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# ON A NEW SUBFAMILIES OF ANALYTIC AND UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENT WITH RESPECT TO OTHER POINTS

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OLATUNJI S.O. AND OLADIPO A. T.

Abstract. In this work, the authors introduced new subfamilies of $\omega-starlike$ and $\omega-convex$ functions with negative coefficient with respect to other points. The coefficient estimates for these classes are obtained. Also relevant connection to classical Fekete-Zsego theorem is briefly discussed.

## 1. Introduction

In the recent time, precisely in 1999, Kanas and Ronning [3] introduced a new concept of analytic functions denoted by $A(\omega)$ and of the form

$$f(z) = (z-\omega) + \sum_{k=2}^{\infty} a_k (z-\omega)^k \tag{1.1}$$

which are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$ and normalized by $f(\omega) = 0$ and $f'(\omega) - 1 = 0$ and $\omega$ is a fixed point in $U$. Also they denoted by $S(\omega)$ a subclass of $A(\omega)$ the class of functions analytic and univalent. They use (1.1) to define the following classes

$$ST(\omega) = S^*(\omega) = \left\{ f(z) \in S(\omega) : Re\frac{(z-\omega)f'(z)}{f(z)} > 0, z \in U \right\}$$

$$CV(\omega) = S^c(\omega) = \left\{ f(z) \in S(\omega) : 1 + Re\frac{(z-\omega)f'(z)}{f'(z)} > 0, z \in U \right\}$$

and $\omega$ is a fixed point in $U$. The above two classes are known as $\omega - starlike$ and $\omega - convex$ functions. Several other authors, the likes of Acu and Owa [1], Oladipo [4], Oladipo and Breaz [5] has worked on these classes, and they view them from

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different perspective and they obtained many interesting result.
Let $H(\omega)$ be the subfamily of $S(\omega)$ and of the form

$$f(z) = (z-\omega) - \sum_{k=2}^{\infty} a_k (z-\omega)^k \tag{1.2}$$

which are analytic and normalized as in the above.
Let $f$ be defined by (1.2) and $f \in H(\omega)$ satisfy

$$Re \frac{(z-\omega)f'(z)}{f(z)} > 0$$

then $f(z) \in T^*(\omega)$ where $T^*(\omega)$ is a subfamily of $S^*(\omega)$ and $\omega$ is a fixed point in $U$.
Also, let $f$ be defined as in (2) and $f \in H(\omega)$ satisfy

$$Re \left\{ 1 + \frac{(z-\omega)f'(z)}{f'(z)} \right\} > 0$$

then $f \in K^c(\omega)$ and $K^c(\omega)$ is a subfamily of $S^c(\omega)$ and $\omega$ is a fixed point in $U$. The classes are respectively subfamilies of $\omega - starlike$ and $\omega - convex$.
The authors here wish to give the following preliminaries which shall be well dealt with in our subsequent section.
We let $T_s^*(\omega)$ be the subclass of $S$ consisting

$$Re \left\{ \frac{(z-\omega)f'(z)}{f(z) - f(-z)} \right\} > 0, z \in U.$$

We shall referred to this class of functions as $\omega - starlike$ with respect to symmetric points.
Also, $T_c^*(\omega)$ consisting of functions $\omega - starlike$ with respect to conjugate points. The class $T_c^*(\omega)$ a subclass $S(\omega)$ consisting of functions $f$ defined by (1.2) satisfying the condition

$$Re \left\{ \frac{(z-\omega)f'(z)}{f(z) + \overline{f(\overline{z})}} \right\} > 0, z \in U.$$

and $\omega$ is a fixed point in $U$.
Furthermore, we let $K_s^c(\omega)$ be the subclass of $S(\omega)$ consisting of functions given by (1.2) satisfying the condition

$$Re \left\{ \frac{((z-\omega)f'(z))'}{(f(z) - f(-z))'} \right\} > 0, \quad z \in U$$

and $\omega$ is a fixed point in $U$. This is the class $\omega - convex$ with respect to symmetric point.
Moreover, in term of subordination, we recalled that in 1982 Goel and Mehrok [2], C. Selvaraj and N. Vasanthi [6] introduced a subclasses of $S_s^*$ denoted by $S_s^*(A, B)$ and $f$ is of the form (1.1). We shall employ the analogue of their definition. That is, we let $T_s^*(\omega, A, B)$ be the class of functions $f$ of the form (1.2) and satisfying the condition

$$\frac{2(z-\omega)f'(z)}{f(z) - f(-z)} \prec \frac{1 + A(z-\omega)}{1 + B(z-\omega)}, -1 \leq B < A \leq 1, z \in U$$

Also, we let $T_c^*(\omega, A, B)$ be the class of functions of the form (1.2) and satisfying

$$\frac{2\left((z-\omega)f'(z)\right)'}{\left(f(z)+\overline{f(\overline{z})}\right)} \prec \frac{1+A(z-\omega)}{1+B(z-\omega)}, -1 \leq B < A \leq 1, z \in U,$$

Let $K_s^c(\omega, A, B)$ be the class of functions of the form (1.2) and satisfying the condition

$$\frac{2\left((z-\omega)f'(z)\right)'}{\left(f(z)-f(-z)\right)'} \prec \frac{1+A(z-\omega)}{1+B(z-\omega)}, -1 \leq B < A \leq 1, z \in U,$$

Also, we let $K_c^c(\omega, A, B)$ be the class of functions of the form (1.2) and satisfying the condition

$$\frac{2\left((z-\omega)f'(z)\right)'}{\left(f(z)+\overline{f(\overline{z})}\right)'} \prec \frac{1+A(z-\omega)}{1+B(z-\omega)}, -1 \leq B < A \leq 1, z \in U,$$

and $\omega$ is a fixed point in $U$.
In this paper, the authors introduced the class $\Phi_s(\omega, \alpha, A, B)$ consisting of analytic functions $f$ of the form (1.2) and satisfying

$$\frac{2(z-\omega)f'(z)+2\alpha(z-\omega)^2 f''(z)}{(1-\alpha)\left(f(z)-f(-z)\right)+\alpha(z-\omega)\left(f(z)-f(-z)\right)'} \prec \frac{1+A(z-\omega)}{1+B(z-\omega)} \tag{1.3}$$

$-1 \leq B < A \leq 1$, $0 \leq \alpha \leq 1$, $z \in U$, and $\omega$ is a fixed point in $U$.
Also we introduce the class $\Phi_c(\omega, \alpha, A, B)$ consisting of analytic functions $f$ of the form (1.2) and satisfying

$$\frac{2(z-\omega)f'(z)+2\alpha(z-\omega)^2 f''(z)}{(1-\alpha)\left(f(z)+\overline{f(\overline{z})}\right)+\alpha(z-\omega)\left(f(z)+\overline{f(\overline{z})}\right)'} \prec \frac{1+A(z-\omega)}{1+B(z-\omega)} \tag{1.4}$$

$-1 \leq B < A \leq 1$, $0 \leq \alpha \leq 1$, $z \in U$,and $\omega$ is a fixed point in $U$.
By the definition of subordination it follows that $f \in \Phi_s(\omega, \alpha, A, B)$ if and only if

$$\frac{2(z-\omega)f'(z)+2\alpha(z-\omega)^2 f''(z)}{(1-\alpha)\left(f(z)-f(-z)\right)+\alpha(z-\omega)\left(f(z)-f(-z)\right)'} = \frac{1+Ah(z)}{1+Bh(z)} = p(z) \tag{1.5}$$

$h \in U$ and $h$ is of the form

$$h(\omega) = (z-\omega) + \sum_{k=2}^{\infty} b_k (z-\omega)^k$$

$h(\omega) = 0$ and $|h(\omega)| < 1$, $h$ is analytic and univalent and that $f \in \Phi_c(\omega, \alpha, A, B)$ if and only if

$$\frac{2(z-\omega)f'(z)+2\alpha(z-\omega)^2 f''(z)}{(1-\alpha)\left(f(z)+\overline{f(\overline{z})}\right)+\alpha(z-\omega)\left(f(z)+\overline{f(\overline{z})}\right)'} = \frac{1+Ah(z)}{1+Bh(z)} = p(z) \tag{1.6}$$

where $p(z)$ in our case is given as

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k (z-\omega)^k$$

and

$$|p_k| \leq \frac{(A-B)}{(1+d)(1-d)^k}, k \geq 1, |\omega| = d \tag{1.7}$$

In the next section, we study the classes $\Phi_s(\omega,\alpha,A,B)$ and $\Phi_c(\omega,\alpha,A,B)$, the coefficient estimates for functions belonging to these classes are obtained

## 2. Coefficient estimates

**Theorem 2.1:** Let $f\in\Phi_s(\omega,\alpha,A,B)$. Then for $k=2,3,4,5.,\ 0\le\alpha\le 1$

$$|a_2|\le -\frac{A-B}{2(1+\alpha)(1-d^2)}$$

$$|a_3|\le -\frac{A-B}{2(1+2\alpha)(1-d^2)(1-d)} \tag{2.1}$$

$$|a_4|\le -\frac{(A-B)\left[A-B+2(1+d)\right]}{2.4(1+3\alpha)(1-d^2)^2(1-d)}$$

$$|a_5|\le -\frac{(A-B)\left[A-B+2(1+d)\right]}{2.4(1+4\alpha)(1-d^2)^2(1-d)^2}$$

**Proof:**From (1.5) and (1.7), we have

$$\Big[(z-\omega)-2a_2(z-\omega)^2-3a_3(z-\omega)^3-4a_4(z-\omega)^4-5a_5(z-\omega)^5- 6a_6(z-\omega)^6-7a_7(z-\omega)^7-...\Big]$$
$$+\alpha\Big[-2a_2(z-\omega)^2-6a_3(z-\omega)^3-12a_4(z-\omega)^4-20a_5(z-\omega)^5- 30a_6(z-\omega)^6-42a_7(z-\omega)^7-...\Big]=$$
$$(1-\alpha)\Big[(z-\omega)-a_3(z-\omega)^3-a_5(z-\omega)^5-a_7(z-\omega)^7-...\Big]\Big(1+p_1(z-\omega)+ p_2(z-\omega)^2+p_3(z-\omega)^3+p_4(z-\omega)^4+...\Big)$$
$$+\alpha\Big[(z-\omega)-3a_3(z-\omega)^3-5a_5(z-\omega)^5-7a_7(z-\omega)^7-...\Big]\Big(1+p_1(z-\omega)+ p_2(z-\omega)^2+p_3(z-\omega)^3+p_4(z-\omega)^4+...\Big)$$

Equating the coefficient of the like powers of $(z-\omega)$, we have
$-2(1+\alpha)a_2=p_1$
$-2(1+2\alpha)a_3=p_2$
$-4(1+3\alpha)a_4=p_3-(1+2\alpha)p_1a_3$
$-4(1+4\alpha)a_5=p_4-(1+2\alpha)p_2a_3$
Using (1.7) on the above we have

$$|a_2|\le -\frac{A-B}{2(1+\alpha)(1-d^2)}$$

$$|a_3|\le -\frac{A-B}{2(1+2\alpha)(1-d^2)(1-d)}$$

$$|a_4|\le -\frac{(A-B)\left[A-B+2(1+d)\right]}{2.4(1+3\alpha)(1-d^2)^2(1-d)}$$

$$|a_5| \leq -\frac{(A-B)\left[A-B+2(1+d)\right]}{2.4(1+4\alpha)(1-d^2)^2(1-d)^2}$$

and this complete the proof of Theorem 2.1.
If we set $d=0$ in Theorem 2.1 we have

**Corollary 2.1.** *Let $f \in \Phi_s(\omega, \alpha, A, B)$. Then for $k = 2, 3, 4, 5.$, $0 \leq \alpha \leq 1$*

$$|a_2| \leq -\frac{A-B}{2(1+\alpha)}$$

$$|a_3| \leq -\frac{A-B}{2(1+2\alpha)}$$

$$|a_4| \leq -\frac{(A-B)\left[A-B+2\right]}{2.4(1+3\alpha)}$$

$$|a_5| \leq -\frac{(A-B)\left[A-B+2\right]}{2.4(1+4\alpha)}$$

If we set $\alpha = 1$ in corollary A, we have

**Corollary 2.2.** *Let $f \in \Phi_s(\omega, \alpha, A, B)$. Then for $k = 2, 3, 4, 5.$, $0 \leq \alpha \leq 1$*

$$|a_2| \leq -\frac{A-B}{2.2}$$

$$|a_3| \leq -\frac{A-B}{2.3}$$

$$|a_4| \leq -\frac{(A-B)\left[A-B+2\right]}{2.4.4}$$

$$|a_5| \leq -\frac{(A-B)\left[A-B+2\right]}{2.4.5}$$

**Theorem 2.2:**Let $f \in \Phi_c(\omega, \alpha, A, B)$. Then for $k = 2, 3, 4, 5.$, $0 \leq \alpha \leq 1$

$$|a_2| \leq -\frac{A-B}{(1+\alpha)(1-d^2)}$$

$$|a_3| \leq -\frac{(A-B)\left[A-B+(1+d)\right]}{2(1+2\alpha)(1-d^2)^2} \tag{2.2}$$

$$|a_4| \leq -\frac{(A-B)\left[(A-B)^2+3(A-B)(1+d)+2(1+d)^2\right]}{2.3(1+3\alpha)(1-d^2)^3}$$

$$|a_5| \leq -\frac{(A-B)\left[(A-B)^3+6(1+d)(A-B)^2+11(1+d)^2(A-B)+6(1+d)^3\right]}{2.3.4(1+4\alpha)(1-d^2)^4}$$

**Proof:**
From (1.6) and (1.7), we have

$$\Big[(z-\omega)-2a_2(z-\omega)^2-3a_3(z-\omega)^3-4a_4(z-\omega)^4-5a_5(z-\omega)^5-$$
$$6a_6(z-\omega)^6-7a_7(z-\omega)^7-...\Big]$$
$$+\alpha\Big[-2a_2(z-\omega)^2-6a_3(z-\omega)^3-12a_4(z-\omega)^4-20a_5(z-\omega)^5-$$
$$30a_6(z-\omega)^6-42a_7(z-\omega)^7-...\Big]=$$
$$(1-\alpha)\Big[(z-\omega)-a_2(z-\omega)^2-a_3(z-\omega)^3a_4(z-\omega)^4-$$
$$a_5(z-\omega)^5-a_6(z-\omega)^6-a_7(z-\omega)^7-...\Big]$$
$$\Big(1+p_1(z-\omega)+p_2(z-\omega)^2+p_3(z-\omega)^3+p_4(z-\omega)^4+...\Big)$$
$$+\alpha\Big[(z-\omega)-2a_2(z-\omega)^2-3a_3(z-\omega)^3-4a_4(z-\omega)^4-$$
$$5a_5(z-\omega)^5-6a_6(z-\omega)^6-7a_7(z-\omega)^7-...\Big]$$
$$\Big(1+p_1(z-\omega)+p_2(z-\omega)^2+p_3(z-\omega)^3+p_4(z-\omega)^4+...\Big)$$

Equating the coefficient of the like powers of $(z-\omega)$, we have $-(1+\alpha)a_2=p_1$
$-2(1+2\alpha)a_3=p_2-(1+\alpha)a_2p_1$
$-3(1+3\alpha)a_4=p_3-(1+\alpha)a_2p_2-(1+2\alpha)a_3p_1$
$-4(1+4\alpha)a_5=p_4-(1+\alpha)a_2p_3-(1+2\alpha)a_3p_2-(1+3\alpha)a_4p_1$
using (1.7) above we have

$$|a_2|\le-\frac{A-B}{(1+\alpha)(1-d^2)}$$

$$|a_3|\le-\frac{(A-B)\left[A-B+(1+d)\right]}{2(1+2\alpha)(1-d^2)^2}$$

$$|a_4|\le-\frac{(A-B)\left[(A-B)^2+3(A-B)(1+d)+2(1+d)^2\right]}{2.3(1+3\alpha)(1-d^2)^3}$$

$$|a_5|\le-\frac{(A-B)\left[(A-B)^3+6(1+d)(A-B)^2+11(1+d)^2(A-B)+6(1+d)^3\right]}{2.3.4(1+4\alpha)(1-d^2)^4}$$

If we set $d=0$ in Theorem 2.2, we have

**Corollary 2.3.** *Let $f\in\Phi_c(\omega,\alpha,A,B)$. Then for $k=2,3,4,5.$, $0\le\alpha\le1$*

$$|a_2|\le-\frac{A-B}{(1+\alpha)}$$

$$|a_3|\le-\frac{(A-B)\left[A-B+1)\right]}{2(1+2\alpha)}$$

$$|a_4|\le-\frac{(A-B)\left[(A-B)^2+3(A-B)+2\right]}{2.3(1+3\alpha)}$$

$$|a_5| \leq -\frac{(A-B)\left[(A-B)^3+6(A-B)^2+11(A-B)+6\right]}{2.3.4(1+4\alpha)}$$

If we set $\alpha = 1$ in Theorem 2.2, we have

**Corollary 2.4.** *Let $f \in \Phi_c(\omega, \alpha, A, B)$. Then for $k = 2, 3, 4, 5.$, $0 \leq \alpha \leq 1$*

$$|a_2| \leq -\frac{A-B}{2}$$

$$|a_3| \leq -\frac{(A-B)\left[A-B+1)\right]}{2.3}$$

$$|a_4| \leq -\frac{(A-B)\left[(A-B)^2+3(A-B)+2\right]}{2.3.4}$$

$$|a_5| \leq -\frac{(A-B)\left[(A-B)^3+6(A-B)^2+11(A-B)+6\right]}{2.3.4.5}$$

Our next results are the relevant connection of our classes to the classical Fekete-Zsego Theorem.

**Theorem 2.3:** Let $f \in \Phi_s(\omega, \alpha, A, B)$. Then

$$\left|a_3-\mu a_2^2\right| \leq \frac{-(A-B)(1-d)\left[2(1+\alpha)^2(1+d)+\mu(A-B)(2\alpha+1)\right]}{4(1+\alpha)^2(1+2\alpha)(1-d^2)^2(1-d)}, \quad \mu \leq 0 \tag{2.3}$$

$$\left|a_2a_4-a_3^2\right| \leq \frac{(A-B)^2(1-d)\left[(1+2\alpha)^2\left((A-B)+2(1+d)\right)-4(1+d)(1+\alpha)(1+3\alpha)\right]}{16(1+\alpha)(1+2\alpha)^2(1+3\alpha)(1-d)^2(1-d^2)^3} \tag{2.4}$$

**Proof:** The proof follows from Theorem 2.1. With various choices of the parameters involved, various connection of our class to the classical Fekete-Zsego Theorem could be obtained.

**Theorem 2.4:** Let $f \in \Phi_c(\omega, \alpha, A, B)$. Then

$$\left|a_3-\mu a_2^2\right| \leq -\frac{(A-B)\left\{\alpha^2\left((A-B)+(1+d)\right)+(2\alpha+1)\left[(A-B)(2\mu+1)+(1+d)\right]\right\}}{2(1+\alpha)^2(1+2\alpha)(1-d^2)^2} \tag{2.5}$$

$$\left|a_2a_4-a_3^2\right| \leq \frac{(A-B)^2\left[(A-B)^2+3(A-B)(1+d)+2(1+d)^2\right]}{6(1+\alpha)(1+3\alpha)(1-d^2)^4}-\frac{(A-B)^2\left[(A-B)^2+(1+d)^2\right]}{4(1+2\alpha)^2(1-d^2)^4} \tag{2.6}$$

**Proof:** Also, the proof follows from Theorem 2.2.

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Department of Pure and Applied Mathematics,, Ladoke Akintola University of Technology, Ogbomoso, P. M. B. 4000, Ogbomoso, Nigeria.
*E-mail address*: olatfem_80@yahoo.com; atlab_3@yahoo.com