Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 3 Issue 2(2011), Pages 102-109.

EXISTENCE OF SOLUTIONS OF MIXED VECTOR VARIATIONAL-LIKE INEQUALITIES

(COMMUNICATED BY NASEER SHAHZAD)

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Abstract. In this paper, we consider mixed vector variational-like inequality problems in the setting of topological vector spaces. We extend the concept of upper sign continuity for vector-valued mappings. Further, by exploiting KKM-Fan lemma, we establish some existence results for solutions of mixed vector variational-like inequality problems and show that the solution sets of these problems are compact.

1. Introduction

Variational inequalities were introduced and studied by Stampacchia [11] in early sixties. It has been shown that a wide class of linear and nonlinear problems arising in various branches of mathematical and engineering sciences can be studied in the unified and general framework of variational inequalities. Variational inequalities have been generalized and extended in several directions using new techniques. As a useful and important branch of variational inequality theory, vector variational inequalities were initially introduced and studied by Giannessi [5] in a finite-dimensional Euclidean space in 1980. Since then it becomes a powerful tool in the study of vector optimization and traffic equilibrium problems; (see Refs.[6, 9, 10]. Due to its wide range of applications, the vector variational inequality has been generalized in different directions and the existence results and algorithms for a number of classes of vector variational inequality problems have been established under various conditions; For details, we refer to [3, 4, 5, 6, 8, 10, 12, 13, 14] and references therein.

Recently, Fang and Huang [3, 8] considered generalized vector variational inequality problems for a fixed cone. They provided the existence of solutions of these problems under different kinds of pseudomonotonicity and hemicontinuity conditions. They have also provided applications of these problems to vector f-complementarity problems.

2000 *Mathematics Subject Classification.* 49J40, 47H10, 47H17.

Key words and phrases. Mixed vector variational-like inequalities; KKM mapping; P_x-η-pseudomonotonicity; P_x-η-upper sign continuity.

Submitted September 9, 2010. Accepted April 24, 2011.

Inspired and motivated by the recent research activities going on in this direction, we introduce the concept of P_x-η-upper sign continuity which extend the previous concept of upper sign continuity introduced by Hadjisavvas [7] and considered two classes of generalized vector variational-like inequalities. We investigate the solvability of vector variational-like inequalities by means of the KKM-Fan lemma under P_x-η-upper sign continuity with or without pseudomonotonicity assumptions. The results presented in this paper generalize the results given in [3, 4, 7, 8, 13] and enrich the theory of the vector variational inequality.

2. Preliminaries

Let X and Y be two topological vector spaces, $K \subset X$ be a nonempty and convex subset of X. Let $P \subset Y$ be a closed convex and pointed cone with apex at the origin and let $P : K \to 2^Y$ be a set-valued mapping such that for each $x \in K$, $P(x)$ is a proper, closed, convex cone with $\operatorname{int} P(x) \neq \emptyset$, where $\operatorname{int} P(x)$ denotes the interior of $P(x)$. Let $L(X, Y)$ denote the space of all continuous linear mappings from X into Y and $\langle t, x\rangle$ the evaluation of $t \in L(X, Y)$ at $x \in X$. Let $f : K \times K \to Y$ and $\eta : K \times K \to X$ be two bi-mappings.

We consider the following *mixed vector variational-like inequality problem* (in short, MVVLIP): Find $x \in K$ such that

$$\langle Tx, \eta(y, x)\rangle + f(y, x) \notin -\operatorname{int} P(x), \ \forall y \in K. \tag{2.1}$$

The set of solutions of MVVLIP is denoted by S_1.

Another problem which is closely related to MVVLIP is the following *Minty-type mixed vector variational-like inequality problem* (in short, MMVVLIP): Find $x \in K$ such that

$$\langle Ty, \eta(y, x)\rangle + f(y, x) \notin -\operatorname{int} P(x), \ \forall y \in K. \tag{2.2}$$

We denote the set of solutions of MMVVLIP by S_2.

Throughout the paper, unless otherwise specified, let $P_- = \bigcap_{x\in K} P(x)$ is a proper, closed, solid and convex cone in Y. Now, we recall the following concepts and results which are needed in the sequel.

Definition 2.1. *A mapping $f : K \to Y$ is said to be*

(i) *P_--convex, if $f(tx+(1-t)y) \leq_{P_-} tf(x)+(1-t)f(y), \ \forall x, y \in K, \ t \in [0, 1]$;*
(ii) *P_--concave, if $-f$ is P_--convex.*

Definition 2.2. *A mapping $T : K \to L(X, Y)$ is said to be η-hemicontinuous, if for any $x, y \in K$, the mapping $t \to \langle T(x + t(y - x)), \eta(y, x)\rangle$ is continuous at 0^+.*

Definition 2.3. *A mapping $T : K \to L(X, Y)$ is said to be P_x-η-pseudomonotone with respect to f, if for any $x \in K$*

$$\langle Tx, \eta(y, x)\rangle + f(y, x) \notin -int P(x) \Rightarrow \langle Ty, \eta(y, x)\rangle + f(y, x) \notin -int P(x), \ \forall y \in K.$$

Example 2.4. *Let $X=\mathbb{R}$, $K=\mathbb{R}_+$, $Y=\mathbb{R}^2$, $P(x)=\mathbb{R}^2_+$, $\eta(y, x) = y - x$, for all $x, y \in K$ and*

$$T(x) = \begin{pmatrix} 0 \\ 1.5 + \sin x \end{pmatrix} \text{ and } f(y, x) = \begin{pmatrix} y - x \\ y - x \end{pmatrix} \ \forall x, y \in K.$$

Now

$$\langle Tx, \eta(y,x)\rangle + f(y,x) = \begin{pmatrix} y-x \\ (2.5+\sin x)(y-x) \end{pmatrix} \notin -int P(x),$$

we have $y \geq x$. *It follows that*

$$\langle Ty, \eta(y,x)\rangle + f(y,x) = \begin{pmatrix} y-x \\ (2.5+\sin y)(y-x) \end{pmatrix} \notin -int P(x).$$

So, T *is* P_x-η-*pseudomonotone with respect to* f.

Definition 2.5. *A mapping* $T : K \to L(X,Y)$ *is said to be* P_x-η-*upper sign continuous with respect to* f, *if for any* $x, y \in K$ *and* $t \in]0,1[$

$$\langle T(x+t(y-x)), \eta(y,x)\rangle + f(y,x) \notin -int P(x),\ \forall t \in]0,1[$$
$$\Rightarrow \langle Tx, \eta(y,x)\rangle + f(y,x) \notin -int P(x).$$

Remark.

(i) *It is remarked that the concept of upper sign continuity for set-valued vector valued mappings is extended and used in proving the existence of solutions to some generalized classes of vector variational-like Inequalities ; See for details* [4].

(ii) *For* $f \equiv 0$, *it is easy to see that the* η-*hemicontinuity of* T *implies* P_x-η-*upper sign continuity of* T. *If* X=Y=$\mathbb{R}$, K=$P(x)$=$[0,\infty)$ *and* $f \equiv 0$, $\eta(y,x)$=$y-x$, *for all* $x, y \in K$, *then any positive mapping* $T : K \to L(X,Y) = \mathbb{R}$ *is* P_x-η-*upper sign continuous while it is not hemicontinuous. In this case, the concept of* P_x-η-*upper sign continuity reduces to upper sign continuity introduced by Hadjisavvas* [7].

Definition 2.6. [14]. *Let* K *be a nonempty subset of a topological space* X. *A set-valued mapping* $\Gamma : K \to 2^K$ *is said to be transfer closed-valued on* K, *if for all* $x \in K, y \notin \Gamma(x)$ *implies that there exists a point* $x' \in K$ *such that* $y \notin cl_K\Gamma(x')$, *where* $cl_K\Gamma(x)$ *denotes the closure of* $\Gamma(x) \subset K$. *It is clear that this definition is equivalent to:*

$$\bigcap_{x\in K} cl_K\Gamma(x) = \bigcap_{x\in K} \Gamma(x).$$

Definition 2.7. *Let* X *and* Y *be two topological vector spaces. A set-valued mapping* $T : X \to 2^Y$ *is said to be:*

(i) *upper semi-continuous at* $x \in X$, *if for each open set* V *containing* $T(x)$, *there is an open set* U *containing* x *such that for all* $t \in U$, $T(t) \in V$ *and* T *is said to be upper semi-continuous on* X, *if it is upper semi-continuous at every point* $x \in X$;

(ii) *closed, if the graph* $G_r(T) = \{(x,y) \in X \times Y : x \in X,\ y \in T(x)\}$ *of* T *is a closed set;*

(iii) *compact, if the closure of range* T, *that is,* $clT(X)$ *is compact, where* $T(X) = \bigcup_{x\in X} T(x)$.

Definition 2.8. *Let* K_0 *be a nonempty subset of* K. *A set-valued mapping* $\Gamma : K_0 \to 2^K$ *is said to be a KKM mapping, if* $coA \subseteq \bigcup_{x\in A} \Gamma(x)$ *for very finite subset* A *of* K_0, *where co denotes the convex hull.*

Lemma 2.9. [3]. *Let K be a nonempty subset of a topological vector space X and $\Gamma : K \to 2^X$ be a KKM mapping with closed values. Assume that there exist a nonempty compact convex subset $D \subseteq K$ such that $B = \bigcap_{x\in D} \Gamma(x)$ is compact. Then $\bigcap_{x\in K} \Gamma(x) \neq \emptyset$.*

3. Existence of Solutions of MVVLIP

In order to establish existence results for the solutions of MVVLIP, we prove the following lemma.

Lemma 3.1. *Let $K \subset X$ be a nonempty and convex subset of X. Let $f : K \times K \to Y$ and $\eta : K \times K \to X$ be two bi-mappings. Suppose following conditions hold:*

(i) *$f : K \times K \to Y$ is P_--convex in first argument with the condition $f(x,x) = 0,\ \forall x \in K$;*
(ii) *$\eta : K \times K \to X$ is an affine mapping in first argument with the condition $\eta(x,x) = 0,\ \forall x \in K$;*
(iii) *$T : K \to L(X,Y)$ is P_x-η-upper sign continuous and P_x-η-pseudomonotone mapping with respect to f.*

Then, the solution sets of MVVLIP and MMVVLIP are equivalent.

Proof. By P_x-η-pseudomonotonicity of T with respect to f, every solution of MVVLIP is a solution of MMVLIP.

Conversely, let $x \in K$ be the solution of MMVVLIP. For any given $y \in K$, we know that $y_t = ty + (1-t)x \in K,\ \forall t \in]0,1[$, as K is convex. Since $x \in K$ is a solution of MMVVLIP, so for each $x \in K$, it follows that

$$\langle Ty_t, \eta(y_t,x)\rangle + f(y_t,x) \notin -\mathrm{int}\, P(x). \tag{3.1}$$

Since f is P_--convex in first argument and $f(x,x) = 0,\ \forall x \in K$, therefore

$$f(y_t,x) \leq_{P_-} tf(y,x) + (1-t)f(x,x) = tf(y,x). \tag{3.2}$$

From assumption (ii) on η, we have

$$\begin{aligned}\langle Ty_t, \eta(y_t,x)\rangle &= \langle Ty_t, \eta(ty + (1-t)x, x)\rangle. \\ &= t\langle Ty_t, \eta(y,x)\rangle + (1-t)\langle Ty_t, \eta(x,x)\rangle \\ &= t\langle Ty_t, \eta(y,x)\rangle.\end{aligned} \tag{3.3}$$

From inclusions (3.1)-(3.3), we have

$$t\langle Ty_t, \eta(y,x)\rangle + tf(y,x) \notin -\mathrm{int}\, P(x).$$

Since $Y \backslash \{-\mathrm{int}\ P(x)\}$ is closed, therefore $\forall t \in]0,1]$, we have

$$\langle Ty_t, \eta(y,x)\rangle + f(y,x) \notin -\mathrm{int}\, P(x).$$

From P_x-η-upper sign continuity of T with respect to f, we get

$$\langle Tx, \eta(y,x)\rangle\ + f(y,x) \notin -\mathrm{int}\, P(x),\ \forall y \in K.$$

Therefore, $x \in K$ is solution of MVVLIP. This completes the proof. □

We now establish an existence result for MVVLIP under P_x-η-upper sign continuity.

Theorem 3.2. *Let $K \subset X$ be a nonempty and convex subset of X. Let $f : K\times K \to Y$ and $\eta : K \times K \to X$ be two bi-mappings. Suppose following conditions hold:*

(i) $f : K \times K \to Y$ *is* P_-*-convex in first argument with the condition* $f(x,x) = 0,\ \forall x \in K$*;*
(ii) $\eta : K \times K \to X$ *is an affine mapping in first argument with the condition* $\eta(x,x) = 0,\ \forall x \in K$*;*
(iii) *The set-valued mapping* $y \mapsto \{x \in K : \langle Ty, \eta(y,x)\rangle + f(y,x) \notin -int\, P(x)\}$ *is transfer closed-valued on* K*;*
(iv) $T : K \to L(X,Y)$ *is* P_x*-*η*-upper sign continuous and* P_x*-*η*-pseudomonotone mapping with respect to* f*.*
(v) *There exist compact subset* $B \subseteq K$ *and compact convex subset* $D \subseteq K$ *such that* $\forall x \in K \backslash B, \exists\ y \in D$ *such that* $\langle Ty, \eta(y,x)\rangle + f(y,x) \in -int\, P(x)$*.*

Then, the solution set S_1 *of MVVLIP is nonempty and compact.*

Proof. Define a set-valued mapping $\Gamma : K \to 2^K$ as follows:

$$\Gamma(y) = \{x \in K : \langle Ty, \eta(y,x)\rangle + f(y,x) \notin -\text{int}\, P(x)\},\ \forall y \in K.$$

We claim that Γ is a KKM mapping. If this is not true, then there exist a finite set $\{y_1, ..., y_n\} \subset K$ and $z \in \text{co}(\{y_1, ..., y_n\})$ such that $z \notin \bigcup_{i=1}^{n} \Gamma(y_i)$. Then

$$\langle T(y_i), \eta(y_i, z)\rangle + f(y_i, z) \in -\text{int}\, P(z),\ i = 1, ..., n.$$

Since T is P_x-η-pseudomonotone with respect to f, we have

$$\langle Tz, \eta(y_i, z)\rangle + f(y_i, z) \in -\text{int}\, P(z),\ i = 1, ..., n. \tag{3.4}$$

For each $i = 1, ..., n$, let $t_i \in]0,1[$ with $\sum_{i=1}^{n} t_i = 1$. Multiplying inclusion (3.4) by t_i and summing, we obtain

$$\sum_{i=1}^{n} t_i \langle Tz, \eta(y_i, z)\rangle + \sum_{i=1}^{n} t_i f(y_i, z) \in -\text{int}\, P(z),\ i = 1, ..., n. \tag{3.5}$$

From assumptions (i) and (ii), inclusion (3.5) becomes

$$\langle Tz, \eta(\sum_{i=1}^{n} t_i y_i, z)\rangle + f(\sum_{i=1}^{n} t_i y_i, z) \in -\text{int}\, P(z),$$

and thus, $0 = \langle Tz, \eta(z,z)\rangle + f(z,z) \in -\text{int}\, P(z)$, which leads a contradiction to our assumption that $P(z) \neq Y$. Thus our claim is verified. So Γ is a KKM mapping.

From the assumption (v),

$$cl_K(\bigcap_{y \in D} \Gamma(y)) \subseteq B.$$

Consequently, set-valued mapping $cl\Gamma : K \to 2^K$ satisfies all the conditions of Lemma 2.9 and so

$$\bigcap_{x \in K} \Gamma(x) \neq \emptyset.$$

By condition (iii), we get

$$S_1 = \bigcap_{x \in K} cl\Gamma(x) = \bigcap_{x \in K} \Gamma(x),$$

which implies that the solution set S_2 of MMVVLIP is nonempty. Moreover, since T is P_x-η-upper sign continuous with respect to f and $f(.,y)$ is P_--convex, by using Lemma 3.1, we get

$$S_1 = \bigcap_{y\in K} \Gamma(y) = \bigcap_{y\in K} \{x \in K : \langle Tx, \eta(y,x)\rangle + f(y,x) \notin -\text{int}\, P(x)\}.$$

This and conditions (iii) and (v) imply that the solution set of MVVLIP is a nonempty and compact set of B. This completes the proof. □

Example 3.3. *Let $X=\mathbb{R}$, $K=[0,1]$, $Y=\mathbb{R}^2$ and $P(x)=P=\{(u,v)\in\mathbb{R}^2 : u\geq 0, v\geq 0\}$ for all $x\in K$, be a fixed closed convex cone in Y. Let us define $T(x)(t) = \langle T(x), t\rangle = t(x,x^2)$, $\eta(y,x) = y-x$, $\forall x,y\in K$ and $f\equiv 0$, for all $x\in K$ and $t\in X$. Then, f is P_--convex and T is P_x-η-pseudomonotone and P_x-η-upper sign continuous with respect to f and $\langle T(x),\eta(y,x)\rangle + f(y,x) = (y-x)(x,x^2) = ((y-x)x,(y-x)x^2)$. It is easy to see that the set $\{x\in K : \langle T(y),\eta(y,x) \notin -int\, P(x)\} = [0,y]$ is closed and so the mapping $y\mapsto \{x\in K : \langle T(y),\eta(y,x)\rangle \notin -int\, P(x)\}$ is transfer closed valued on K. Since K is compact, condition (v) of Theorem 3.2 trivially holds. Therefore, T satisfies all the assumptions of Theorem 3.2 and so the solution set of MVVLIP is nonempty and compact. It is clear that only $x=0$ satisfies the following relation*

$$\langle T(x),\eta(y,x)\rangle \notin -int\, P(x),\ \forall y\in K.$$

Similarly, only $x=0$ satisfies the following relation

$$\langle T(y),\eta(y,x)\rangle \notin -int\, P(x),\ \forall y\in K.$$

Hence the solution sets of MVVLIP and MMVVLIP are equal to the singleton set $\{0\}$.

Remark.

(a) *If X is a real reflexive Banach space and K is a nonempty, bounded, closed and convex subset of X, then K is weakly compact. In this case, condition (v) of Theorem 3.2 can be removed.*

(b) *It is obvious that if $f(y,.)$ is continuous and the set-valued mapping $W(x) = Y\backslash(-int\, P(x))$ for all $x\in K$, is closed, then condition (iii) of Theorem 3.2 trivially holds.*

Now we prove the existence of a solutions of MVVLIP without any kind of pseudomonotonicty assumption.

Theorem 3.4. *Let K, X, Y and P be the same as in Theorem 3.2 and let $f : K\times K\to Y$ and $\eta : K\times K\to X$ are two bi-mappings with the conditions $f(x,x) = 0$, $\eta(x,x)=0$, $\forall x\in K$. Assume that the set-valued mapping $T : K\to 2^K$ satisfies the following conditions:*

(i) *for all $y\in K$, the set $\{x\in K : \langle Tx,\eta(y,x)\rangle + f(y,x) \notin -int\, P(x)\}$ is P_--convex;*

(ii) *The set-valued mapping $y\mapsto \{x\in K : \langle Tx,\eta(y,x)\rangle + f(y,x) \notin -int\, P(x)\}$ is transfer closed-valued on K;*

(iii) *There exist compact subset $B\subseteq K$ and compact convex subset $D\subseteq K$ such that $\forall x\in K\backslash B, \exists\, y\in D$ such that $\langle Tx,\eta(y,x)\rangle + f(y,x) \in -int\, P(x)$.*

Then the solution set S_1 of MVVLIP is nonempty and compact.

Proof. For all $y \in K$, define $\Gamma : K \to 2^K$ as

$$\Gamma(y) = \{x \in K : \langle T(x), \eta(y,x)\rangle + f(y,x) \notin -\mathrm{int}\, P(x)\}.$$

By the same argument as in the proof of Theorem 3.2, it is easy to see that $cl_K\Gamma$, satisfies all the conditions of Lemma 2.9, hence $\bigcap_{x\in K} cl_K\Gamma(x) \neq \emptyset$. Since $S_1 = \bigcap_{x\in K} \Gamma(x)$, condition (ii) implies that S_1 is nonempty and again by conditions (ii) and (iii), S_1 is compact. □

Remark. *Condition (ii) of Theorem 3.4 holds when $f(y,.)$ is continuous and the mapping $W(x) = Y\backslash(-int\, P(x))$ is closed.*

Example 3.5. *Let X=Y=$\mathbb{R}$, $K = [0,1]$, and $P(x)$=$[0,\infty)$, for all $x \in K$. Also $\eta(y,x) = y - x$, for all $x, y \in K$ and $f(y,x) = y - x$, for all $x, y \in K$. Let us define $T : K \to L(X,Y) = \mathbb{R}$ by*

$$T(x) = \begin{cases} 1, & \text{if } x \text{ rational}, \\ 0, & \text{if } x \text{ irrational}. \end{cases}$$

It is easy to see that T is P_x-η-upper sign continuous with respect to f (note that T is a non-negative mapping and f is continuous) while T is not upper semicontinuous (if x is an irrational number and $\{x_n\}$ is a sequence of rational numbers in [0,1], then the relation $\lim \sup T(x_n) \leq T(x)$ *does not hold). For all $y \in K$ we have*

$$\{x \in K : \langle T(x), \eta(y,x) + f(y,x) \notin -int\, P(x)\} = [0,y]$$

is closed and convex. Then T satisfies all the conditions of Theorem 3.4 and so the solution set of MVVLIP is nonempty and compact. We claim that the solution set of MVVLIP is the singleton set $\{0\}$.
If x is a rational number belongs to [0,1] and a solution, then the following relation does not hold.

$$\langle Tx, \eta(y,x)\rangle + f(y,x) = f(y,x) = y - x \notin -int\, P(x),\ \forall y \in K = [0,1].$$

Similarly, if $x \in]0,1]$ is a rational number then the previous relation also does not hold. Finally, if $x = 0$, then

$$\langle Tx, \eta(y,x)\rangle + f(y,x) = 2y \notin -int\, P(x),\ \text{for all } y \in K = [0,1] \text{ holds.}$$

Similarly, we can easily see that the solution set of MMVVLIP is the singleton set $\{0\}$.

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

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