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SPECTRUM OF A kth-ORDER SLANT HANKEL OPERATOR

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Abstract. For an integer $k \geq 2$, a kth-order slant Hankel operator D_ϕ with symbol ϕ in L^∞ is an operator on L^2 whose representing matrix is $(a_{-ki-j})_{i,j=-\infty}^{\infty}$, where a_t's are the Fourier coefficients of ϕ. In this paper, we discuss some spectral properties of the operator D_ϕ and identify its spectrum and essential spectrum.

1. Introduction

Ever since the introduction of the class of Hankel operators in the middle of twentieth century, it has assumed tremendous importance due to its far reaching applications to problems of rational approximation, information and control theory, interpolation and prediction problems etc. Hankel operators and their matrix variants have occured in realization problem for certain discrete time linear systems and in determining which systems are exactly controllable [10].

Various generalizations of the notion of Hankel operators have been the subject of research of many mathematicians. R.A.M. Avendano [4], [6] initiated the notion of λ-Hankel operators in the year 2000. Later in the year 2002, motivated by the work of Barria and Halmos, Avendano [5] brought the concept of essentially Hankel operators into picture. S.C. Arora [1], along with his research associates, discussed a new variant of the class of Hankel operators and introduced the class of slant Hankel operators. Subsequently, the notion of slant Hankel operators was generalized to kth-order slant Hankel operators ($k \geq 2, k$ an integer) by Arora and Bhola [2], [3].

In this paper, we carry this study further and investigate some spectral properties of kth-order slant Hankel operators on the space $L^2(\mathbb{T})$, $\mathbb{T}$ denoting the unit circle. We begin with the following:

Let $\phi(z) = \sum\limits_{i=-\infty}^{\infty} a_i z^i$ be a bounded measurable function on the unit circle $\mathbb{T}$, where $a_i = \langle \phi, z^i \rangle$ is the ith Fourier coefficient of ϕ and $\{z^i : i \in \mathbb{Z}\}$ is the standard basis of the space $L^2(\mathbb{T})$.

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For any integer $k \geq 2$, let W_k be an operator on L^2 such that

$$W_k(z^i) = \begin{cases} z^{i/k}, & \text{if } i \text{ is divisible by } k \\ 0 & \text{otherwise} \end{cases}$$

In particular for $k = 2$,the operator W_2 denoted by W is

$$W(z^{2i}) = z^i$$
$$W(z^{2i-1}) = 0$$

for all i in $\mathbb{Z}$. Let J denote the reflection operator on L^2, that is $J : L^2 \to L^2$, $J(f(z)) = f(\bar{z})$. It is easy to see the following:

(i) $\|W_k\| = 1$, $\|J\| = 1$
(ii) $JW_k = W_k J$.
(iii) $J = J^*$, $J^2 = I$
(iv) $W_k^* z^i = z^{ki}$, for all i in $\mathbb{Z}$.
(v) $W_k W_k^* = I$ and $W_k^* W_k = P_k$, where P_k denotes the projection of L^2 onto the closed linear span of $\{z^{ki} : i \in \mathbb{Z}\}$

Definition 1 ([1])**.** A slant Hankel operator K_ϕ induced by ϕ in L^∞ on the space L^2 is defined as

$$K_\phi = JWM_\phi$$

where J and $W = W_2$ are as defined above and M_ϕ denotes the multiplication operator induced by ϕ in L^∞ on the space L^2.

It is known that [1] an operator A on L^2 is a slant Hankel operator if and only if $M_{\bar{z}} A = A M_{z^2}$.

Arora and Bhola [2] generalized the notion of slant Hankel operators to kth-order slant Hankel operators as follows:

Definition 2 ([2])**.** For an integer $k \geq 2$, a kth-order slant Hankel operator D_ϕ on the space L^2 is defined as

$$D_\phi = JW_k M_\phi$$

where J and W_k are as defined above and M_ϕ is the multiplication operator on L^2 induced by ϕ in L^∞. It is clear that a slant Hankel operator is a particular case of a kth-order slant Hankel operator (for $k = 2$).

For $\phi(z) = \sum_{i=-\infty}^{\infty} a_i z^i \in L^\infty$, the matrix representation of D_ϕ with respect to the standard basis $\{z^i : i \in \mathbb{Z}\}$ of L^2 is given by:

$$\left[\begin{array}{cc|cccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & a_{2k+1} & a_{2k} & a_{2k-1} & a_{2k-2} & \cdots & a_k & \cdots \\
\cdots & a_{k+1} & a_k & a_{k-1} & a_{k-2} & \cdots & a_0 & \cdots \\
\hline
\cdots & a_1 & a_0 & a_{-1} & a_{-2} & \cdots & a_{-k} & \cdots \\
\cdots & a_{-k+1} & a_{-k} & a_{-k-1} & a_{-k-2} & \cdots & a_{-2k} & \cdots \\
\cdots & a_{-2k+1} & a_{-2k} & a_{-2k-1} & a_{-2k-2} & \cdots & a_{-3k} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}\right]$$

So,

$$\begin{aligned}\langle D_\phi z^j, z^i\rangle &= \langle JW_kM_\phi z^j, z^i\rangle\\ &= \langle W_kM_\phi z^j, z^{-i}\rangle\\ &= \langle \phi, z^{-ki-j}\rangle\\ &= a_{-ki-j}\end{aligned}$$

Thus, $D_\phi : L^2 \to L^2$ is defined as

$$D_\phi(z^j) = \sum_{t=-\infty}^{\infty} a_{-kt-j} z^t, \text{ for all } \ j \text{ in } \ \mathbb{Z}.$$

For a fixed integer $k \geq 2$, the set of all kth-order slant Hankel operators is denoted by $\mathcal{D}_k$[2]. It is proved that [2] a bounded linear operator D on the space L^2 is a kth-order slant Hankel operator if and only if $M_{\bar{z}}D = DM_{z^k}$. Also it is shown that the set $\mathcal{D}_k$ is neither an algebra, nor a self adjoint set and contains no non-zero hyponormal operator, no non-zero compact operator and no isometry.In continuation we also observe here that D_ϕ in $\mathcal{D}_k$ is a coisometry if and only if

$$\left|\phi\left(\frac{\theta}{k}\right)\right|^2 + \left|\phi\left(\frac{\theta+2\pi}{k}\right)\right|^2 + \ldots + \left|\phi\left(\frac{\theta+(k-1)2\pi}{k}\right)\right|^2 = k.$$

For if f is in L^2, then

$$\begin{aligned}\|D_\phi^* f\|_2^2 &= \|M_{\bar{\phi}}W_k^*Jf\|_2^2\\ &= \int_0^{2\pi} |\phi(\theta)|^2|f(-k\theta)|^2\frac{d\theta}{2\pi}\\ &= \int_0^{2k\pi}\left|\phi\left(\frac{\theta}{k}\right)\right|^2|f(-\theta)|^2\frac{d\theta}{2k\pi}\\ &= \frac{1}{k}\int_0^{2\pi}\left\{\left|\phi\left(\frac{\theta}{k}\right)\right|^2+\left|\phi\left(\frac{\theta+2\pi}{k}\right)\right|^2+\ldots\right.\\ &\qquad \left. +\left|\phi\left(\frac{\theta+(k-1)2\pi}{k}\right)\right|^2\right\}|f(-\theta)|^2\frac{d\theta}{2\pi}\\ &= \|M_\psi \hat{f}\|_2^2\end{aligned}$$

where

$$\psi = \left[\frac{1}{k}\left\{\left|\phi\left(\frac{\theta}{k}\right)\right|^2 + \left|\phi\left(\frac{\theta+2\pi}{k}\right)\right|^2 + \ldots + \left|\phi\left(\frac{\theta+(k-1)2\pi}{k}\right)\right|^2\right\}\right]^{\frac{1}{2}}$$

and $\hat{f} = Jf$. Since for any f in $L^2, \|M_\psi f\|_2 = \|f\|_2$ if and only if $|\psi| = 1$ a.e on $\mathbb{T}$, it follows that D_ϕ is a coisometry iff

$$\left|\phi\left(\frac{\theta}{k}\right)\right|^2 + \left|\phi\left(\frac{\theta+2\pi}{k}\right)\right|^2 + \ldots + \left|\phi\left(\frac{\theta+(k-1)2\pi}{k}\right)\right|^2 = k$$

So far, only the algebraic properties and some topological properties of the set $\mathcal{D}_k$ have been studied. In this paper we intend to discuss the spectral properties of the operator D_ϕ and determine the spectrum and the essential spectrum of the operator D_ϕ for some special symbols ϕ in L^∞. To determine the desired ones we firstly make the following observations:

Observation 1: Any operator unitarily equivalent to a slant Toeplitz operator is also a slant Toeplitz operator.

If T is a slant Toeplitz operator, then $M_zT = TM_{z^2}$. Now, for any unitary operator U on L^2, we have

$$\begin{aligned}(U^*M_zU)(U^*TU) &= U^*M_zTU \\ &= U^*TM_{z^2}U \\ &= (U^*TU)(U^*M_zU)^2.\end{aligned}$$

Since any operator unitarily equivalent to a shift operator is again a shift operator of the same multiplicity, it follows that U^*TU is also a slant Toeplitz operator. Observation 2: A non-zero slant Toeplitz operator cannot be a slant Hankel operator.

For if, A is a slant Toeplitz operator on L^2 which is also slant Hankel, then the matrix (α_{ij}) of A with respect to the standard orthonormal basis satisfies

$$\begin{aligned}\alpha_{ij} &= \alpha_{i+1,j+2} \\ \alpha_{ij} &= \alpha_{i-1,j+2}\end{aligned}$$

for all $i, j \in \mathbb{Z}$. As A is a bounded operator on L^2, we have $\alpha_{ij} = 0$ for all $i, j \in \mathbb{Z}$.

From the above two observations, it follows that a non-zero slant Toeplitz operator cannot be unitarily equivalent to a slant Hankel operator.

2. Spectrum of D_ϕ

It is interesting to note that though there is no unitary equivalence between a slant Toeplitz operator and a kth-order slant Hankel operator, yet some of the spectral properties are the same as is shown in this section. In fact here, we aim at proving that if ϕ in L^∞ is invertible then the spectrum of D_ϕ contains a closed disc. More specifically, if ϕ is an inner function then the spectrum of D_ϕ contains the closed unit disc. To accomplish the task we begin with the following:

Lemma 3. *For an invertible ϕ in $L^\infty, \sigma_p(D_\phi) = \sigma_p(D_{\phi(z^k)})$, where $\sigma_p(D_\phi)$ denotes the point spectrum of D_ϕ.*

Proof. Suppose that $\lambda \in \sigma_p(D_\phi)$. Then there exists a non-zero f in L^2 such that $D_\phi f = \lambda f$. Let $F = J(\phi f)$. Then F is in L^2 and

$$\begin{aligned}D_{\phi(z^k)}F &= D_{\phi(z^k)}(J\phi f) \\ &= JW_k(\phi(z^k)J\phi f) \\ &= J\phi(z)\cdot JW_k(\phi f) \\ &= J\phi(z)\cdot D_\phi f \\ &= J\phi(z)\cdot \lambda f \\ &= \lambda J\phi f \\ &= \lambda F.\end{aligned}$$

Therefore, $\lambda \in \sigma_p(D_{\phi(z^k)})$. Conversely, let $\mu \in \sigma_p(D_{\phi(z^k)})$. Then there exists a non-zero g in L^2 satisfying $D_{\phi(z^k)}g = \mu g$. Let $G = \phi^{-1}Jg$. Then G is in L^2 and G

satisfies

$$\begin{aligned} D_\phi G &= JW_kM_\phi(\phi^{-1}Jg) \\ &= JW_kJg \\ &= \phi^{-1}\phi W_kg \\ &= \phi^{-1}W_k(\phi(z^k)g) \\ &= \phi^{-1}JD_{\phi(z^k)}g \\ &= \phi^{-1}J\mu g \\ &= \mu G. \end{aligned}$$

Thus, μ is in $\sigma_p(D_\phi)$. This proves the lemma.

Lemma 4. *For any ϕ in $L^\infty, \sigma(D_\phi) = \sigma(D_{\phi(z^k)})$, where $\sigma(D_\phi)$ denotes the spectrum of D_ϕ.*

Proof. It is known that [7] for any two bounded linear operators A and B on a Hilbert space H,

$$\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}.$$

It follows that

$$\begin{aligned} \sigma(D_\phi^*) \cup \{0\} &= \sigma(M_{\bar\phi}W_k^*J) \cup \{0\} \\ &= \sigma(W_k^*JM_{\bar\phi}) \cup \{0\} \\ &= \sigma(JW_k^*M_{\bar\phi}) \cup \{0\} \\ &= \sigma(J^2U_{\phi(z^k)}^*J) \cup \{0\} \\ &= \sigma(D_{\phi(z^k)}^*) \cup \{0\}. \end{aligned}$$

Here, U_ϕ denotes the operator W_kM_ϕ on the space L^2. From this we get,

$$\sigma(D_\phi) \cup \{0\} = \sigma\overline{(D_\phi^*)} \cup \{0\} = \sigma\overline{(D_{\phi(z^k)}^*)} \cup \{0\} = \sigma(D_{\phi(z^k)}) \cup \{0\}$$

Also we see that the operator JW_k^* is not onto as the range of JW_k^* is the closed linear span of $\{\bar z^{kn} : n \in \mathbb{Z}\}$ in L^2. Thus,

$$cl.\{\text{Range } (JW_k^*M_{\bar\phi})\} \neq L^2.$$

Equivalently, there is a non-zero f in Ker $(W_k^*M_{\bar\phi})$. That is, $M_{\bar\phi(z^k)}W_k^*J(Jf) = 0$, where $0 \neq Jf$ is in L^2. Thus, $0 \in \sigma_p(D_{\phi(z^k)}^*)$ which in turn implies $0 \in \sigma_p(D_{\phi(z^k)})$. Now, if ϕ is invertible, then by Lemma 3, $0 \in \sigma_p(D_\phi)$ and the conclusion is immediate. If ϕ is not invertible then we can have a sequence $\{\phi_n\}$ of invertible functions in L^∞ such that $\|\phi_n - \phi\|_\infty \to 0$ as $n \to \infty$, for the set $\{\phi \in L^\infty : \phi^{-1} \in L^\infty\}$ is dense in L^∞ [8]. Since each ϕ_n is invertible, therefore $0 \in \sigma_p(D_{\phi_n})$ for every n. Hence there exists $f_n \neq 0$, corresponding to each n, such that $D_{\phi_n}f_n = 0$. Without loss of generality we assume that $\|f_n\| = 1$ and we have

$$\begin{aligned} \|D_\phi f_n\| &= \|D_\phi f_n - D_{\phi_n}f_n + D_{\phi_n}f_n\| \\ &\leq \|D_\phi f_n - D_{\phi_n}f_n\| + \|D_{\phi_n}f_n\| \\ &\leq \|\phi - \phi_n\|_\infty \\ &\to 0 \end{aligned}$$

as $n \to \infty$. Therefore, $0 \in \pi(D_\phi)$, the approximate point spectrum of D_ϕ and hence $0 \in \sigma(D_\phi)$. This completes the proof.

Theorem 5. *Let $D_\phi \in \mathcal{D}_k$, where ϕ in L^∞ is invertible. Then $\sigma(D_\phi)$ contains a closed disc. Furthermore, the radius of this closed disc is $\dfrac{1}{r(D_\psi)}$, where $\psi = (J\bar{\phi})^{-1}$ and $r(D_\psi)$ denotes the spectral radius of D_ψ.*

Proof. Let $\lambda \neq 0$ and assume that $D^*_{\phi(z^k)} - \lambda I$ is onto. For any f in L^2 we have:

$$\begin{aligned}
(D^*_{\phi(z^k)} - \lambda)f &= (JW_kM_{\phi(z^k)})^*f - \lambda f \\
&= \bar{\phi}(z^k).W_k^*Jf - \lambda f \\
&= W_k^*(\bar{\phi}Jf) - \lambda f \\
&= W_k^*(\bar{\phi}Jf) - \lambda(P_kf \oplus P_k'f) \\
&= (W_k^*(\bar{\phi}Jf) - \lambda P_kf) \oplus (-\lambda P_k'f) \\
&= \lambda W_k^*M_{\bar{\phi}}(\lambda^{-1} - M_{\bar{\phi}^{-1}}W_k)(Jf) \oplus (-\lambda P_k'f) \\
&= \lambda W_k^*M_{\bar{\phi}}(\lambda^{-1} - J(W_kJM_{\bar{\phi}^{-1}(z^k)})(Jf) \oplus (-\lambda P_k'f)) \\
&= \lambda W_k^*M_{\bar{\phi}}(\lambda^{-1} - JW_kM_{\bar{\phi}^{-1}(z^k)})(Jf) \oplus (-\lambda P_k'f) \\
&= \lambda W_k^*M_{\bar{\phi}}(\lambda^{-1} - D_{\bar{\phi}^{-1}(\bar{z}^k)})(Jf) \oplus (-\lambda P_k'f),
\end{aligned}$$

where P_k denotes the projection of L^2 on the closed linear span of $\{z^{kn} : n \in \mathbb{Z}\}$ and $P_k' = I - P_k$.

Now let $0 \neq g_0 \in P_k'(L^2)$. Since $D^*_{\phi(z^k)} - \lambda I$ is onto, therefore there exists a non-zero f in L^2 such that

$$(D^*_{\phi(z^k)} - \lambda I)f = g_0.$$

This implies that

$$\lambda W_k^*M_{\bar{\phi}}(\lambda^{-1} - D_{\bar{\phi}^{-1}(\bar{z}^k)})(Jf) \oplus (-\lambda P_k'f) = g_0.$$

Since $g_0 \in P_k'(L^2)$, we have

$$\lambda W_k^*M_{\bar{\phi}}(\lambda^{-1} - D_{\bar{\phi}^{-1}(\bar{z}^k)})(Jf) = 0.$$

As $\lambda \neq 0, W_k^*$ is an isometry and $M_{\bar{\phi}}$ is invertible, therefore we have

$$\begin{aligned}
&(\lambda^{-1} - D_{\bar{\phi}^{-1}(\bar{z}^k)})(Jf) = 0 \\
&\Rightarrow \lambda^{-1} \in \sigma_p(D_{\bar{\phi}^{-1}(\bar{z}^k)}).
\end{aligned}$$

Since for any $\lambda \in \rho(D^*_{\phi(z^k)})$, the resolvent set of $D^*_{\phi(z^k)}$, we have $D^*_{\phi(z^k)} - \lambda I$ is invertible and hence onto, above calculations yield

$$\begin{aligned}
\mathfrak{D} = \{\lambda^{-1} : \lambda \in \rho(D^*_{\phi(z^k)})\} &\subseteq \sigma_p(D_{\bar{\phi}^{-1}(\bar{z}^k)}) \\
&= \sigma_p(D_{(J\bar{\phi})^{-1}(z^k)}).
\end{aligned}$$

Replacing $(J\bar{\phi})^{-1}$ by ϕ we get that for any invertible ϕ in L^∞, the spectrum of D_ϕ contains a disc of eigenvalues of D_ϕ. Since the spectrum of any bounded linear operator is compact, $\sigma(D_\phi)$ contains a closed disc. Also,

$$\begin{aligned}\max\{|\lambda^{-1}| : \lambda \in \rho(D^*_{\phi(z^k)})\} &= [\min\{|\lambda| : \lambda \in \rho(D^*_{\phi(z^k)})\}]^{-1} \\ &= [r(D^*_{\phi(z^k)})]^{-1} \\ &= (r(D_{\phi(z^k)}))^{-1} \\ &= [r(D_\phi)]^{-1}\end{aligned}$$

Replacing ϕ by $(J\bar{\phi})^{-1}$, we have that the radius of the closed disc contained in $\sigma(D_\phi)$ is $\dfrac{1}{r(D_\psi)}$, where $\psi = (J\bar{\phi})^{-1}$.

From the proof of the above theorem, it follows that

$$[r(D_\psi)]^{-1} \leq r(D_\phi), \text{ where } \psi = (J\bar{\phi})^{-1}.$$

Now if ϕ in L^∞ is unimodular then

$$\begin{aligned}\|D^n_\phi\|^2 &= \|D^n_\phi D^{*n}_\phi\| \\ &= \|I\| \\ &= 1,\end{aligned}$$

so that $r(D_\phi) = \lim_{n\to\infty} \|D^n_\phi\|^{1/n} = 1$. Moreover, if $|\phi| = 1$ then $|\bar{\phi}^{-1}| = 1$. Therefore in this case

$$[r(D_\psi)]^{-1} \leq 1 \text{ and } 1 \leq [r(D_\psi)]^{-1}.$$

Therefore $\sigma(D_\phi)$ contains the closed unit disc. □

Hence we have the following:

Theorem 6. *If $|\phi| = 1$, then $\sigma(D_\phi) = \bar{\mathfrak{D}}$ where $\bar{\mathfrak{D}}$ denotes the closed unit disc in the complex plane. In particular if ϕ is an inner function then $\sigma(D_\phi)$ contains $\bar{\mathfrak{D}}$.*

Remark 7. From the proof of Theorem 5, it may be seen that if ϕ is invertible in L^∞ then $\sigma(D_\phi)$ contains a disc of eigenvalues of infinite multiplicity. Therefore in this case $\sigma_e(D_\phi)$, the essential spectrum of D_ϕ also contains a closed disc. So we have the following:

(i) For any invertible ϕ in L^∞,

$$\mathfrak{D} \subset \sigma_e(D_\phi) \subset \sigma(D_\phi)$$

(ii) If ϕ is unimodular, then

$$\sigma_e(D_\phi) = \sigma(D_\phi) = \bar{\mathfrak{D}}$$

where $\bar{\mathfrak{D}}$ denotes the closed unit disc in the complex plane.

Theorem 8. *Let $D_\phi \in \mathcal{D}_k$, where ϕ in L^∞ is unimodular. If $|\lambda| < 1$, then λ is an eigenvalue of D_ϕ such that the corresponding eigenspace is generated by*

$$\sum_{n=0}^{\infty} \lambda^n D^{*^n}_\phi f$$

where $f \in \operatorname{span}\{\bar{\phi}\bar{z}^{nk-j} : n \in \mathbb{Z}, j = 1, 2, \ldots, k-1\}$

Proof. Let $\phi \in L^\infty$ be unimodular. Then D_ϕ^* is an isometry. Therefore, by von Neumann-Wold decomposition for isometries [9], we can write

$$D_\phi^* \simeq U \oplus V$$

where U is unitary on $\bigcap_{n=0}^{\infty} D_\phi^{*^n}(L^2)$ and V is unilateral shift on $\bigoplus_{n=0}^{\infty} D_\phi^{*^n}(L^2)$.

Since $D_\phi \simeq U^* \oplus V^*$ and the eigenvalues of V^* are $\{\lambda : |\lambda| < 1\}$, it follows that $\{\lambda : |\lambda| < 1\}$ is a set of eigenvalues of D_ϕ also.

For $|\lambda| < 1$, consider the function $g = \sum_{n=0}^{\infty} \lambda^n D_\phi^{*^n} f$, where $0 \neq f \in (D_\phi^*(L^2))^\perp$. We have,

$$\begin{aligned} |g(z)| &= \left| \sum_{n=0}^{\infty} \lambda^n D_\phi^{*^n} f \right| \\ &\leq \sum_{n=0}^{\infty} |\lambda^n| \|D_\phi^{*^n} f\| \\ &< \infty \end{aligned}$$

for $|\lambda| < 1$ and $\|D_\phi^{*^n} f\| = \|f\| < \infty$.

Also,

$$\begin{aligned} D_\phi g &= D_\phi f + \lambda \sum_{n=1}^{\infty} \lambda^{n-1} D_\phi^{*^{n-1}} f \\ &= 0 + \lambda g \\ &= \lambda g. \end{aligned}$$

Therefore, g is an eigenvector corresponding to the eigenvalue λ of D_ϕ. Further,

$$\begin{aligned} D_\phi^*(L^2) &= (JW_k M_\phi)^*(L^2) \\ &= M_{\bar{\phi}} W_k^* J(L^2) \\ &= \operatorname{span}\{\bar{\phi}\bar{z}^{nk} : n \in \mathbb{Z}\} \end{aligned}$$

If follows that $(D_\phi^*(L^2))^\perp \supseteq$ span $\{\bar{\phi}\bar{z}^{nk-j} : n \in \mathbb{Z}, j = 1, 2, \ldots, k-1\}$ because for any m, n in $\mathbb{Z}$ and $j = 1, 2, \ldots, k-1$,

$$\begin{aligned} \langle \bar{\phi}\bar{z}^{nk}, \bar{\phi}\bar{z}^{mk-j} \rangle &= \langle \phi\bar{\phi}, \bar{z}^{mk-nk-j} \rangle \\ &= \langle 1, \bar{z}^{(m-n)k-j} \rangle \\ &= 0 \end{aligned}$$

for $(m-n)k - j \neq 0$ for any m and n in $\mathbb{Z}$, $j = 1, 2, \ldots, k-1$. On the other hand, since $\bar{\phi}$ is invertible, $\bar{\phi}L^2 = L^2$. Therefore,

$$\operatorname{span}\{\bar{\phi}\bar{z}^{nk} : n \in \mathbb{Z}\} \oplus \operatorname{span}\{\bar{\phi}\bar{z}^{nk-j} : n \in \mathbb{Z}, j = 1, 2, \ldots, k-1\} = \bar{\phi}L^2 = L^2.$$

so that

$$(D_\phi^*(L^2))^\perp \subseteq \operatorname{span}\{\bar{\phi}\bar{z}^{nk-j} : n \in \mathbb{Z}, j = 1, 2, \ldots k-1\}.$$

Hence,

$$(D_\phi^*(L^2))^\perp = \operatorname{span}\{\bar{\phi}\bar{z}^{nk-j} : n \in \mathbb{Z}, j = 1, 2, \ldots k-1\}.$$

It follows that the eigenspace corresponding to the eigenvalue λ (where $|\lambda| < 1$) of D_ϕ is generated by

$$\sum_{n=0}^{\infty} \lambda^n D_\phi^{*^n} f$$

where $0 \neq f \in \text{span}\,\{\bar{\phi}\bar{z}^{nk-j} : n \in \mathbb{Z}, j = 1, 2, \ldots k-1\}$.

This completes the proof.

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