(p+\ell)}{p+\ell}\right|}{p^{m+1}-\alpha} b_k \leq 1,
\end{aligned}$$

since $0 \leq \beta < \alpha < p$ and $f_m \in \overline{H}_p^m(n, \ell; \lambda, \alpha)$. Therefore $f_m * F_m \in \overline{H}_p^m(n, \ell; \lambda, \alpha) \subset \overline{H}_p^m(n, \ell; \lambda, \beta)$.

Acknowledgments: The authors thank the referees for their valuable suggestions to improve the paper.

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