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RIEMANNIAN MANIFOLDS WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION SATISFYING SOME SEMISYMMETRY CONDITIONS

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YUSUF DOĞRU, CIHAN ÖZGÜR AND CENGIZHAN MURATHAN

Abstract. We study on a Riemannian manifold (M, g) with a semi-symmetric non-metric connection. We obtain some characterizations for (M, g) satisfying some semisymmety conditions.

1. Introduction

Let $\tilde{\nabla}$ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T and the curvature tensor $\tilde{R}$ of $\tilde{\nabla}$ are given respectively by

$$T(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y]$$

$$\tilde{R}(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z.$$

The connection $\tilde{\nabla}$ is symmetric if its torsion tensor T vanishes, otherwise it is not symmetric. The connection $\tilde{\nabla}$ is a metric connection if there is a Riemannian metric g in M such that $\tilde{\nabla} g = 0$, otherwise it is non-metric [15]. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection.

In 1932, H. A. Hayden [8] introduced a metric connection $\tilde{\nabla}$ with a non-zero torsion on a Riemannian manifold. Such a connection is called *Hayden connection*. In [7, 11], Friedmann and Schouten introduced the idea of a semi-symmetric linear connection in a differentiable manifold. A linear connection $\tilde{\nabla}$ is said to be a *semi-symmetric connection* if its torsion tensor T is of the form

$$T(X, Y) = \omega(Y)X - \omega(X)Y, \tag{1.1}$$

where the 1-form ω is defined by

$$\omega(X) = g(X, U),$$

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and U is a vector field. Hayden connection with the torsion tensor of the form (1.1) is a semi-symmetric metric connection [10].

In [1], Agashe and Chafle introduced the idea of a semi-symmetric non-metric connection on a Riemannian manifold. This was further developed by Agashe and Chafle [2], De and Kamilya [5], De, Sengupta and Binh [12].

In [13, 14], Szabó studied semisymmetric Riemannian manifolds, that is Riemannian manifolds satisfying the condition $R \cdot R = 0$. It is well known that locally symmetric manifolds (i. e. Riemannian manifolds satisfying the condition $\nabla R = 0$) are trivially semisymmetric. But the converse statement is not true. If $R \cdot S = 0$ then the manifold is called Ricci-semisymmetric. It is trivial that every semisymmetric manifold is Ricci-semisymmetric but the converse statement is not true.

In this paper, we consider Riemannian manifolds admitting a semi-symmetric non-metric connection such that U is a unit parallel vector field with respect to the Levi-Civita connection ∇. We investigate the conditions $R \cdot \tilde{R} = 0$, $\tilde{R} \cdot R = 0$, $R \cdot \tilde{R} - \tilde{R} \cdot R = 0$, $R \cdot \tilde{S} = 0$, $\tilde{R} \cdot S = 0$, $R \cdot \tilde{S} - \tilde{R} \cdot S = 0$ and $\tilde{R} \cdot \tilde{S} = 0$ on M, where R and $\tilde{R}$ (resp. S and $\tilde{S}$) denote the curvature tensors (resp. Ricci tensors) of ∇ and $\tilde{\nabla}$.

The paper is organized as follows. In Section 2 and Section 3, we give the necessary notions and results which will be used in the next sections. In Section 4, we prove that $R \cdot \tilde{R} = 0$ holds on M if and only if M is semisymmetric. Furthermore, we show that M is a quasi-Einstein manifold under certain conditions.

2. Preliminaries

An n-dimensional Riemannian manifold (M, g), $(n \geq 3)$, is said to be an *Einstein manifold* if its Ricci tensor S satisfies the condition $S = \frac{r}{n} g$, where r denotes the scalar curvature of M. If the Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + bD(X)D(Y), \tag{2.1}$$

where a, b are scalars of which $b \neq 0$ and D is a non zero 1-form, then M is called a *quasi-Einstein manifold* [4].

For a $(0,k)$-tensor field T, $k \geq 1$, on (M,g) we define the tensor $R \cdot T$ (see [6]) by

$$\begin{aligned}(R(X,Y) \cdot T)(X_1, ..., X_k) &= -T(R(X,Y)X_1, X_2,...,X_k) \\ &\quad - ... - T(X_1, ..., X_{k-1}, R(X,Y)X_k).\end{aligned} \tag{2.2}$$

If $R \cdot R = 0$ then M is called *semisymmetric* [13]. In addition, if E is a symmetric $(0,2)$-tensor field then we define the $(0, k+2)$-tensor $Q(E,T)$ (see [6]) by

$$\begin{aligned}Q(E,T)(X_1, ..., X_k; X, Y) &= -T((X \wedge_E Y)X_1, X_2, ..., X_k) \\ &\quad - ... - T(X_1, ..., X_{k-1}, (X \wedge_E Y)X_k),\end{aligned} \tag{2.3}$$

where $X \wedge_E Y$ is defined by

$$(X \wedge_E Y)Z = E(Y,Z)X - E(X,Z)Y.$$

3. Semi-symmetric non-metric connection

Let ∇ be the Levi-Civita connection of a Riemannian manifold M. The semi-symmetric non-metric connection $\tilde{\nabla}$ is defined by

$$\tilde{\nabla}_X Y = \nabla_X Y + \omega(Y)X, \tag{3.1}$$

where

$$\omega(X) = g(X, U),$$

and X, Y, U are vector fields on M [1]. Let R and $\tilde{R}$ denote the Riemannian curvature tensors of ∇ and $\tilde{\nabla}$, respectively. Then we know that [1]

$$\begin{aligned} \tilde{R}(X, Y, Z, W) &= R(X, Y, Z, W) - \theta(Y, Z)g(X, W) \\ &\quad + \theta(X, Z)g(Y, W), \end{aligned} \tag{3.2}$$

where

$$\theta(X, Y) = g(AX, Y) = (\nabla_X \omega)Y - \omega(X)\omega(Y). \tag{3.3}$$

Here A is a (1,1)-tensor field which is metrically equivalent to θ. Now assume that U is a parallel unit vector field with respect to the Levi-Civita connection, i.e., $\nabla U = 0$ and $\|U\| = 1$. Then

$$(\nabla_X \omega)Y = \nabla_X \omega(Y) - \omega(\nabla_X Y) = 0. \tag{3.4}$$

So θ is a symmetric $(0, 2)$-tensor field. Since U is a parallel unit vector field, it is easy to see that $\tilde{R}$ is a generalized curvature tensor and it is trivial that $R(X, Y)U = 0$. Hence by a contraction, we find $S(Y, U) = \omega(QY) = 0$, where S denotes the Ricci tensor of ∇ and Q is the Ricci operator defined by $g(QX, Y) = S(X, Y)$. It is easy to see that we have also the following relations:

$$\tilde{\nabla}_X U = X, \tag{3.5}$$

$$\tilde{R}(X, Y)U = \omega(Y)X - \omega(X)Y, \qquad \tilde{R} \cdot \theta = 0, \tag{3.6}$$

$$\tilde{S} = S + (n-1)(\omega \otimes \omega), \tag{3.7}$$

and

$$\tilde{r} = r + (n-1), \tag{3.8}$$

where $\tilde{S}$ and $\tilde{r}$ denote the Ricci tensor and the scalar curvature of M with respect to semi-symmetric non-metric connection $\tilde{\nabla}$.

4. Main Results

The tensors $\tilde{R} \cdot R$ and $Q(\theta, T)$ are defined in the same way as in (2.2) and (2.3). Let $(R \cdot \tilde{R})_{hijklm}$ and $(\tilde{R} \cdot R)_{hijklm}$ denote the local components of the tensors $R \cdot \tilde{R}$ and $\tilde{R} \cdot R$, respectively. Hence, we have the following Proposition:

Proposition 4.1. *Let (M, g) be an $(n \geq 3)$-dimensional Riemannian manifold admitting a semi-symmetric non-metric connection. If U is a parallel unit vector field with respect to the Levi-Civita connection ∇ then*

$$(R \cdot \tilde{R})_{hijklm} = (R \cdot R)_{hijklm} \tag{4.1}$$

and

$$(\tilde{R} \cdot R)_{hijklm} = (R \cdot R)_{hijklm} - Q(-\omega \otimes \omega, R)_{hijklm}. \tag{4.2}$$

Proof. Applying (3.2) in (2.2) and using (2.3), we obtain

$$R \cdot \tilde{R} = R \cdot R \tag{4.3}$$

and

$$\begin{aligned} (\tilde{R} \cdot R)_{hijklm} &= (R \cdot R)_{hijklm} - Q(\theta, R)_{hijklm} \\ &= (R \cdot R)_{hijklm} - Q(-\omega \otimes \omega, R)_{hijklm}. \end{aligned}$$

This completes the proof of the proposition. $\square$

As an immediate consequence of Proposition 4.1, we have the following theorem:

Theorem 4.2. *Let (M, g) be an $(n \geq 3)$-dimensional Riemannian manifold admitting a semi-symmetric non-metric connection and U a parallel unit vector field with respect to the Levi-Civita connection ∇. Then $R \cdot \tilde{R} = 0$ if and only if M is semisymmetric.*

Theorem 4.3. *Let (M, g) be an $(n \geq 3)$-dimensional semisymmetric Riemannian manifold admitting a semi-symmetric non-metric connection. If U is a parallel unit vector field with respect to the Levi-Civita connection ∇ and $\tilde{R} \cdot R = 0$ then M is a quasi-Einstein manifold.*

Proof. Since M is semisymmetric and the condition $\tilde{R} \cdot R = 0$ holds on M, from Proposition 4.1, we have

$$Q(\omega \otimes \omega, R)_{hijklm} = 0. \tag{4.4}$$

Contracting (4.4) with g^{ij} we get

$$Q(\omega \otimes \omega, S)_{hklm} = 0,$$

which gives us

$$S = r(\omega \otimes \omega),$$

where $r : M \to \mathbb{R}$ is a function. So by virtue of (2.1), M is a quasi-Einstein manifold. Thus the proof of the theorem is completed. □

Theorem 4.4. *Let (M, g) be an $(n \geq 3)$-dimensional Riemannian manifold admitting a semi-symmetric non-metric connection. If U is a parallel unit vector field with respect to the Levi-Civita connection ∇ and $R \cdot \tilde{R} - \tilde{R} \cdot R = 0$, then M is a quasi-Einstein manifold.*

Proof. Using (4.1) and(4.2) we get

$$Q(\omega \otimes \omega, R)_{hijklm} = 0.$$

Using the same method as in the proof of Theorem 4.3, we obtain that M is a quasi-Einstein manifold. So we get the result as required. □

Proposition 4.5. *Let (M, g) be an $(n \geq 3)$-dimensional Riemannian manifold admitting a semi-symmetric non-metric connection. If U is a parallel unit vector field with respect to the Levi-Civita connection ∇ then*

$$(R \cdot \tilde{S})_{hklm} = (R \cdot S)_{hklm}, \tag{4.5}$$

$$(\tilde{R} \cdot S)_{hklm} = (R \cdot S)_{hklm} - Q(-\omega \otimes \omega, S)_{hklm}. \tag{4.6}$$

Proof. Applying (3.7) and (3.2) in (2.2) and using (2.3), we obtain

$$R \cdot \tilde{S} = R \cdot S$$

and

$$\begin{aligned}(\tilde{R} \cdot S)_{hklm} &= (R \cdot S)_{hklm} - Q(\theta, S)_{hklm} \\ &= (R \cdot S)_{hklm} - Q(-\omega \otimes \omega, S)_{hklm}.\end{aligned}$$

This completes the proof of the proposition. □

As an immediate consequence of Proposition 4.5, we have the following theorem:

Theorem 4.6. *Let (M,g) be an $(n \geq 3)$-dimensional Riemannian manifold admitting a semi-symmetric non-metric connection and U a parallel unit vector field with respect to the Levi-Civita connection ∇. Then $R \cdot \tilde{S} = 0$ if and only if M is Ricci-semisymmetric.*

Theorem 4.7. *Let (M,g) be an $(n \geq 3)$-dimensional Ricci-semisymmetric Riemannian manifold admitting a semi-symmetric non-metric connection. If U is a parallel unit vector field with respect to the Levi-Civita connection ∇ and $\tilde{R} \cdot S = 0$ then M is a quasi-Einstein manifold.*

Proof. Since the condition $\tilde{R} \cdot S = 0$ holds on M, from Proposition 4.5, we have

$$Q(\omega \otimes \omega, S)_{hklm} = 0.$$

So by the same reason as in the proof of Theorem 4.3, M is a quasi-Einstein manifold. Thus the proof of the theorem is completed. □

Theorem 4.8. *Let (M,g) be an $(n \geq 3)$-dimensional Riemannian manifold admitting a semi-symmetric non-metric connection. If U is a parallel unit vector field with respect to Levi-Civita connection ∇ and $R \cdot \tilde{S} - \tilde{R} \cdot S = 0$, then M is a quasi-Einstein manifold.*

Proof. Using (4.5) and(4.6) we get

$$Q(\omega \otimes \omega, S)_{hklm} = 0.$$

Using the same method as in the proof of Theorem 4.3, we obtain that M is a quasi-Einstein manifold. This proves the theorem. □

Theorem 4.9. *Let (M,g) be an $(n \geq 3)$-dimensional Ricci-semisymmetric Riemannian manifold admitting a semi-symmetric non-metric connection. If U is a parallel unit vector field with respect to the Levi-Civita connection ∇ and $\tilde{R} \cdot \tilde{S} = 0$, then M is a quasi-Einstein manifold.*

Proof. Applying (3.7) and (3.2) in (2.2) and using (2.3) we obtain,

$$(\tilde{R} \cdot \tilde{S})_{hklm} = (R \cdot S)_{hklm} - Q(-\omega \otimes \omega, S)_{hklm}.$$

We suppose that $\tilde{R} \cdot \tilde{S} = 0$ and $R \cdot S = 0$. So using the same method as in the proof of Theorem 4.3, we obtain that M is a quasi Einstein manifold. Thus the proof of the theorem is completed.

The following example shows that there is a Riemannian manifold with a semi-symmetric non-metric connection having a parallel vector field associated to the 1-form satisfying $R \cdot \tilde{R} = R \cdot R$. □

Example. Let M^{2m+1} be a $(2m+1)$-dimensional almost contact manifold endowed with an almost contact structure (φ, ξ, η), that is, φ is a $(1,1)$-tensor field, ξ is a vector field and η is a 1-form such that

$$\varphi^2 = -I + \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1.$$

Then

$$\varphi(\xi) = 0 \quad \text{and} \quad \eta \circ \varphi = 0.$$

Let g be a compatible Riemannian metric with (φ, ξ, η), that is,

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$$

or equivalently,

$$g(X,\varphi Y) = -g(\varphi X; Y) \quad \text{and} \quad g(X,\xi) = \eta(X)$$

for all $X, Y \in \chi(M)$. Then, M^{2m+1} becomes an almost contact metric manifold equipped with an almost contact metric structure (φ, ξ, η, g). An almost contact metric manifold is *cosymplectic* [3] if $\nabla_X \varphi = 0$. From the formula $\nabla_X \varphi = 0$, it follows that

$$\nabla_X \xi = 0, \quad \nabla_X \eta = 0, \quad \text{and} \quad R(X,Y)\xi = 0.$$

If we define a connection

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X$$

on the above manifold, then we obtain

$$T(X,Y) = \eta(Y)X - \eta(X)Y$$

and

$$\theta = -\eta \otimes \eta,$$

which shows that $\tilde{\nabla}$ is a semi-symmetric non-metric connection and by virtue of Proposition 4.1, we have $R \cdot \tilde{R} = R \cdot R$.

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Yusuf Doğru
Işiklar Air Force High School, 16039 Bursa, TURKEY
E-mail address: yusufdogru1@yahoo.com

Cihan Özgür
Department of Mathematics, Balikesir University, 10145, Çağiş, Balikesir, TURKEY
E-mail address: cozgur@balikesir.edu.tr

Cengizhan Murathan
Department of Mathematics, Uludağ University, 16059, Görükle, Bursa, TURKEY
E-mail address: cengiz@uludag.edu.tr