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ON THE CLASS OF n-POWER QUASI-NORMAL OPERATORS ON HILBERT SPACE

(COMMUNICATED BY SALAH MECHERI)

OULD AHMED MAHMOUD SID AHMED

Abstract. Let T be a bounded linear operator on a complex Hilbert space H. In this paper we investigate some, properties of the class of n-power quasi-normal operators , denoted $[nQN]$, satisfying $T^n|T|^2 - |T|^2T^n = 0$ and some relations between n-normal operators and n-quasinormal operators.

1. INTRODUCTION AND TERMINOLOGIES

A bounded linear operator on a complex Hilbert space, is quasi-normal if T and T^*T commute. The class of quasi-normal operators was first introduced and studied by A.Brown [5] in 1953. From the definition, it is easily seen that this class contains normal operators and isometries. In [9] the author introduce the class of n-power normal operators as a generalization of the class of normal operators and study sum properties of such class for different values of the parameter n. In particular for $n = 2$ and $n = 3$ (see for instance [9,10]). In this paper, we study the bounded linear transformations T of complex Hilbert space H that satisfy an identity of the form

$$T^nT^*T = T^*TT^n, \tag{1.1}$$

for some integer n. Operators T satisfying (1.1) are said to be n-power quasi-normal.

Let $\mathcal{L}(H) = \mathcal{L}(H, H)$ be the Banach algebra of all bounded linear operators on a complex Hilbert space H. For $T \in \mathcal{L}(H)$, we use symbols $R(T)$, $N(T)$ and T^* the range , the kernel and the adjoint of T respectively,.

Let $W(T) = \{ \langle Tx \mid x\rangle \ : x \in H, \|x\| = 1\}$ the numerical range of T. A subspace $M \subset H$ is said to be invariant for an operator $T \in \mathcal{L}(H)$ if $TM \subset M$, and in this situation we denote by $T|M$ the restriction of T to M. Let $\sigma(T), \sigma_a(T)$ and $\sigma_p(T)$, respectively denote the spectrum, the approximate point spectrum and point spectrum of the operator T.

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For any arbitrary operator $T \in \mathcal{L}(H)$, $|T| = (T^*T)^{\frac{1}{2}}$ and

$$[T^*, T] = T^*T - TT^* = |T|^2 - |T^*|^2$$

(the self-commutator of T).

An operator T is normal if $T^*T = TT^*$, positive-normal (posinormal) il there exits a positive operator $P \in \mathcal{L}(H)$ such that $TT^* = T^*PT$, hyponormal if $[T^*, T]$ is nonnegative(i.e. $|T^*|^2 \leq |T|^2$, equivalently $\|T^*x\| \leq \|Tx\|$, $\forall\ x \in H$), quasi-hyponormal if $T^*[T^*, T]T$ is nonnegative, paranormal if $\|Tx\|^2 \leq \|T^2x\|$ for all $x \in H$, n-isometry if

$$T^{*n}T^n - \binom{n}{1}T^{*n-1}T^{n-1} + \binom{n}{2}T^{*n-2}T^{n-2}... + (-1)^n I = 0,$$

m-hyponormal if there exists a positive number m, such that

$$m^2(T - \lambda I)^*(T - \lambda I) - (T - \lambda I)(T - \lambda I)^* \leq 0; \text{for all } \lambda \in \mathbb{C}, \quad .$$

Let $[N]; [QN]; [H]$; and $(m - H)$ denote the classes constituting of normal, quasi-normal, hyponormal, and m-hyponormal operators. Then

$$[N] \subset [QN] \subset [H] \subset [m - H].$$

For more details see [1, 2, 3, 11, 14 ,15].

Definition 1.1. *([7]) An operator $T \in \mathcal{L}(H)$ is called (α, β)-normal $(0 \leq \alpha \leq 1 \leq \beta)$ if*

$$\alpha^2 T^*T \leq TT^* \leq \beta^2 T^*T.$$

or equivalently

$$\alpha\|Tx\| \leq \|T^*x\| \leq \beta\|Tx\| \text{ for all } x \in H.$$

Definition 1.2. *([9]) Let $T \in \mathcal{L}(H)$. T is said n-power normal operator for a positive integer n if*

$$T^nT^* = T^*T^n.$$

The class of all n-normal operators is denoted by $[nN]$.

Proposition 1.3. *([9]) Let $T \in \mathcal{L}(H)$, then T is of class $[nN]$ if and only if T^n is normal for any positif integer n.*

Remark. *T is n-power normal if and only if T^n is (1,1)-normal.*

The outline of the paper is as follows: Introduction and terminologies are described in first section. In the second section we introduce the class of n-power quasi-normal operators in Hilbert spaces and we develop some basic properties of this class. In section three we investigate some properties of a class of operators denoted by $(\mathbb{Z}^n)$ contained the class $[nQN.]$

2. BASIC PROPERTIES OF THE CLASS $[nQN]$

In this section, we will study some property which are applied for the $n-$power quasi-normal operators.

Definition 2.1. *For $n \in \mathbb{N}$, an operator $T \in \mathcal{L}(H)$ is said to be n-power quasi-normal operator if*

$$T^nT^*T = T^*T^{n+1}.$$

We denote the set of n-power quasi-normal operators by $[nQN]$. It is obvious that the class of all n-power quasi-normal operators properly contained classes of n-normal operators and quasi-normal operators, i.e., the following inclusions holds

$$[nN] \subset [nQN] \quad \text{and} \quad [QN] \subset [nQN].$$

Remark.

(1) *A 1-power quasi-normal operator is quasi-normal.*
(2) *Every quasi-normal operator is n-power quasi-normal for each n.*
(3) *It is clear that a n-power normal operator is also n-power quasi-normal. That the converse need not hold can be seen by choosing T to be the unilateral shift, that is, if $H = l^2$, the matrix $T = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$. It is easily verified that $T^2T^* - T^*T^2 \neq 0$ and $(T^2T^* - T^*T^2)T = 0$. So that T is not 2-power normal but is a 2-power quasi-normal.*

Remark. *An operator T is n-power quasi-normal if and only if*

$$[T^n, T^*T] = [T^n, T^*]T = 0.$$

Remark. *An operator T is n-power quasi-normal if and only if*

$$T^n|T|^2 = |T|^2T^n.$$

First we record some elementary properties of $[nNQ]$

Theorem 2.2. *If $T \in [nQN]$, then*

(1) *T is of class $[2nQN]$.*
(2) *if T has a dense range in H , T is of class $[nN]$. In particular, if T is invertible, then T^{-1} is of class $[nQN]$.*
(3) *If T and S are of class $[nQN]$ such that $[T,\ S] = [T,\ S^*] = 0$, then TS is of class $[nQN]$.*
(4) *If S and T are of class $[nQN]$ such that $ST = TS = T^*S = ST^* = 0$,, then $S + T$ is of class $[nQN]$.*

Proof.

(1) Since T is of $[nQN]$, then

$$T^nT^*T = T^*TT^n. \tag{2.1}$$

Multiplying (2.1) to the left by T^n, we obtain

$$T^{2n}T^*T = T^*TT^{2n}.$$

Thus T is of class $[2nQN]$.

(2) Since T is of class $[nQN]$, we have for $y \in R(T) : y = Tx,\ x \in H$, and

$$\|(T^nT^* - T^*T^n)y\| = \|(T^nT^* - T^*T^n)Tx\| = \|(T^nT^*T - T^*T^{n+1})x\| = 0.$$

Thus, T is n-power normal on $R(T)$ and hence T is of class $[nN]$. In case T invertible, then it is an invertible operator of class $[nN]$ and so

$$T^nT^* = T^*T^n.$$

This in turn shows that

$$T^{-n}(T^{*-1}T^{-1}) = [(TT^*)T^n]^{-1} = [T^{n+1}T^*]^{-1} = [T^{*-1}T^{-1}]T^{-n},$$

which prove the result.

(3)

$$\begin{aligned}(TS)^n(TS)^*TS &= T^nS^nT^*S^*TS = T^nT^*TS^nS^*S \\ &= T^*T^{n+1}S^*S^{n+1} = (TS)^*(TS)^{n+1}.\end{aligned}$$

Hence, TS is of class $[nQN]$.

(4)

$$\begin{aligned}(T+S)^n(T+S)^*(T+S) &= (T^n+S^n)(T^*T+S^*S) \\ &= T^nT^*T+S^nS^*S \\ &= T^*T^{n+1}+S^*S^{n+1} \\ &= (T+S)^*(T+S)^{n+1}.\end{aligned}$$

Which implies that $T+S$ is of class $[nQN]$.

Proposition 2.3. *If T is of class $[nQN]$ such that T is a partial isometry, then T is of class $[(n+1)QN]$.*

Proof. Since T is a partial isometry, therefore

$$TT^*T = T \quad [4], p.153). \tag{2.2}$$

Multiplying (2.2) to the left by T^*T^{n+1} and using the fact that T is of class $[nNQ]$, we get

$$\begin{aligned}T^*T^{n+2} &= T^*T^{n+2}T^*T \\ &= T^nT^*T.TT^*T \\ &= T^{n+1}T^*T,\end{aligned}$$

which implies that T is of class $[(n+1)QN]$.

The following examples show that the two classes $[2NQ]$ and $[3NQ]$ are not the same.

Example 2.4. *Let $H=\mathbb{C}^3$ and let $T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^3)$. Then by simple calculations we see that T is not of class $[3QN]$ but of class $[2QN]$.*

Example 2.5. *Let $H=\mathbb{C}^3$ and let $S = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^3)$. Then by simple calculations we see that S is not of class $[2QN]$ but of class $[3QN]$.*

Proposition 2.6. *Let $T \in \mathcal{L}(H)$ such that T is of class $[2QN]\bigcap[3QN]$, then T is of class $[nQN]$ for all positive integer $n \geq 4$.*

Proof. We proof the assertion by using the mathematical induction. For $n=4$ it is a consequence of Theorem 2.2. 1 .

We prove this for $n=5$. Since $T \in [2QN]$,

$$T^2T^*T = T^*T^3, \tag{2.3}$$

multiplying (2.3) to the left by T^3 we get

$$T^5T^*T = T^3T^*T^3.$$

Thus we have

$$\begin{aligned} T^5T^*T &= T^3T^*T^3 \\ &= T^*T^4T^2 \\ &= T^*T^6. \end{aligned}$$

Now assume that the result is true for $n \geq 5$ i.e

$$T^nT^*T = T^*TT^n,$$

then

$$\begin{aligned} T^{n+1}T^*T &= TT^*T^{n+1} \\ &= TT^*T^3T^{n-2} \\ &= T^3T^*TT^{n-2} \\ &= T^*T^4T^{*(n-2)} \\ &= T^*T^{n+2}. \end{aligned}$$

Thus T is of class $[(n+1)QN]$.

Proposition 2.7. *If T is of class $[nQN]$ such that $N(T^*) \subset N(T)$, then T is of class $[nN]$.*

Proof. In view of the inclusion $N(T^*) \subset N(T)$, it is not difficult to verify the normality of T^n.

Next couple of results shows that $[nQN]$ is not translation invariant

Theorem 2.8. *If T and $T - I$ are of class $[2QN]$, then T is normal.*

Proof. First we see that the condition on $T - I$ implies

$$T^2(T^*T) - T^2T^* - 2T(T^*T) + 2TT^* = (T^*T)T^2 - T^*T^2 - 2(T^*T)T + 2T^*T.$$

Since T is of class $[2QN]$, we have

$$-T^2T^* - 2T(T^*T) + 2TT^* = -T^*T^2 - 2(T^*T)T + 2T^*T,$$

or

$$-TT^{*2} - 2(T^*T)T^* + 2TT^* = -T^{*2}T - 2T^*(T^*T) + 2T^*T \tag{2.4}$$

We first show that (2.4) implies

$$N(T^*) \subset N(T) \tag{2.5}$$

Suppose $T^*x = 0$. From (2.4), we get

$$-3T^{*2}Tx + 2T^*Tx = 0. \tag{2.6}$$

Then

$$-3T^{*3}Tx + 2T^{*2}Tx = 0.$$

Therefore, as T is of class $[2QN]$,

$$-3T^*TT^{*2}x + 2T^{*2}Tx = 0$$

and hence

$$2T^{*2}Tx = 0.$$

Consequently, (2.6) gives $2T^*Tx = 0$ or $Tx = 0$.This proves (2.5). As observe in Proposition 2.7 and Proposition 1.3 T^2 is normal. This along with (2.4) gives

$$-T(T^*T) + TT^* = -(T^*T)T + T^*T,$$

or

$$T^*(T^*T - TT^*) = T^*T - TT^*. \tag{2.7}$$

If $N(T^* - I) = \{0\}$, then (2.7) implies T is normal.

Now assume that $N(T^* - I)$ is non trivial. Let $T^*x = x$. Then (2.6) gives

$$T^{*2}Tx - T^*Tx = T^*Tx - Tx.$$

Since $T^{*2}T = TT^{*2}$, we have

$$T^*Tx = Tx.$$

Therefore

$$||Tx||^2 =< T^*Tx \mid x >=< Tx \mid x >=< x \mid T^*x >= ||x||^2.$$

Hence

$$\begin{aligned} ||Tx - x||^2 &= ||Tx||^2 + ||x||^2 - 2Re < Tx \mid x > \\ &= ||Tx||^2 - ||x||^2 \\ &= 0. \end{aligned}$$

Or $Tx = x$. Thus $N(T^* - I) \subset N(T - I)$.This along with (2.7), yields

$$T(T^*T - TT^*) = T^*T - TT^*$$

and so

$$T(T^*T - TT^*)T = (T^*T - TT^*)T$$

or

$$TT^*T^2 - T^2T^*T = T^*T^2 - TT^*T.$$

Since $T^2T^* = T^*T^2$ and $T^3T^* = T^*T^3$ we deduce that $T^*T^2 = TT^*T$. Thus T is quasinormal. From (2.5), the normality of T follows.

In attempt to extend the above result for operators of class $[nQN]$, we prove

Theorem 2.9. *If T is of class $[2QN] \cap [3QN]$ such that $T - I$ is of class $[nQN]$, then T is normal.*

Proof. Since $T - I$ is of class $[nQN]$, we have

$$\sum_{k=1}^{n} a_k T^k T^* T - \sum_{k=1}^{n} a_k T^k T^* = T^*T \sum_{k=1}^{n} a_k T^k - T^* \sum_{k=1}^{n} a_k T^k,\ a_k = (-1)^{n-k}\tbinom{n}{k}.$$

Under the condition on T, we have by Proposition 2.6

$$a_1 T(T^*T) - (\sum_{k=1}^{n} a_k T^k)T^* = a_1(T^*T)T - T^*(\sum_{k=1}^{n} a_k T^k)$$

or

$$a_1(T^*T)T^* - T(\sum_{k=1}^{n} a_k T^{*k}) = a_1 T^*(T^*T) - (\sum_{k=1}^{n} a_k T^{*k})T. \tag{2.8}$$

(2.8) implies that $N(T^*) \subset N(T)$. In fact, let $T^*x = 0$. From (2.8),we have

$$a_1 T^{*2}Tx - (\sum_{k=1}^{n} a_k T^{*k})Tx = 0.$$

T is of class $[2QN]$ and of class $[3QN]$, we deduce that

$$a_1 T^{*2}Tx - a_1 T^*Tx - a_2 T^{*2}Tx = 0 \tag{2.9}$$

and hence

$$a_1T^{*3}Tx - a_1T^{*2}T - a_2T^{*3}Tx = 0$$

Hence

$$a_1T^{*2}Tx.$$

Consequently (2.9) gives $T^*Tx = 0$, which implies that $Tx = 0$.
It follows by Proposition 2.7 that T^k is normal for $k = 2, 3, ..., n$ and hence

$$T(T^*T) - TT^* = (T^*T)T - T^*T$$

or

$$T^*(TT^* - T^*T) = TT^* - T^*T.$$

Hence,

$$(T^* - I)(TT^* - T^*T) = 0.$$

A similar argument given in as in the proof of Theorem 2.8 gives the desired result.

Theorem 2.10. *If T and T^* are of class $[nQN]$, then T^n is normal.*

First we establish

Lemma 2.11. *If T is of class $[nQN]$, then $N(T^n) \subset N(T^{*n})$ for $n \geq 2$.*

Proof. Suppose $T^n x = 0$. Then

$$T^{*n}(T^*T)T^{n-1}x = 0.$$

By hypotheses,

$$T^*TT^{*n}T^{n-1}x = 0,$$

which implies

$$TT^{*n}T^{n-1}x = 0.$$

Hence

$$T^{*n}T^{n-1}x = 0.$$

Under the condition on T, we have

$$T^*TT^{*n}T^{n-2}x = 0$$

Hence

$$T^{*n}T^{n-2}x = 0.$$

By repeating this process we can find

$$T^{*n}x = 0.$$

Proof of Theorem 2.10. By hypotheses and Lemma 2.11

$$N(T^{*n}) = N(T^n).$$

Since T is of $[nQN]$, $[T^nT^* - T^*T^n]T^n = 0$,i.e. $[T^nT^* - T^*T^n] = 0$ on $clR(T)$.
also the fact that $N(T^*)$ is a subset of $N(T^n)$ gives $[T^nT^* - T^*T^n] = 0$ on $N(T^*)$.
Hence the result follows.

Theorem 2.12. *If T and T^2 are of class $[2QN]$, and T is of class $[3QN]$, then T^2 is quasinormal.*

Proof. The condition that T^2 is of class $[2QN]$ gives

$$T^{*4}(T^{*2}T^2) = (T^{*2}T^2)T^{*4}$$

Implies

$$T^{*5}(T^*T)T = (T^{*2}T^2)T^{*4}$$

Since T if of class $[3QN]$, we have

$$T^{*2}(T^*T)T^{*3}T = (T^{*2}T^2)T^{*4} \quad .$$

And hence

$$T^{*2}(T^*T)^2T^{*2} = (T^{*2}T^2)T^{*4} \quad [T \text{ is of class } \quad [2QN]].$$

Implies

$$(T^*T)^2T^{*4} = (T^{*2}T^2)T^{*4} \quad [T\text{is of class } [2QN]]$$

or

$$T^4((T^*T)^2 - T^{*2}T^2) = 0.$$

By Lemma 2.11,

$$T^{*2}T^2((T^*T)^2 - T^{*2}T^2) = 0$$

or

$$T^2[(T^*T)^2 - T^{*2}T^2)] = 0. \tag{2.10}$$

Hence

$$T^{*2}[((T^*T)^2 - T^{*2}T^2)] = 0, \quad [N(T^2) \text{ is a subset of } \ N(T^{*2})].$$

Or

$$[((T^*T)^2 - T^{*2}T^2)]T^2 = 0. \tag{2.11}$$

Since T is of class $[2QN]$, T^2 commutes with $(T^*T)^2$. Hence from (2.10) and (2.11), we get the desired conclusion.

Theorem 2.13. *If T and T^2 are of class $[2QN]$ and $N(T) \subset N(T^*)$, then T^2 is quasinormal.*

Proof. By the condition that T^2 is of class $[2QN]$, we have

$$\begin{aligned}
(T^{*2}T^2)T^{*4} &= T^{*4}(T^{*2}T^2) \\
&= T^*T^{*4}(T^*T)T \\
&= T^*(T^*T)T^{*4}T \quad [T \text{ is of class } \ [2QN]] \\
&= T^*(T^*T)T^*(T^*T)T^{*2}
\end{aligned}$$

Thus we have

$$\{(T^{*2}T^2)T^{*2} - [T^*(T^*T)]^2\}T^{*2} = 0$$

or

$$T^2\{T^2(T^{*2}T^2) - [(T^*T)T]^2\} = 0.$$

Then under the kernel condition

$$T\{T^2(T^{*2}T^2) - [(T^*T)T]^2\} = 0$$

or

$$\{(T^{*2}T^2)T^{*2} - [T^*(T^*T)]^2\}x = 0 \ \text{ for } \ x \in clR(T^*).$$

Since $N(T) \subset N(T^*)$,

$$\{(T^{*2}T^2)T^{*2} - [T^*(T^*T)]^2\}y = 0 \text{ for y } \in N(T).$$

Thus

$$\{(T^{*2}T^2)T^{*2} - [T^*(T^*T)]^2\} = 0$$

or

$$\begin{aligned} T^2(T^{*2}T^2) &= [(T^*T)T]^2 \\ &= T^*T^2T^*T^2 \\ &= T^*T^2(T^*T)T \\ &= T^*(T^*T)T^3 \quad [T \text{is of class} \quad [2QN] \\ &= (T^{*2}T^2)T^2. \end{aligned}$$

This proves the result.

Theorem 2.14. *Let T be an operator of class $[2QN]$ with polar decomposition $T = U|T|$. If $N(T^*) \subset N(T)$, then the operator S with polar decomposition $U^2|T|$ is normal.*

Proof. It follows by Proposition 2.7 that T^2 is normal and $N(T^*) = N(T^{*2})$ and by Lemma 2.11 we have

$$N(T) = N(T^*). \tag{2.12}$$

As a consequence, U turns out to be normal and it is easy to verify that

$$|T|U|T|^2U^*|T| = |T|U^*|T|^2U|T|.$$

Since

$$N(|T|) = N(U) = N(U^*),$$

$$|T|U|T|^2U^* = |T|U^*|T|^2U$$

and hence

$$U|T|^2U^* = U^*|T|^2U.$$

Again by the normality of U , we have

$$U|T|U^* = U^*|T|U \tag{2.13}$$

Also $U^{*2}U^2 = U^*U$, showing U^2 to be normal partial isometry with $N(U^2) = N(|T|)$. Thus $U^2|T|$ is the polar decomposition Note that (2.13) the normality shows that U^2 and $|T|$ are commuting. Consequently

$$\begin{aligned} (U^2|T|)^*(U^2|T|) &= |T|U^{*2}U^2|T| \\ &= |T|U^2U^{*2}|T| \\ &= (U^2|T|)(U^2|T|)^*. \end{aligned}$$

This completes the proof.

Corollary 2.15. *If T is of class $[2QN]$ and $0 \notin W(T)$, then T is normal*

Proof. Since $0 \notin W(T)$ gives $N(T) = N(T^*) = \{0\}$ and so by our Proposition 2.7 , T^2 is normal. Then $[T^*T, TT^*] = 0$. Now the conclusion follows form [8].

Theorem 2.16. *Let T is of class $[2QN]$ such that $[T^*T, TT^*] = 0$. Then T^2 is quasinormal.*

Proof.

$$\begin{aligned}(T^{*2}T^2)T^2 &= T^*(T^*T)T^3\\ &= T^*T^2T^*T^2\\ &= (T^*T)(TT^*)T^2\\ &= (TT^*)(T^*T)T^2\\ &= TT^*T^2T^*T\\ &= T(T^*T)(TT^*)T\\ &= T(TT^*)(T^*T)T\\ &= T^2(T^{*2}T^2).\end{aligned}$$

This proves the result.

Theorem 2.17. *If T is of class $[2QN]$ and $[3QN]$ with $N(T) \subset N(T^*)$, then T is quasinormal.*

Proof.

$$\begin{aligned}T^{*3}(T^*T) &= T^*(T^*T)T^{*2} \quad [T \text{ is of class } [2QN]]\\ &= (T^*T)T^{*3}\end{aligned}$$

Hence

$$[T^{*2}T - T^*TT^*]T^{*2} = 0$$

or

$$T^2[T^*T^2 - TT^*T] = 0,$$

Since $N(T) \subset N(T^*), N(T) = N(T^2)$ and therefore

$$T[T^*T^2 - TT^*T] = 0, \text{ or } [T^{*2}T - T^*TT^*]T^* = 0.$$

Again by $N(T) \subset N(T^*)$,we get the desired conclusion.

Theorem 2.18. *If an operator T of class [2QN] is a 2-isometry , then it is an isometry.*

Proof. By the definition of a 2-isometry,

$$(T^{*2}T^2)(T^*T) - 2(T^*T)^2 + T^*T = 0.$$

Since T is of class $[2QN]$

$$T^{*2}(T^*T)T^2 - 2(T^*T)^2 + T^*T = 0,$$

that is

$$T^{*3}T^3 - 2(T^*T)^2 + T^*T = 0. \tag{2.14}$$

Also

$$T^*[T^{*2}T^2 - 2T^*T + I]T = 0$$

i.e.

$$T^{*3}T^3 - 2T^{*2}T^2 + T^*T = 0. \tag{2.15}$$

From (2-14) and (2-15) $T^{*2}T^2 = (T^*T)^2$ and hence

$$(T^*T)^2 - 2(T^*T) + I = T^{*2}T^2 - 2T^*T + I = (T^*T - I)^2 = 0$$

or

$$T^*T = I.$$

Theorem 2.19. *If An operator T is of class $[2QN]\cap[3QN]$ is an n-isometry, then T is an isometry.*

Proof. By the definition of n-isometry,

$T^{*n}T^nT^*T-\binom{n}{1}T^{*n-1}T^{n-1}T^*T+...+(-1)^{n-2}\binom{n}{n-2}T^{*2}T^2T^*T+(-1)^{n-1}\binom{n}{n-1}T^*TT^*T+(-1)^nT^*T=0.$

Since T is of class $[2QN]\cap[3QN,]$ we have by Proposition 2.6

$$T^{*n+1}T^{n+1}-\binom{n}{1}T^{*n}T^n+...+(-1)^{n-2}\binom{n}{n-2}T^{*3}T^3+(-1)^n\binom{n}{n-1}(T^*T)^2+(-1)^nT^*T=0. \tag{2.16}$$

Also

$$T^*[T^{*n}T^n-\binom{n}{1}T^{*n-1}T^{n-1}+...+(-1)^{n-1}\binom{n}{n-1}T^*T+(-1)^nI]T=0$$

i.e.

$$T^{*n+1}T^{n+1}-\binom{n}{1}T^{*n}T^n+....+(-1)^{n-1}\binom{n}{n-1}T^{*2}T^2+(-1)^nT^*T=0 \tag{2.17}$$

From (2.16) and (2.17) $T^{2*}T^2=(T^*T)^2$. Consequently $(T^*)^kT^k=(T^*T)^k,\ \forall\ k\in\mathbb{N}$, and hence

$$(T^*T)^n-\binom{n}{1}(T^*T)^{n-1}+....+(-1)^{n-1}\binom{n}{n-1}(T^*T)+(-1)^nI=0=(I-T^*T)^n.$$

This completes the proof.

Definition 2.20. *An operator $A\in\mathcal{L}(H)$ is said to be quasi-invertible if A has zero kernel and dense range.*

Definition 2.21. *([18]) Two operators S and T in $\mathcal{L}(H)$ are quasi-similar if there are quasi-invertible operators A and B in $\mathcal{L}(H)$ which satisfy the equations*

$$AS=TA \ \textit{and}\ \ BT=SB.$$

If M is a closed subspace of H,$H=M\oplus M^\perp$. If T is in $\mathcal{L}(H)$, then T can be written as a 2×2 matrix with operators entries,

$$T=\begin{pmatrix} W & X\\ Y & Z\end{pmatrix}$$

where $W\in\mathcal{L}(M)$, $X\in\mathcal{L}(M^\perp,M)$, $Y\in\mathcal{L}(M,M^\perp)$, and $Z\in\mathcal{L}(M^\perp)$ (cf. Conway [6]).

Proposition 2.22. *If S and T are quasi-similar n-power quasi-normal operators in $\mathcal{L}(H)$ such that $N(S)=N(T)$,$N(T)$ and $N(S)$ are reducing respectively for T and S, then $S_1=S|_{N(S)^\perp}$ and $T_1=T|_{N(T)^\perp}$ are quasi-similar n-power quasi-normal operators.*

Proof. Since S and T are quasi-similar, there exists quasi-invertible operators A and B such that $AS=TA$ and $SB=BT$. the $N(S)$ is invariant under both A and B . Thus the matrices of S,T,A and B with respect to decomposition $H=N(S)\oplus N(S)^\perp$ are

$$\begin{pmatrix} S_1 & O\\ O & O\end{pmatrix},\begin{pmatrix} T_1 & O\\ O & O\end{pmatrix},\begin{pmatrix} A_1 & O\\ A_2 & A_3\end{pmatrix},\begin{pmatrix} B_1 & O\\ B_2 & B_3\end{pmatrix}$$

respectively. It is easy to verify that the ranges of A_1 and B_1 are dense in $N(S)^\perp$. We now show that $N(A_1)=N(B_1)=\{0\}$.

Suppose that $x\in N(A_1)$. Then $TA(x\oplus 0)=0$. The equation $AS=TA$ implies that $x\in N(S_1)$. This implies that $x=0$, and so $N(A_1)=\{0\}$. Likewise $N(B_1)=$

$\{0\}$. Therefore A_1 and B_1 are quasi-invertible operators on $N(S)^{\perp}$ and equations $AS = TA$ and $SB = TB$ imply that $A_1S_1 = T_1A_1$ and $S_1B_1 = B_1T_1$. Hence S_1 and T_1 are quasi-similar. By a similar way as in [10, Proposition 2.1.(iv)] we can see that the operators S_1 and T_1 are n-power quasi-normal.

3. THE $(\mathbb{Z}^n)$ -CLASS OPERATORS

In this section we consider the class $(\mathbb{Z}^n_\alpha)$ of operators T satisfies

$$|T^nT^*T - T^*TT^n|^\alpha \leq c_\alpha^2(T-\lambda I)^{*n}(T-\lambda I)^n, \text{ for all } \lambda \in \mathbb{C}$$

and for a positive α. The motivation is due to S. Mecheri [13] who considered the class of operators T satisfying

$$|TT^* - T^*T|^\alpha \leq c_\alpha^2(T-\lambda I)^*(T-\lambda I)$$

and A. Uchiyama and T. Yoshino [19] who discussed the class of operators T satisfying

$$|TT^* - T^*T|^\alpha \leq c_\alpha^2(T-\lambda I)(T-\lambda I)^*.$$

Definition 3.1. *For $T \in \mathcal{L}(H)$ we say that T belongs to the class $(\mathbb{Z}^n_\alpha)$ for some $\alpha \geq 1$ if there is a positive number c_α such that*

$$|T^nT^*T - T^*T^{n+1}|^\alpha \leq c_\alpha^2(T-\lambda I)^{*n}(T-\lambda I)^n \text{ for all } \lambda \in \mathbb{C},$$

or equivalently, if there is a positive number c_α such that

$$\||T^nT^*T - T^*TT^n|^{\frac{\alpha}{2}}x\| \leq c_\alpha\|(T-\lambda I)^nx\|,$$

for all x in H and $\lambda \in \mathbb{C}$. Also, let

$$\mathbb{Z}^n = \bigcup_{\alpha\geq 1} \mathbb{Z}^n_\alpha.$$

Remark. *An operator T of class $[nQN]$, it is of class $(\mathbb{Z}^n)$.*

In the following examples we give an example of an operator not in the classes $\mathbb{Z}^n$,and an operator of these classes, which are not of class $[nQN]$.

Example 3.2. *If f is a sequence of complex numbers, $f = \langle f(0), f(1), f(2), \dots \rangle^T$. The p-Cesáro operators C_p acting on the Hilbert space l^2 of square-summable complex sequences f is defined by*

$$(C_pf)(k) = \frac{1}{(k+1)^p}\sum_{i=1}^{k} f(i) \quad \textit{for fixed real } p > 1 \quad \textit{and } k = 0, 1, 2, \dots.$$

These operators was studied extensively in [16] where it was shown, that these operators are bounded and $(C_p^*f)(k) = \sum_{i=k}^{\infty} \frac{1}{(i+1)^p} f(i).$

In matrix form, we have

$$C_p = \begin{pmatrix} 1 & 0 & 0 & \dots \\ (\frac{1}{2})^p & (\frac{1}{2})^p & 0 & \dots \\ (\frac{1}{3})^p & (\frac{1}{3})^p & (\frac{1}{3})^p & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We consider the sequence f defined as follows

$$f(0) = 1 \text{ and } f(k) = \prod_{j=1}^{k} \frac{j^p}{(1+j)^p - 1} \text{ for } k \geq 1.$$

In [16] it is verified that $f \in l^2$, is eigenvector for C_p associated with eigenvalue 1, so $f \in N(C_p - I)$, but $f \notin N(C_p^* - I)$. It follows that $\|(C_p - I)^n f)\| = 0$.

On the other hand, we have

$$(C_p^n C_p^* C_p - C_p^* C_p C_p^n) f = (C_p^n - I) C_p^* f \neq 0.$$

Hence, C_p is a bounded operator but not of classes $\mathbb{Z}^n$.

Example 3.3. *Let T be a weighted shift operator on l^2 with weights $\alpha_1 = 2, \alpha_k = 1$ for all $k \geq 2$. That is*

$$T_\alpha(x_1, x_2, x_3, ...) = (0, \alpha_1 x_1, \alpha_2 x_2, ...) \text{ and } T^*(x_1, x_2, ...) = (\alpha_1 x_2, \alpha_2 x_3, ...).$$

A simple computation shows that

$$(T^n T^* T - T^* T T^n)(x) = (0, 0, ..., 0, 6x_1, 0, 0, ...)$$

with $6x_1$ at the $(n+1)$th place.
Morover

$$(T^{*n} T^* T - T^* T T^{*n})(x) = (-6x_{n+1}, 0, 0, ...) \text{ and } |T^n T^* T - T^* T T^n|^2 x = (-36x_1, 0, 0, ...).$$

Therefore T is not of class $[nQN]$ and however T is of class $\mathbb{Z}_4^n \subseteq \mathbb{Z}^n$.

Lemma 3.4. *For each α, β such as $1 \leq \alpha \leq \beta$, we have $\mathbb{Z}_\alpha^n \subseteq \mathbb{Z}_\beta^n$.*

Proof.

$$\begin{aligned}
|T^n T^* T - T^* T^{n+1}|^\beta &= |T^n T^* T - T^* T^{n+1}|^{\frac{\alpha}{2}} |T^n T^* T - T^* T^{n+1}|^{\beta-\alpha} |T^n T^* T - T^* T^{n+1}|^{\frac{\alpha}{2}} \\
&\leq \|T^n T^* T - T^* T^{n+1}\|^{\beta-\alpha} |T^n T^* T - T^* T^{n+1}|^\alpha \\
&\leq (2\|T\|^{n+2})^{\beta-\alpha} c_\alpha^2 (T - \lambda I)^{*n} (T - \lambda I)^n \\
&= c_\beta^2 (T - \lambda I)^{*n} (T - \lambda I)^n
\end{aligned}$$

where

$$C_\beta^2 = (2\|T\|^{n+2})^{\beta-\alpha} c_\alpha^2.$$

There exists an Hilbert space H°: $H \subset H^\circ$, and an isometric *-homomorphism preserving order , i.e, for all $T, S \in \mathcal{L}(H)$ and $\lambda, \mu \in \mathbb{C}$, we have

Proposition 3.5. *([6],[13] Berberian technique) Let H be a complex Hilbert space. Then there exists a Hilbert space $H^\circ \supset H$ and a map*

$$\Phi : \mathcal{L}(H) \to \mathcal{L}(H^\circ) : T \longmapsto T^\circ$$

*satisfying: Φ is an *-isometric isomorphism preserving the order such that*

1. $\Phi(T^*) = \Phi(T)^*$.
2. $\Phi(\lambda T + \mu S) = \lambda \Phi(T) + \mu \Phi(S)$.
3. $\Phi(I_H) = I_{H^\circ}$.
4. $\Phi(TS) = \Phi(T)\Phi(S)$.
5. $\|\Phi(T)\| = \|T\|$.
6. $\Phi(T) \leq \Phi(S)$ *if* $T \leq S$.
7. $\sigma(\Phi(T)) = \sigma(T)$, $\sigma_a(T) = \sigma_a(\Phi(T)) = \sigma_p(\Phi(T))$.
8. *If T is a positive operator, then* $\Phi(T^\alpha) = |\Phi(T)|^\alpha$ *for all* $\alpha > 0$.

Lemma 3.6. *If an operator T is of class $[nQN]$, then $\Phi(T)$ is of class $[nQN]$.*

Lemma 3.7. *If an operator T is of class $\mathbb{Z}^n$, then $\Phi(T)$ is of class $\mathbb{Z}^n$.*

Proof. Since T is of class $\mathbb{Z}^n$, there exists $\alpha \geq 1$ and $c_\alpha > 0$ such that

$$|T^nT^*T - T^*T^{n+1}|^\alpha \leq c_\alpha^2(T-\lambda)^{*n}(T-\lambda I)^n \text{ for all } \lambda \in \mathbb{C}.$$

It follows from the properties of the map Φ that

$$\Phi(|T^nT^*T - T^*T^{n+1}|^\alpha) \leq \Phi(c_\alpha^2(T-\lambda)^{*n}(T-\lambda I)^n) \text{ for all } \lambda \in \mathbb{C}.$$

By the condition 8. above we have

$$\Phi(|T^nT^*T - T^*T^{n+11}|^\alpha) = |\Phi(|T^nT^*T - T^*T^{n+1}|)|^\alpha, \quad \text{for all } \alpha > 0.$$

Therefore

$$|\Phi(T)^n\Phi(T^*)\Phi(T) - \Phi(T^*)\Phi(T)^{n+1}| \leq \Phi(c_\alpha^2(T-\lambda)^{*n}(T-\lambda I)^n) \text{ for all } \lambda \in \mathbb{C}.$$

Hence $\Phi(T)$ is of class $\mathbb{Z}^n$.

Proposition 3.8. *Let T be a class $\mathbb{Z}^n$ operator and assume that there exists a subspace $\mathbb{M}$ that reduces T, then $T|\mathbb{M}$ is of class $\mathbb{Z}^n$ operator.*

Proof. Since T is of class $\mathbb{Z}^n$, there exists an integer $p \geq 1$ and $c_p > 0$ such that

$$\||T^nT^*T - T^*T^{n+1}|^{2^{p-1}}x\| \leq c_{2^p}\|(T-\lambda I)^n x\|, \text{ for all x } \in H, \text{ for all } \lambda \in \mathbb{C}.$$

$\mathbb{M}$ reduces T, T can be written respect to the composition $H = \mathbb{M} \oplus \mathbb{M}^\perp$ as follows:

$$T = \begin{pmatrix} A & O \\ O & B \end{pmatrix},$$

By a simple calculation we get

$$T^nT^*T - T^*T^{n+1} = \begin{pmatrix} A^nA^*A - A^*A^{n+1} & O \\ O & B^nB^*B - B^*B^{n+1} \end{pmatrix}$$

By the uniqueness of the square root, we obtain

$$|T^nT^*T - T^*T^{n+1}| = \begin{pmatrix} |A^nA^*A - A^*A^{n+1}| & O \\ O & |B^nB^*B - B^*B^{n+1}| \end{pmatrix}.$$

Now by iteration to the order 2^p, it results that

$$|T^nT^*T - T^*T^{n+1}|^{2^{p-1}} = \begin{pmatrix} |A^nA^*A - A^*A^{n+1}|^{2^{p-1}} & 0 \\ 0 & |B^nB^*B - B^*B^{n+1}|^{2^{p-1}} \end{pmatrix}.$$

Therefore for all $x \in \mathbb{M}$, we have

$$\||T^nT^*T - T^*T^{n+1}|^{2^{p-1}}x\| = \||A^nA^*A - A^*A^{n+1}|^{2^{p-1}}x\| \leq c_{2^p}\|(T-\lambda I)x\| = \|(A-\lambda I)^n x\|.$$

Hence A is of class $\mathbb{Z}_{2^p}^n \subset \mathbb{Z}^n$.

Theorem 3.9. *Let T of class $\mathbb{Z}^1$.*

(1) *If $\lambda \in \sigma_p(T)$, $\lambda \neq 0$, then $\overline{\lambda} \in \sigma_p(T^*)$, furthermore if $\lambda \neq \mu$, then E_λ (the proper subspace associated with λ) is orthogonal to E_μ.*
(2) *If $\lambda \in \sigma_a(T)$, then $\overline{\lambda} \in \sigma_a(T^*)$.*
(3) *$TT^*T - T^*T^2$ is not invertible.*

Proof.

(1) If $T \in \mathbb{Z}^1$, then $T \in \mathbb{Z}^1_\alpha$ for some $\alpha \geq 1$ and there exists a positive constant c_α such that

$$|TT^*T - T^*T^2|^\alpha \leq c_\alpha (T - \lambda I)^*(T - \lambda I) \text{ for all } \lambda \in \mathbb{C}.$$

As $Tx = \lambda x$ implies $|TT^*T - T^*T^2|^{\frac{\alpha}{2}} x = 0$ and $(TT^* - T^*T)x = 0$ and hence

$$\|(T - \lambda)^* x\| = \|(T - \lambda)x\|$$

$$\lambda \langle x \,| y\rangle = \langle \lambda x \,| y\rangle = \langle Tx \,| y\rangle = \langle x \,| T^* y\rangle = \langle x \,| \overline{\mu} y\rangle = \mu \langle x \,| y\rangle.$$

Hence

$$\langle x \,| y\rangle = 0.$$

(2) Let $\lambda \in \sigma_a(T)$ from the condition 7. above, we have

$$\sigma_a(T) = \sigma_a(\Phi(T)) = \sigma_p(\phi(T)).$$

Therefore $\lambda \in \sigma_p(\phi(T))$. By applying Lemma 3.7 and the above condition 1., we get

$$\overline{\lambda} \in \sigma_p(\Phi(T)^*) = \sigma_p(\Phi(T^*)).$$

(3) Let $T \in \mathbb{Z}^1$. then there exists an integer $p \geq 1$ and $c_p > 0$ such that

$$\| |TT^*T - T^*T^2|^{2^{p-1}} x\| \leq c_p^2 \|(T - \lambda I)x\|^2 \quad \text{for all} \quad x \in H \text{ and for all } \lambda \in \mathbb{C}.$$

It is know that $\sigma_a(T) \neq \emptyset$. If $\lambda \in \sigma_a(T)$, then there exists a normed sequence (x_m) in H such that $\|(T - \lambda I)x_m\| \longrightarrow 0$ as $m \longrightarrow \infty$. Then

$$(TT^*T - T^*T^2)x_m \longrightarrow 0 \text{ as } m \longrightarrow \infty$$

and so, $(TT^*T - T^*T^2)$ is not invertible.

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Mathematics Department , College of Science. Aljouf University Aljouf 2014. Saudi Arabia
E-mail address: sidahmed@ju.edu.sa