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DETERMINATION OF THE $\eta-$DUAL OF SOME CLASSICAL SETS OF n-SEQUENCES OF FUZZY NUMBERS

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Abstract. The main aim of the present paper is to introduce the notion of $\eta-$dual for sets of n-sequences of fuzzy numbers and compute the $\eta-$dual of some classical sets of n-sequences of fuzzy numbers.

1. Introduction

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [23] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [15] introduced bounded and convergent sequences of fuzzy numbers and studied their some properties. Later on sequences of fuzzy numbers have been discussed by Diamond and Kloeden [11], Nanda [16], Esi [9], Dutta [4,5,6] and many others.

A fuzzy number is a fuzzy set on the real axis, i.e., a mapping $u : R \longrightarrow [0,1]$ which satisfies the following four conditions:
(i) u is normal, i.e., there exists an $x_0 \in R$ such that $u(x_0) = 1$.
(ii) u is fuzzy convex, i.e., $u[\lambda x + (1-\lambda)y] \geq min\{u(x), u(y)\}$ for all $x, y \in R$ and for all $\lambda \in [0,1]$.
(iii) u is upper semi-continuous.
(iv) The set $[u]_0 = \overline{\{x \in R : u(x) > 0\}}$ is compact, where $\overline{\{x \in R : u(x) > 0\}}$ denotes the closure of the set $\{x \in R : u(x) > 0\}$ in the usual topology of R.
We denote the set of all fuzzy numbers on R by E^1 and called it as the space of fuzzy numbers. $\lambda-$level set $[u]_\lambda$ of $u \in E^1$ is defined by

$$[u]_\lambda = \{t \in R : u(t) \geq \lambda\},\ (0 < \lambda \leq 1),$$
$$= \overline{\{t \in R : u(t) > \lambda\}},\ (\lambda = 0).$$

The set $[u]_\lambda$ is a closed, bounded and non-empty interval for each $\lambda \in [0,1]$ which is defined by $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$. R can be embedded in E^1, since each $r \in R$

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can be regarded as a fuzzy number

$$\overline{r}(t) = 1,\ t = r,$$
$$= 0,\ t \neq r.$$

Let W be the set of all closed bounded intervals A of real numbers such that $A = [A_1, A_2]$. Define the relation d on W as follows:

$$d(A, B) = max\{|A_1 - B_1|, |A_2 - B_2|\}.$$

Then (W, d) is a complete metric space (see Diamond and Kloeden [10], Nanda [16]). Then Talo and Başar [18] defined the metric D on E^1 by means of Hausdorff metric d as

$$d(u, v) = \sup_{\lambda \in [0,1]} d([u]_\lambda, [v]_\lambda) = \sup_{\lambda \in [0,1]} max\ d(|u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)|).$$

Lemma 1.1. *(Talo and Başar[18]) Let $u, v, w, z \in E^1$ and $k \in R$. Then*
(i) $)(E^1, D)$ is a complete metric space.
(ii) $D(ku, kv) = |k|\ D(u, v)$.
(iii) $D(u + v, w + v) = D(u, w)$.
(iv) $D(u + v, w + z) \leq D(u, w) + D(v, z)$.
(v) $|D(u, \overline{0}) - D(v, \overline{0})| \leq D(u, v) \leq D(u, \overline{0}) + D(v, \overline{0})$.

Lemma 1.2. *(Talo and Başar[18]) The following statements hold:*
(i) $D(uv, \overline{0}) \leq D(u, \overline{0})D(v, \overline{0})$ for all $u, v \in E^1$.
(ii) If $u_k \longrightarrow u$, as $k \longrightarrow \infty$ then $D(u_k, \overline{0}) \longrightarrow D(u, \overline{0})$ as $k \longrightarrow \infty$.

Let $n \geq 2$ be an integer. Then we define a fuzzy n-sequence is an n-infinite array of fuzzy real numbers. We denote a fuzzy n-sequence by $(a_{m_1 m_2 \ldots m_n})$, where $a_{m_1 m_2 \ldots m_n}$ are fuzzy real numbers for each $m_1, m_2, \ldots, m_n \in N$. For $n = 2$, we known it as double sequence of fuzzy real numbers. The initial works on double sequences of real or complex terms is found in Bromwich [1]. Hardy [12] introduced the notion of regular convergence for double sequences of real or complex terms. The works on double sequence was further investigated by Basarir and Solancan [2], Moricz [17], Tripathy and Dutta [21], Tripathy and Sarma [22] and many others. Talo and Başar [18] introduced the notion of $\alpha-$, $\beta-$ and $\gamma-$dual of single sequences of fuzzy numbers and computed $\alpha-$, $\beta-$ and $\gamma-$duals of some classical sets of sequences of fuzzy numbers. Recently Dutta and Surender Reddy [8] has computed $\alpha-$dual for some sets of sequences of fuzzy numbers defined using on Orlicz function. Let us denote the set of all n-sequences of fuzzy numbers as ${}_nw^F$.

Definition 1.1. *Let $(a_{m_1 m_2 \ldots m_n}) \in {}_nw^F$. Then the expression $\sum\limits_{m_1} \sum\limits_{m_2} \ldots \sum\limits_{m_n} a_{m_1 m_2 \ldots m_n}$ is called a series corresponding to the n-sequence $(a_{m_1 m_2 \ldots m_n})$ of fuzzy number. Denote*

$$S_{k_1 k_2 \ldots k_n} = \sum_{m_1=1}^{k_1} \sum_{m_2=1}^{k_2} \ldots \sum_{m_n=1}^{k_n} a_{m_1 m_2 \ldots m_n}$$

for all $k_1, k_2, \ldots, k_n \in N$. If the sequence $(S_{k_1 k_2 \ldots k_n})$ converges to a fuzzy number u, then we say that the series $\sum\limits_{m_1} \sum\limits_{m_2} \ldots \sum\limits_{m_n} a_{m_1 m_2 \ldots m_n}$ converges to u and write $\sum\limits_{m_1} \sum\limits_{m_2} \ldots \sum\limits_{m_n} a_{m_1 m_2 \ldots m_n} = u$.

We denote by ${}_n cs^F$ and ${}_n bs^F$, the set of all convergent and bounded series of fuzzy numbers respectively.

We define the classical sets ${}_n c^F$, ${}_n c_0^F$, ${}_n \ell_1^F$, ${}_n \ell_p^F$, ${}_n \ell_\infty^F$, ${}_n \sigma^F$ and ${}_n w_p^F$ consisting of the convergent in Pringsheim's sense, null in Pringsheim's sense, absolutely summable, p-absolutely summable, bounded, eventually alternating and strongly p-Cesàro summable n-sequences of fuzzy numbers, as follows:

$$ {}_n\ell_\infty^F = \{(a_{m_1m_2\dots m_n}) \in {}_n w^F : \sup_{m_1,m_2,\dots,m_n} D(a_{m_1m_2\dots m_n}, \overline{0}) < \infty\}, $$

$$ {}_n\ell_p^F = \{(a_{m_1m_2\dots m_n}) \in {}_n w^F : \sum_{m_1}\sum_{m_2}\dots\sum_{m_n} D(a_{m_1m_2\dots m_n}, \overline{0})^p < \infty\}, $$

$$ {}_n c^F = \{(a_{m_1m_2\dots m_n}) \in {}_n w^F : D(a_{m_1m_2\dots m_n}, L) = 0, \ as\ min(m_1, m_2, \dots, m_n) \longrightarrow \infty, \ for\ some\ fuzzy\ number\ L\}, $$

$$ {}_n c_0^F = \{(a_{m_1m_2\dots m_n}) \in {}_n w^F : D(a_{m_1m_2\dots m_n}, \overline{0}) = 0, \ as\ min(m_1, m_2, \dots, m_n) \longrightarrow \infty\}, $$

$$ {}_n w_p^F = \{(a_{m_1m_2\dots m_n}) \in {}_n w^F : \lim_{i_1,i_2,\dots,i_n \longrightarrow \infty} \frac{1}{i_1 i_2 \dots i_n} \sum_{m_1=1}^{i_1}\sum_{m_2=1}^{i_2}\dots\sum_{m_n=1}^{i_n} D(a_{m_1m_2\dots m_n}, L)^p = 0\}, $$

$$ {}_n \sigma^F = \{(a_{m_1m_2\dots m_n}) \in {}_n w^F : a_{m_1m_2\dots m_n} = -a_{m_1+1,m_2,\dots,m_n} \ for\ all\ m_1 \geq l_1, \dots, \ a_{m_1m_2\dots m_n} = -a_{m_1,m_2,\dots,m_n+1}, \ for\ all\ m_n \geq l_n\}. $$

An Orlicz function is a function $M : [0,\infty) \longrightarrow [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \longrightarrow \infty$, as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [14] used the Orlicz function and introduced the sequence space ℓ_M as follows:

$$ \ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}. $$

They proved that ℓ_M is a Banach space normed by

$$ \|(x_k)\| = inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}. $$

Now we further generalize the above spaces using an Orlicz function as follows:

$$ {}_n\ell_\infty^F(M) = \left\{ (a_{m_1m_2\dots m_n}) \in {}_n w^F : \sup_{m_1,m_2,\dots,m_n} M\left(\frac{D\left(a_{m_1m_2\dots m_n}, \overline{0}\right)}{\rho}\right) < \infty, \ for\ some\ \rho > 0 \right\}, $$

$$ {}_n\ell_p^F(M) = \left\{ (a_{m_1m_2\dots m_n}) \in {}_n w^F : \sum_{m_1}\sum_{m_2}\dots\sum_{m_n} \left(M\left(\frac{D\left(a_{m_1m_2\dots m_n}, \overline{0}\right)}{\rho}\right)\right)^p < \infty, \ for\ some\ \rho > 0 \right\}, $$

$$ {}_n c^F(M) = \{(a_{m_1m_2\dots m_n}) \in {}_n w^F : M\left(\frac{D\left(a_{m_1m_2\dots m_n}, L\right)}{\rho}\right) = 0, \ as\ min(m_1, m_2, \dots, m_n) \longrightarrow \infty, \ for\ some\ fuzzy\ number\ L\ and\ \rho > 0\}, $$

$$ {}_n c_0^F(M) = \{(a_{m_1m_2\dots m_n}) \in {}_n w^F : M\left(\frac{D\left(a_{m_1m_2\dots m_n}, \overline{0}\right)}{\rho}\right) = 0, \ as\ min(m_1, m_2, \dots, m_n) \longrightarrow \infty \ \text{and for some } \rho > 0\}, $$

$$ {}_nw_p^F(M) = \{(a_{m_1m_2\ldots m_n}) \in {}_nw^F : \lim_{i_1,i_2,\ldots,i_n \longrightarrow \infty} \frac{1}{i_1, i_2, \ldots, i_n} \sum_{m_1=1}^{i_1} \sum_{m_2=1}^{i_2} \cdots \sum_{m_n=1}^{i_n} $$

$$ \left(M \left(\frac{D\left(a_{m_1m_2\ldots m_n}, L\right)}{\rho} \right) \right)^p = 0, \ for\ some\ \rho > 0\}, $$

Let $r \geq 1$. Then we define $\eta-$dual, $\beta-$dual and $\gamma-$duals of a set ${}_nE^F \subset {}_nw^F$ which are respectively denoted by $\{{}_nE^F\}^\eta$, $\{{}_nE^F\}^\beta$ and $\{{}_nE^F\}^\gamma$, as follows:

$\{{}_nE^F\}^\alpha = \{(a_{m_1m_2\ldots m_n}) \in {}_nw^F : (a_{m_1m_2\ldots m_n} b_{m_1m_2\ldots m_n}) \in {}_n\ell_r^F, \ for\ all$ $(b_{m_1m_2\ldots m_n}) \in {}_nE^F\}$,

$\{{}_nE^F\}^\beta = \{(a_{m_1m_2\ldots m_n}) \in {}_nw^F : (a_{m_1m_2\ldots m_n} b_{m_1m_2\ldots m_n}) \in {}_ncs^F, \ for\ all$ $(b_{m_1m_2\ldots m_n}) \in {}_nE^F\}$,

$\{{}_nE^F\}^\gamma = \{(a_{m_1m_2\ldots m_n}) \in {}_nw^F : (a_{m_1m_2\ldots m_n} b_{m_1m_2\ldots m_n}) \in {}_nbs^F, \ for\ all$ $(b_{m_1m_2\ldots m_n}) \in {}_nE^F\}$.

For $r = 1$, we write $\{{}_nE^F\}^\eta = \{{}_nE^F\}^\alpha$, $\alpha-$dual of the set ${}_nE^F$. The notion of $\eta-$dual for classical sets of single sequence of scalars was introduced by Chandra and Tripathy [3] and extended to classical sets of double sequences of scalars by Sarma [19].

The proof of the following results is obvious in view of the definition of $\eta-$dual of n-sequences.

Lemma 1.3. *Let ${}_nE^F$ and ${}_nZ^F$ be any two non-empty subsets of ${}_nw^F$. Then*
(i) $\{{}_nE^F\}^\eta$ is a linear subspace of ${}_nw^F$.
(ii) ${}_nE^F \subset {}_nZ^F$ implies $\{{}_nZ^F\}^\eta \subset \{{}_nE^F\}^\eta$.
(iii) ${}_nE^F \subseteq \{{}_nE^F\}^{\eta\eta}$.

2. Main Results

In this section we give $\eta-$, $\beta-$ and $\gamma-$duals of the sets ${}_nc^F(M)$, ${}_nc_0^F(M)$, ${}_n\ell_\infty^F(M)$, ${}_n\sigma^F$ and ${}_nw_p^F(M) \cap {}_n\ell_\infty^F(M)$.

Lemma 2.1. *Let $(a_{m_1m_2\ldots m_n})$ be any element belong to any one of the sets ${}_nc^F(M)$, ${}_nc_0^F(M)$, ${}_n\ell_\infty^F(M)$ and ${}_nw_p^F(M) \cap {}_n\ell_\infty^F(M)$. Then $(a_{m_1m_2\ldots m_n}) \in {}_n\ell_\infty^F$.*

Proof. Let $(a_{m_1m_2\ldots m_n})$ belongs to any one of the sets of the statement of the Lemma. Then there exists some $\rho > 0$ such that

$$ \sup_{m_1,m_2,\ldots,m_n} M\left(\frac{D\left(a_{m_1m_2\ldots m_n}, \overline{0}\right)}{\rho} \right) < \infty $$

It follows that

$$ M\left(\frac{D\left(a_{m_1m_2\ldots m_n}, \overline{0}\right)}{\rho} \right) < \infty, \text{ for all } m_1, m_2, \ldots, m_n \in N $$

Since M is an Orlicz function ($M(x) \longrightarrow \infty, \text{ as } x \longrightarrow \infty$), we have $D\left(a_{m_1m_2\ldots m_n}, \overline{0}\right) < \infty$ for all $m_1, m_2, \ldots, m_n \in N$. Thus $(a_{m_1m_2\ldots m_n}) \in {}_n\ell_\infty^F$. $\square$

Theorem 2.2. *Let M be any Orlicz function. Then the $\eta-$dual of the sets ${}_nc_0^F(M)$, ${}_nc^F(M)$ and ${}_n\ell_\infty^F(M)$ of n-sequences of fuzzy numbers is the set ${}_n\ell_r^F$.*

Proof. Since the proof is similar for the sets ${}_nc_0^F(M)$ and ${}_nc^F(M)$, we give the proof only for the set ${}_n\ell_\infty^F(M)$. Let $(a_{m_1m_2\ldots m_n}) \in {}_n\ell_\infty^F(M)$. By Lemma 2.1, there exists a $K > 0$ such that $D\left(a_{m_1m_2\ldots m_n}, \overline{0}\right) \le K$ for all $m_1, m_2, \ldots, m_n \in N$. Now we have

$$\sum_{m_1}\sum_{m_2}\cdots\sum_{m_n}\left[D\left(a_{m_1m_2\ldots m_n}b_{m_1m_2\ldots m_n}, \overline{0}\right)\right]^r \le$$

$$\sum_{m_1}\sum_{m_2}\cdots\sum_{m_n}\left[D\left(a_{m_1m_2\ldots m_n}, \overline{0}\right)\right]^r\left[D\left(b_{m_1m_2\ldots m_n}, \overline{0}\right)\right]^r \le K^r\sum_{m_1}\sum_{m_2}\cdots\sum_{m_n}\left[D\left(b_{m_1m_2\ldots m_n}, \overline{0}\right)\right]^r$$

If $(b_{m_1m_2\ldots m_n}) \in {}_n\ell_r^F$, then from the above inequality we have $(b_{m_1m_2\ldots m_n}) \in \left\{{}_n\ell_\infty^F(M)\right\}^\eta$. Hence ${}_n\ell_r^F \subseteq \left\{{}_n\ell_\infty^F(M)\right\}^\eta$. Conversely let $(b_{m_1m_2\ldots m_n}) \in \left\{{}_n\ell_\infty^F(M)\right\}^\eta$. If we consider the n-sequence $(a_{m_1m_2\ldots m_n})$, where $a_{m_1m_2\ldots m_n} = \overline{1}$, for all $m_1, m_2, \ldots, m_n \in N$. Then

$$\sum_{m_1}\sum_{m_2}\cdots\sum_{m_n}\left[D\left(b_{m_1m_2\ldots m_n}, \overline{0}\right)\right]^r = \sum_{m_1}\sum_{m_2}\cdots\sum_{m_n}\left[D\left(a_{m_1m_2\ldots m_n}b_{m_1m_2\ldots m_n}, \overline{0}\right)\right]^r < \infty$$

Thus $(b_{m_1m_2\ldots m_n}) \in {}_n\ell_r^F$.. Hence $\left\{{}_n\ell_\infty^F(M)\right\}^\eta \subseteq {}_n\ell_r^F$. This completes the proof of the Theorem. □

Taking $M(x) = x$ for all $x \in [0,1]$ in Theorem 2.2, we get the results for the sets ${}_nc_0^F$, ${}_nc^F$ and ${}_n\ell_\infty^F$.

Corollary 2.3. *The $\eta-$dual of the sets ${}_nc_0^F$, ${}_nc^F$ and ${}_n\ell_\infty^F$ of n-sequences of fuzzy numbers is the set ${}_n\ell_r^F$.*

Theorem 2.4. *The $\eta-$dual of the sets ${}_n\sigma^F$ of n-sequences of fuzzy numbers is the set ${}_n\ell_r^F$.*

Proof. We have ${}_n\sigma^F \subseteq {}_n\ell_\infty^F$. Hence ${}_n\ell_r^F = \left\{{}_n\ell_\infty^F\right\}^\eta \subseteq \left\{{}_n\sigma^F\right\}^\eta$. For converse part, let $(b_{m_1m_2\ldots m_n}) \in \left\{{}_n\sigma^F\right\}^\eta$. Then

$$\sum_{m_1}\sum_{m_2}\cdots\sum_{m_n}\left[D\left(a_{m_1m_2\ldots m_n}b_{m_1m_2\ldots m_n}, \overline{0}\right)\right]^r < \infty \text{ for all } (a_{m_1m_2\ldots m_n}) \in {}_n\sigma^F.$$

Consider $a_{m_1m_2\ldots m_n} = \overline{1} = -a_{m_1+1,m_2,\ldots,m_n} = \cdots = -a_{m_1,m_2,\ldots,m_n+1}$, for all $m_1, m_2, \ldots, m_n \in N$. Then $(a_{m_1m_2\ldots m_n}) \in {}_n\sigma^F$ and

$$\sum_{m_1}\sum_{m_2}\cdots\sum_{m_n}\left[D\left(b_{m_1m_2\ldots m_n}, \overline{0}\right)\right]^r < \infty$$

This implies that $(b_{m_1m_2\ldots m_n}) \in {}_n\ell_r^F$. Hence $\left\{{}_n\sigma^F\right\}^\eta \subseteq {}_n\ell_r^F$. Thus $\left\{{}_n\sigma^F\right\}^\eta = {}_n\ell_r^F$ □

Theorem 2.5. *Let M be any Orlicz function. Then the $\eta-$dual of the set ${}_nw_p^F(M) \cap {}_n\ell_\infty^F(M)$ of n-sequences of fuzzy numbers is the set ${}_n\ell_r^F$.*

Proof. Since ${}_nw_p^F(M) \cap {}_n\ell_\infty^F(M) \subset {}_n\ell_\infty^F(M)$, we have ${}_n\ell_r^F \subseteq \left\{{}_nw_p^F(M) \cap {}_n\ell_\infty^F(M)\right\}^\eta$. Conversely, let $(a_{m_1m_2\ldots m_n}) \notin {}_n\ell_r^F$, then we can write

$$\sum_{m_1}\sum_{m_2}\cdots\sum_{m_n}\left[D\left(a_{m_1m_2\ldots m_n}, \overline{0}\right)\right]^r = \infty.$$

Consider the n-sequence $(b_{m_1m_2\ldots m_n})$, defined by $b_{m_1m_2\ldots m_n} = \overline{1}$ for all $m_1, m_2, \ldots, m_n \in N$. Then $(b_{m_1m_2\ldots m_n}) \in {}_nw_p^F(M) \cap {}_n\ell_\infty^F(M)$, but

$$\sum_{m_1}\sum_{m_2}\cdots\sum_{m_n}\left[D\left(a_{m_1m_2\ldots m_n}b_{m_1m_2\ldots m_n}, \overline{0}\right)\right]^r = \infty.$$

Hence $(a_{m_1m_2\ldots m_n}) \notin \left\{{}_nw_p^F(M) \cap {}_n\ell_\infty^F(M)\right\}^\eta$. It follows that $\left\{{}_nw_p^F(M) \cap {}_n\ell_\infty^F(M)\right\}^\eta \subseteq {}_n\ell_r^F$. Thus $\left\{{}_nw_p^F(M) \cap {}_n\ell_\infty^F(M)\right\}^\eta = {}_n\ell_r^F$. □

Corollary 2.6. *The $\eta-$dual of the set ${}_nw_p^F \cap {}_n\ell_\infty^F$ of n-sequences of fuzzy numbers is the set ${}_n\ell_r^F$.*

Taking $r = 1$ we get the following Corollary.

Corollary 2.7. *Let M be any Orlicz function. Then the $\alpha-$dual of the sets ${}_nc_0^F(M)$, ${}_nc^F(M)$, ${}_n\ell_\infty^F(M)$, ${}_n\sigma^F$ and ${}_nw_p^F(M) \cap {}_n\ell_\infty^F(M)$ of n-sequences of fuzzy numbers is the set ${}_n\ell_1^F$.*

Again taking $M(x) = x$ for all $x \in [0, 1]$ in Corollary 2.7, we get the next result.

Corollary 2.8. *The $\alpha-$dual of the sets ${}_nc_0^F$, ${}_nc^F$, ${}_n\ell_\infty^F$ and ${}_nw_p^F \cap {}_n\ell_\infty^F$ of n-sequences of fuzzy numbers is the set ${}_n\ell_1^F$.*

Definition 2.1. *A set ${}_nE^F \subset {}_nw^F$ is said to be solid if $(b_{m_1m_2\ldots m_n}) \in {}_nE^F$ whenever $D\left(b_{m_1m_2\ldots m_n}, \overline{0}\right) \leq D\left(a_{m_1m_2\ldots m_n}, \overline{0}\right)$ for all $m_1, m_2, \ldots, m_n \in N$ and $(a_{m_1m_2\ldots m_n}) \in {}_nE^F$. Therefore we can conclude that the $\alpha-$, $\beta-$ and $\gamma-$duals of a set of n-sequences of fuzzy numbers are identical if it is solid.*

Proposition 2.9. *The sets ${}_nc_0^F(M)$, ${}_n\ell_\infty^F(M)$ and ${}_nw_p^F(M) \cap {}_n\ell_\infty^F(M)$ are solid.*

Proof. We given the proof only for the set ${}_nc_0^F(M)$ and for other sets it will follow on applying similar arguments. Let $(a_{m_1m_2\ldots m_n}) \in {}_nc_0^F(M)$. Then

$$\lim_{m_1,m_2,\ldots,m_n} M\left(\frac{D\left(a_{m_1m_2\ldots m_n}, \overline{0}\right)}{\rho}\right) = 0, \text{ for some } \rho > 0.$$

Now suppose that for some $(b_{m_1m_2\ldots m_n}) \in {}_nw^F$, we have
$D\left(b_{m_1m_2\ldots m_n}, \overline{0}\right) \leq D\left(a_{m_1m_2\ldots m_n}, \overline{0}\right)$ for all $m_1, m_2, \ldots, m_n \in N$. Since M is non-decreasing, we have

$$\lim_{m_1,m_2,\ldots,m_n} M\left(\frac{D\left(b_{m_1m_2\ldots m_n}, \overline{0}\right)}{\rho}\right) \leq \lim_{m_1,m_2,\ldots,m_n} M\left(\frac{D\left(a_{m_1m_2\ldots m_n}, \overline{0}\right)}{\rho}\right) = 0.$$

It follows that $(b_{m_1m_2\ldots m_n}) \in {}_nc_0^F(M)$. Thus ${}_nc_0^F(M)$ is solid. □

Hence we have the next Proposition.

Proposition 2.10. *Let M be any Orlicz function. Then the $\beta-$ and $\gamma-$duals of the sets ${}_nc_0^F(M)$, ${}_n\ell_\infty^F(M)$ and ${}_nw_p^F(M) \cap {}_n\ell_\infty^F(M)$ of n-sequences of fuzzy numbers is the set ${}_n\ell_1^F$.*

Corollary 2.11. *The $\beta-$ and $\gamma-$duals of the sets ${}_nc_0^F$, ${}_n\ell_\infty^F$ and ${}_nw_p^F \cap {}_n\ell_\infty^F$ of n-sequences of fuzzy numbers is the set ${}_n\ell_1^F$.*

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