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POSITION VECTORS OF GENERAL HELICES IN EUCLIDEAN 3-SPACE

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Abstract. In this paper, position vector of a general helix with respect to Frenet frame is determined. Besides, we deduce the natural representation of a general helix in terms of the curvature and torsion with respect to standard frame of Euclidean 3-space.

1. Introduction

Helix is one of the most fascinating curves in science and nature. Scientists have long held a fascinating, sometimes bordering on mystical obsession, for helical structures in nature. Helices arise in nano-springs, carbon nano-tubes, α-helices, DNA double and collagen triple helix, lipid bilayers, bacterial flagella in salmonella and escherichia coli, aerial hyphae in actinomycetes, bacterial shape in spirochetes, horns, tendrils, vines, screws, springs, helical staircases and sea shells [6, 11]. Helical structures are used in fractal geometry, for instance hyper-helices [15]. In the field of computer aided design and computer graphics, helices can be used for the tool path description, the simulation of kinematic motion or the design of highways, etc. [16].

From the view of differential geometry, a helix is a geometric curve with non-vanishing constant curvature κ and non-vanishing constant torsion τ [3]. The helix is also known as *circular helix* or *W-curve* which is a special case of the *general helix* [1, 5, 9, 10, 12]. The main feature of general helix is that the tangent makes a constant angle with a fixed straight line which is called the axis of the general helix. A classical result stated by Lancret in 1802 and first proved by de Saint Venant in 1845 says that: *A necessary and sufficient condition that a curve be a general helix is that the ratio*

$$\frac{\kappa}{\tau}$$

is constant along the curve, where κ and τ denote the curvature and the torsion, respectively [14].

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Given two functions of one parameter (potentially curvature $\kappa = \kappa(s)$ and torsion $\tau = \tau(s)$ parameterized by arc-length s) one may desire to find an arc-length parameterized curve for which the two functions work as the curvature and the torsion. This problem, known as *solving natural equations*, is generally achieved by solving a *Riccati equation* [8]. Barros et. al. [2, 4] showed that the general helices in Euclidean 3-space $\mathbf{E}^3$ and in the three-sphere $\mathbf{S}^2$ are geodesic either of right cylinders or of Hopf cylinders according to whether the curve lies in $\mathbf{E}^3$ or $\mathbf{S}^2$, respectively.

In [13] the authors establish a system of differential equations whose solution gives the components of the position vector of a curve on the Frenet axis and give some special solutions. However, the mentioned work does not contain the solution of the case of the curve is a general helix. In this work, first, we establish the same system and solve it in the case of a general helix. Since, we obtain position vector of a general helix with respect to Frenet frame. Second, we determine the position vector ψ from intrinsic equations using Frenet frame and standard frame in $\mathbf{E}^3$ for a general helix $\dfrac{\tau}{\kappa} = m$, where the constant $m = \cot[\phi]$, ϕ is the angle between the tangent of the curve ψ and the constant vector U called the axis of a general helix.

2. Preliminaries

In Euclidean space $\mathbf{E}^3$, it is well known that to each unit speed curve with at least four continuous derivatives, one can associate three mutually orthogonal unit vector fields $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ which are respectively called, the tangent, the principal normal and the binormal vector fields. We consider the usual metric in Euclidean 3-space $\mathbf{E}^3$, that is,

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbf{E}^3$.

Let $\psi : I \subset \mathbb{R} \to \mathbf{E}^3$, $\psi = \psi(s)$, be an arbitrary curve in $\mathbf{E}^3$. The curve ψ is said to be of unit speed (or parameterized by the arc-length) if $\langle \psi'(s), \psi'(s) \rangle = 1$ for any $s \in I$. In particular, if $\psi(s) \neq 0$ for any s, then it is possible to re-parameterize ψ, that is, $\alpha = \psi(\phi(s))$ so that α is parameterized by the arc-length. Thus, we will assume throughout this work that ψ is a unit speed curve.

Let $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ be the moving frame along ψ, where the vectors $\mathbf{T}, \mathbf{N}$ and $\mathbf{B}$ are mutually orthogonal vectors satisfying $\langle \mathbf{T}, \mathbf{T} \rangle = \langle \mathbf{N}, \mathbf{N} \rangle = \langle \mathbf{B}, \mathbf{B} \rangle = 1$. The Frenet equations for ψ are given by ([7])

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{B}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}. \tag{2.1}$$

If $\tau(s) = 0$ for any $s \in I$, then $\mathbf{B}(s)$ is a constant vector V and the curve ψ lies in a 2-dimensional affine subspace orthogonal to V, which is isometric to the Euclidean 2-space $\mathbf{E}^2$.

3. Position vectors of a general helix with respect to Frenet frame

Theorem 3.1. *The position vector $\alpha(s)$ of a general helix with respect to Frenet frame is given by:*

$$\psi(s) = \Big[(s+c_3) - \tfrac{\kappa(s)\nu(s)}{\tau(s)}\Big]\mathbf{T}(s) - \tfrac{1}{\tau(s)}\nu'(s)\,\mathbf{N}(s) + \nu(s)\,\mathbf{B}(s), \tag{3.1}$$

where

$$\begin{aligned}\nu(s) = \cos\Big[\tfrac{\sqrt{\kappa^2+\tau^2}}{\kappa}\textstyle\int\kappa\,ds\Big]\Big(c_1 - \tfrac{\tau}{\sqrt{\kappa^2+\tau^2}}\textstyle\int(s+c_3)\,\kappa\,\sin\Big[\tfrac{\sqrt{\kappa^2+\tau^2}}{\kappa}\textstyle\int\kappa\,ds\Big]ds\Big)\\ +\sin\Big[\tfrac{\sqrt{\kappa^2+\tau^2}}{\kappa}\textstyle\int\kappa\,ds\Big]\Big(c_2 + \tfrac{\tau}{\sqrt{\kappa^2+\tau^2}}\textstyle\int(s+c_3)\,\kappa\,\cos\Big[\tfrac{\sqrt{\kappa^2+\tau^2}}{\kappa}\textstyle\int\kappa\,ds\Big]ds\Big),\end{aligned} \tag{3.2}$$

while c_1, c_2, c_3 are arbitrary constants, $\kappa = \kappa(s)$ and $\tau = \tau(s)$.

Proof. Let $\psi(s)$ be an arbitrary curve in Euclidean space $\mathbf{E}^3$, then, we may express its position vector as follows:

$$\psi(s) = \lambda(s)\,\mathbf{T}(s) + \mu(s)\,\mathbf{N}(s) + \nu(s)\,\mathbf{B}(s), \tag{3.3}$$

where λ, μ and γ are differentiable functions of $s \in I \subset R$. Differentiating the above equation with respect to s and using the Frenet equations, we get the following:

$$\begin{cases}\lambda' - \kappa\,\mu - 1 = 0,\\ \mu' + \kappa\,\lambda - \tau\,\nu = 0,\\ \nu' + \tau\,\mu = 0,\end{cases} \tag{3.4}$$

[13]. By means of the change of variables $\theta = \int\kappa(s)ds$, the third equation of (3.4) leads to:

$$\mu(\theta) = -\frac{\dot{\nu}(\theta)}{f(\theta)}, \tag{3.5}$$

where $f(\theta) = \frac{\tau(\theta)}{\kappa(\theta)}$ and dot denote the derivative with respect to θ. The second equation of (3.4) becomes

$$\lambda(\theta) = f(\theta)\nu(\theta) + \Big(\frac{\dot{\nu}(\theta)}{f(\theta)}\Big)^{\cdot}. \tag{3.6}$$

Substituting the equations (3.5) and (3.6) into the first equation of (3.4) we get the following equation of $\nu(\theta)$

$$\Big(\frac{\dot{\nu}(\theta)}{f(\theta)}\Big)^{\cdot\cdot} + \Big(\frac{f^2(\theta)+1}{f(\theta)}\Big)\dot{\nu}(\theta) + \dot{f}(\theta)\nu(\theta) = \frac{1}{\kappa(\theta)}. \tag{3.7}$$

Solving the above equation, we obtain the position vector of an arbitrary curve in the Frenet frame. Here, we take a special case when $f(\theta) = m$, i.e., the curve is general helix. The equation above becomes:

$$\nu^{\cdot\cdot\cdot}(\theta) + (m^2+1)\dot{\nu}(\theta) = \frac{m}{\kappa(\theta)}. \tag{3.8}$$

Integrating both sides of equation (3.8), we get

$$\nu^{\cdot\cdot}(\theta) + (m^2+1)\nu(\theta) = m\int\frac{d\theta}{\kappa(\theta)}. \tag{3.9}$$

The general solution of equation (3.9) is

$$\begin{aligned}\nu(\theta) = \cos[M\theta]\Big[c_1 - \tfrac{m}{M}\textstyle\int\sin[M\theta]\Big(\textstyle\int\tfrac{d\theta}{\kappa(\theta)}\Big)d\theta\Big]\\ +\sin[M\theta]\Big[c_2 + \tfrac{m}{M}\textstyle\int\cos[M\theta]\Big(\textstyle\int\tfrac{d\theta}{\kappa(\theta)}\Big)d\theta\Big].\end{aligned} \tag{3.10}$$

where c_1, c_2 are arbitrary constants and $M=\sqrt{1+m^2}$. From (3.5), the function $\mu(\theta)$ is given by:

$$\begin{aligned}\mu(\theta)=\tfrac{M}{m}\sin[M\theta]\Big[c_1-\tfrac{m}{M}\int\sin[M\theta]\Big(\int\tfrac{d\theta}{\kappa(\theta)}\Big)d\theta\Big]\\ -\tfrac{M}{m}\cos[M\theta]\Big[c_2+\tfrac{m}{M}\int\cos[M\theta]\Big(\int\tfrac{d\theta}{\kappa(\theta)}\Big)d\theta\Big].\end{aligned}\tag{3.11}$$

In view of (3.6) and (3.9), $\lambda(\theta)$ is expressed as

$$\begin{aligned}\lambda(\theta)&=m\nu(\theta)+\tfrac{\nu^{\cdot\cdot}(\theta)}{m}\\&=\tfrac{\nu^{\cdot\cdot}(\theta)+M^2\nu(\theta)}{m}-\tfrac{\nu(\theta)}{m}\\&=\int\tfrac{d\theta}{\kappa(\theta)}-\tfrac{\nu(\theta)}{m}.\end{aligned}\tag{3.12}$$

Setting $\theta=\int\kappa(s)ds$ and substituting equations (3.10), (3.11) and (3.12) into (3.3) we get equation (3.1) which completes the proof. □

As a consequence of he above theorem we have the following lemma:

Lemma 3.2. *The position vector $\psi(s)$ of a circular helix with respect to Frenet frame is given by:*

$$\psi(s)=\lambda(s)\,\mathbf{T}(s)+\mu(s)\,\mathbf{N}(s)+\nu(s)\,\mathbf{B}(s),\tag{3.13}$$

such that

$$\begin{aligned}\lambda(s)&=\tfrac{\tau^2(s+c_3)}{\kappa^2+\tau^2}-\tfrac{\kappa}{\tau}\Big(c_1\,\cos[s\,\sqrt{\kappa^2+\tau^2}]+c_2\,\sin[s\,\sqrt{\kappa^2+\tau^2}]\Big),\\ \mu(s)&=-\tfrac{\kappa}{\kappa^2+\tau^2}+\tfrac{\sqrt{\kappa^2+\tau^2}}{\tau}\Big(c_1\,\sin[s\,\sqrt{\kappa^2+\tau^2}]-c_2\,\cos[s\,\sqrt{\kappa^2+\tau^2}]\Big),\\ \nu(s)&=\tfrac{\kappa\tau(s+c_3)}{\kappa^2+\tau^2}+c_1\,\cos[s\,\sqrt{\kappa^2+\tau^2}]+c_2\,\sin[s\,\sqrt{\kappa^2+\tau^2}],\end{aligned}\tag{3.14}$$

where c_1, c_2, c_3 are arbitrary constants while κ and τ are arbitrary constants representing the curvature and the torsion, respectively.

4. Position vectors of a general helix with respect to standard frame

Theorem 4.1. *Let $\psi=\psi(s)$ be an unit speed curve. Then, position ψ satisfies a vector differential forth order as follows*

$$\frac{d}{ds}\Big[\frac{1}{\tau}\frac{d}{ds}\Big(\frac{1}{\kappa}\frac{d^2\psi}{ds^2}\Big)\Big]+\Big(\frac{\kappa}{\tau}+\frac{\tau}{\kappa}\Big)\frac{d^2\psi}{ds^2}+\frac{d}{ds}\Big(\frac{\kappa}{\tau}\Big)\frac{d\psi}{ds}=0.\tag{4.1}$$

Proof. Let $\psi=\psi(s)$ be an unit speed curve and if we substitute $(2.1)_1$ to $(2.1)_2$ we have

$$\mathbf{B}=\frac{1}{\tau}\frac{d}{ds}\Big(\frac{1}{\kappa}\frac{d\mathbf{T}}{ds}\Big)+\frac{\kappa}{\tau}\mathbf{T}.\tag{4.2}$$

Differentiating of (4.2) and using in $(2.1)_3$, we write

$$\frac{d}{ds}\Big[\frac{1}{\tau}\frac{d}{ds}\Big(\frac{1}{\kappa}\frac{d\mathbf{T}}{ds}\Big)\Big]+\Big(\frac{\kappa}{\tau}+\frac{\tau}{\kappa}\Big)\frac{d\mathbf{T}}{ds}+\frac{d}{ds}\Big(\frac{\kappa}{\tau}\Big)\mathbf{T}=0.\tag{4.3}$$

Denoting $\frac{d\psi}{ds}=\mathbf{T}$, we have a vector differential equation of fourth order (4.1) as desired. □

The equation (4.1) can be rewritten in the following simple form:

$$\frac{d}{d\theta}\Big(\frac{1}{f}\,\frac{d^2\mathbf{T}}{d\theta^2}\Big)+\Big(\frac{f^2+1}{f}\Big)\frac{d\mathbf{T}}{d\theta}-\frac{1}{f^2}\frac{df}{d\theta}\mathbf{T}=0,\tag{4.4}$$

where $f = f(\theta) = \dfrac{\tau(\theta)}{\kappa(\theta)}$ and $\theta = \int \kappa(s)ds$. The solution of this equation gives a position vector of an arbitrary space curve. However, for general helices, we have the following theorem:

Theorem 4.2. *The position vector ψ of a general helix whose tangent vector makes a constant angle with a fixed straight line in the space, is expressed in the natural representation form as follows:*

$$\psi(s) = \sqrt{1-n^2} \int \Big(\cos\Big[\sqrt{1+m^2} \int \kappa(s)ds\Big], \sin\Big[\sqrt{1+m^2} \int \kappa(s)ds\Big], m\Big) ds, \tag{4.5}$$

or in the parametric form as follows:

$$\psi(\xi) = \frac{\sqrt{1-n^2}}{\sqrt{1+m^2}} \int \frac{1}{\kappa(\theta)} \Big(\cos[\xi], \sin[\xi], m\Big) d\xi, \tag{4.6}$$

where $\xi = \sqrt{1+m^2} \int \kappa(s)\, ds$, $m = \frac{n}{\sqrt{1-n^2}}$, $n = \cos[\phi]$ and ϕ is the angle between the fixed straight line $\mathbf{e}_3$ (axis of a general helix) and the tangent vector of the curve ψ.

Proof. If ψ is a general helix whose tangent vector $\mathbf{T}$ makes an angle ϕ with the a straight line U, then we can write $f(\theta) = \cot[\phi] = m$, where $\theta = \int \kappa(s)ds$. Therefore the equation (4.4) becomes

$$\frac{d^3\mathbf{T}}{d\theta^3} + (1+m^2)\frac{d\mathbf{T}}{d\theta} = 0. \tag{4.7}$$

The tangent vector $\mathbf{T}$ can be given by:

$$\mathbf{T} = T_1(\theta)\mathbf{e}_1 + T_2(\theta)\mathbf{e}_2 + T_3(\theta)\mathbf{e}_3. \tag{4.8}$$

Because the curve ψ is a general helix, i.e. the tangent vector $\mathbf{T}$ makes a constant angle ϕ with the constant vector calling the axis of the helix. So, with out loss of generality, we take the axis to be parallel to $\mathbf{e}_3$. Then

$$T_3(\theta) = \langle \mathbf{T}, \mathbf{e}_3 \rangle = \cos[\phi] = n. \tag{4.9}$$

On other hand, the tangent vector $\mathbf{T}$ is a unit vector, so the following condition must be satisfied:

$$T_1^2(\theta) + T_2^2(\theta) = 1 - n^2. \tag{4.10}$$

The general solution of equation (4.10) is given by:

$$T_1(\theta) = \sqrt{1-n^2}\, \cos[t(\theta)], \quad T_2(\theta) = \sqrt{1-n^2}\, \sin[t(\theta)], \tag{4.11}$$

where t is an arbitrary function of θ. Each one of the components of the vector $\mathbf{T}(\theta)$ satisfies the equation (4.8). So, substituting the components $T_1(\theta)$ and $T_2(\theta)$ in the equation (4.8), we get the following differential equations of $t(\theta)$

$$3t't'' \cos[t] + \Big[(1+m^2)t' - t'^3 + t'''\Big] \sin[t] = 0, \tag{4.12}$$

$$3t't'' \sin[t] - \Big[(1+m^2)t' - t'^3 + t'''\Big] \cos[t] = 0, \tag{4.13}$$

which can be reduced to:

$$t'\, t'' = 0, \tag{4.14}$$

$$(1+m^2)t' - t'^3 + t''' = 0. \tag{4.15}$$

Since t is not constant, then $t' \neq 0$ and hence the general solution of the equation (4.14) is

$$t(\theta) = c_2 + c_1\,\theta, \tag{4.16}$$

where c_1 and c_2 are constants of integration. The constant c_2 will disappear in making the change $t \to t + c_2$. Substituting the solution (4.16) in the equation (4.15), we obtain:

$$c_1 = \sqrt{1+m^2}.$$

Now, the tangent vector take the following form:

$$T(\theta) = \sqrt{1-n^2}\Big(\cos[\sqrt{1+m^2}\,\theta], \sin[\sqrt{1+m^2}\,\theta], m\Big). \tag{4.17}$$

If we integrate the equation (4.17), we get the two equations (4.5) and (4.6), which completes the proof. $\square$

The following three lemmas are direct consequences of the above theorem:

Lemma 4.3. *The position vector ψ of a circular helix $\kappa(s) = \kappa$ and $\tau(s) = \tau$ is expressed in the natural representation form:*

$$\psi(s) = \frac{\kappa}{\kappa^2+\tau^2}\Big(\sin[\xi], -\cos[\xi], \frac{\tau}{\kappa}\,\xi\Big), \tag{4.18}$$

where $\xi = \sqrt{\kappa^2+\tau^2}\,s$.

One can see a special example of such curve when ($\kappa = 4$, $\tau = 1$), ($\kappa = \tau = 1$) and ($\kappa = 1$, $\tau = 3$) in the left, middle and right hand sides of figure 1, respectively.

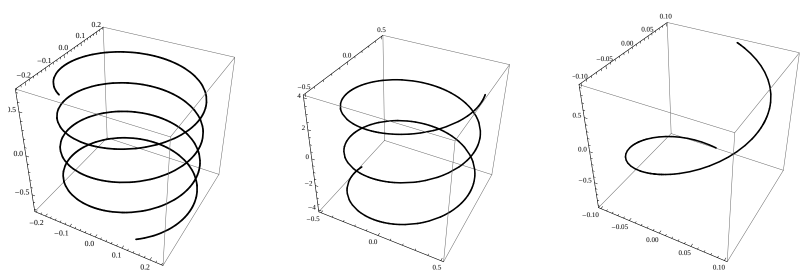

FIGURE 1. Some W-curves.

Lemma 4.4. *The position vector ψ of a general helix with $\kappa(s) = \frac{h}{s}$ and $\tau(s) = \frac{r}{s}$ is expressed in the natural representation form:*

$$\psi(s) = \frac{h\,e^{\frac{\xi}{\sqrt{h^2+r^2}}}}{1+h^2+r^2}\Big(\frac{\cos[\xi]}{\sqrt{h^2+r^2}} + \sin[\xi], \frac{\sin[\xi]}{\sqrt{h^2+r^2}} - \cos[\xi], \frac{r(1+h^2+r^2)}{h}\Big), \tag{4.19}$$

where $\xi = \sqrt{h^2+r^2}\,\ln[s]$.

One can see a special example of such curve when ($h = 3$, $r = 1$), ($h = r = 1$) and ($h = 1$, $r = 2$) in the left, middle and right hand sides of figure 2, respectively.

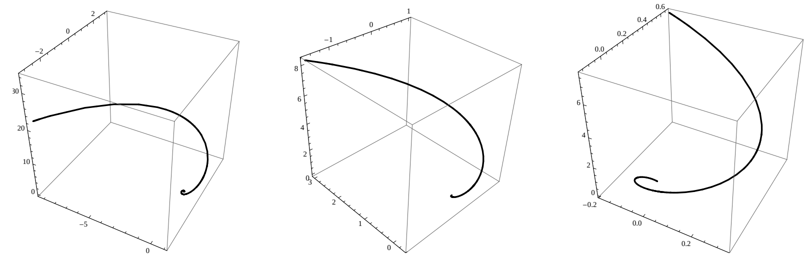

FIGURE 2. Some general helices with $\kappa(s) = \frac{h}{s}$ and $\tau(s) = \frac{r}{s}$.

Lemma 4.5. *The position vector* ψ *of a general helix with* $\kappa(s) = \frac{h}{1+s^2}$ *and* $\tau(s) = \frac{\sqrt{1-h^2}}{1+s^2}$ *is expressed in the natural representation form:*

$$\psi(s) = h\Big(\ln\Big[\sec[\xi] + \tan[\xi]\Big], \sec[\xi], \frac{\sqrt{1-h^2}}{h}\tan[\xi]\Big), \tag{4.20}$$

where $\xi = \arctan[s]$.

One can see a special example of such curve when $h = \frac{1}{2\sqrt{2}}$, $h = \frac{1}{\sqrt{2}}$ and $h = \frac{\sqrt{3}}{2}$ in the left, middle and right hand sides of figure 3, respectively.

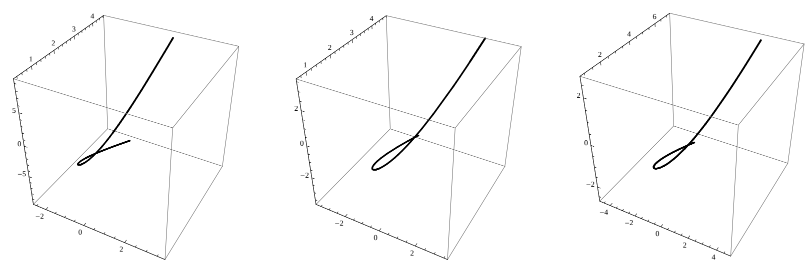

FIGURE 3. Some general helices with $\kappa(s) = \frac{h}{1+s^2}$ and $\tau(s) = \frac{\sqrt{1-h^2}}{1+s^2}$.

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