Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 3 Issue 2(2011), Pages 118-122.

THE HIGHER DUALS OF A BANACH ALGEBRA INDUCED BY A BOUNDED LINEAR FUNCTIONAL

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Abstract. Let A be a Banach space and let $\varphi \in A^*$ be non-zero with $\|\varphi\| \leq 1$. The product $a \cdot b = \langle \varphi, a \rangle\, b$ makes A into a Banach algebra. We denote it by ${}_{\varphi}A$. Some of the properties of ${}_{\varphi}A$ such as Arens regularity, n-weak amenability and semi-simplicity are investigated.

1. Introduction

This paper has its genesis in a simple example of Zhang [10, Page 507]. For an infinite set S he equipped $l^1(S)$ with the algebra product $a \cdot b = a(s_0)b\ \ (a, b \in l^1(S))$, where s_0 is a fixed element of S. He used this as a Banach algebra which is $(2n-1)-$weakly amenable but is not $(2n)-$weakly amenable for any $n \in \mathbb{N}$. Here we study a more general form of this example. Indeed, we equip a non-trivial product on a general Banach space turning it to a Banach algebra. It can serve as a source of (counter-)examples for various purposes in functional analysis.

Let A be a Banach space and fix a non-zero $\varphi \in A^*$ with $\|\varphi\| \leq 1$. Then the product $a \cdot b = \langle \varphi, a \rangle b$ turning A into a Banach algebra which will be denoted by ${}_{\varphi}A$. Some properties of algebras of this type are investigated in [5, 4, 1, 7]. Trivially ${}_{\varphi}A$ has a left identity (indeed, every $e \in {}_{\varphi}A$ with $\langle \varphi, e \rangle = 1$ is a left identity), while it has no bounded approximate identity in the case where $\dim(A) \geq 2$. Now the Zhang's example can be interpreted as an special case of ours. Indeed, he studied ${}_{\varphi_{s_0}}l^1(S)$, where $\varphi_{s_0} \in l^\infty(S)$ is the characteristic function at s_0. Here, among other things, we focus on the higher duals of ${}_{\varphi}A$ and investigate various notions of amenability for ${}_{\varphi}A$. In particular, we prove that for every $n \in \mathbb{N}$, ${}_{\varphi}A$ is $(2n-1)-$weakly amenable but it is not $(2n)-$weakly amenable for any n, in the case where $\dim(\ker \varphi) \geq 2$.

2010 *Mathematics Subject Classification.* 46H20, 46H25.
Key words and phrases. Arens product; n-weak amenability; semi-simplicity.

Submitted February 9, 2011. Accepted April, 30, 2011.

2. The results

Before we proceed for the main results we need some preliminaries. As we shall be concerned with the Arens products $\Box$ and $\Diamond$ on the bidual A^{**} of a Banach algebra A, let us introduce these products.

Let $a, b \in A, f \in A^*$ and $m, n \in A^{**}$.

$$\begin{array}{ll} \langle f \cdot a, b\rangle = \langle f, ab\rangle & \langle b, a \cdot f\rangle = \langle ba, f\rangle \\ \langle n \cdot f, a\rangle = \langle n, f \cdot a\rangle & \langle a, f \cdot n\rangle = \langle a \cdot f, n\rangle \\ \langle m\Box n, f\rangle = \langle m, n \cdot f\rangle & \langle f, m\Diamond n\rangle = \langle f \cdot m, n\rangle. \end{array}$$

If $\Box$ and $\Diamond$ coincide on the whole of A^{**} then A is called Arens regular. For the brevity of notation we use the same symbol " $\cdot$ " for the various module operations linking A, such as A^*, A^{**} and also as well for the n^{th} dual $A^{(n)}$, $(n \in \mathbb{N})$. The main properties of these products and various $A-$module operations are detailed in [2]; see also [9].

A derivation from a Banach algebra A to a Banach A-module X is a bounded linear mapping $D : A \to X$ such that $D(ab) = D(a)b + aD(b) \quad (a, b \in A)$. For each $x \in X$ the mapping $\delta_x : a \to ax - xa$, $(a \in A)$ is a bounded derivation, called an inner derivation. The concept of n-weak amenability was introduced and intensively studied by Dales *et al.* [3]. A Banach algebra $\mathcal{A}$ is said to be n-weakly amenable $(n \in \mathbb{N})$ if every derivation from $\mathcal{A}$ into $\mathcal{A}^{(n)}$ is inner. Trivially, 1-weak amenability is nothing else than weak amenability. A derivation $D : A \to A^*$ is called cyclic if $\langle D(a), b\rangle + \langle D(b), a\rangle = 0$ $(a, b \in A)$. If every bounded cyclic derivation from A to A^* is inner then A is called cyclicly amenable which was studied by Grønbaek [8]. Throughout the paper we usually identify an element of a space with its canonical image in its second dual.

Now we come to ${}_{\varphi}A$. A direct verification reveals that for $a \in A, f \in ({}_{\varphi}A)^*$ and $m, n \in ({}_{\varphi}A)^{**}$,

$$\begin{array}{ll} f \cdot a = \langle \varphi, a\rangle f & a \cdot f = \langle f, a\rangle \varphi \\ n \cdot f = \langle n, f\rangle \varphi & f \cdot n = \langle n, \varphi\rangle f \\ m\Box n = \langle m, \varphi\rangle n & m\Diamond n = \langle m, \varphi\rangle n. \end{array}$$

The same calculation gives the ${}_{\varphi}A-$module operations of $({}_{\varphi}A)^{(2n-1)}$ and $({}_{\varphi}A)^{(2n)}$ as follows,

$$\begin{array}{lll} F \cdot a = \langle \varphi, a\rangle F & a \cdot F = \langle F, a\rangle \varphi & (F \in ({}_{\varphi}A)^{(2n-1)}) \\ G \cdot a = \langle G, \varphi\rangle a & a \cdot G = \langle \varphi, a\rangle G & (G \in ({}_{\varphi}A)^{(2n)}). \end{array}$$

We commence with the next straightforward result, most parts of which are based on the latter observations on the various duals of ${}_{\varphi}A$.

Proposition 2.1. *(i)* *${}_{\varphi}A$ is Arens regular and $({}_{\varphi}A)^{**} = {}_{\varphi}(A^{**})$. Furthermore, for each $n \in \mathbb{N}$, $({}_{\varphi}A)^{(2n)}$ is Arens regular.*

(ii) *$({}_{\varphi}A)^{**} \cdot {}_{\varphi}A = {}_{\varphi}A$ and ${}_{\varphi}A \cdot ({}_{\varphi}A)^{**} = ({}_{\varphi}A)^{**}$; in particular, ${}_{\varphi}A$ is a left ideal of $({}_{\varphi}A)^{**}$.*

(iii) *$({}_{\varphi}A)^* \cdot {}_{\varphi}A = ({}_{\varphi}A)^*$ and ${}_{\varphi}A \cdot ({}_{\varphi}A)^* = \mathbb{C}\varphi$.*

As ${}_{\varphi}A$ has no approximate identity, in general, it is not amenable. The next result investigates $n-$weak amenability of ${}_{\varphi}A$.

Theorem 2.2. *For each* $n \in \mathbb{N}$, ${}_{\varphi}A$ *is* $(2n-1)$*-weakly amenable, while in the case where* $\dim(\ker\varphi) \geq 2$, ${}_{\varphi}A$ *is not* $(2n)-$*weakly amenable for any* $n \in \mathbb{N}$.

Proof. Let $D: {}_{\varphi}A \to ({}_{\varphi}A)^{(2n-1)}$ be a derivation and let $a, b \in {}_{\varphi}A$. Then

$$\langle\varphi, a\rangle D(b) = D(ab) = D(a)b + aD(b) = \langle\varphi, b\rangle D(a) + \langle D(b), a\rangle\varphi.$$

It follows that $\langle\varphi, a\rangle\langle D(b), a\rangle = \langle\varphi, b\rangle\langle D(a), a\rangle + \langle\varphi, a\rangle\langle D(b), a\rangle$, from which we have $\langle D(a), a\rangle = 0$, or equivalently, $\langle D(a+b), a+b\rangle = 0$. Therefore $\langle D(a), b\rangle = -\langle D(b), a\rangle$. Now with e as a left identity for ${}_{\varphi}A$ we have

$$D(b) = D(eb) = \langle\varphi, b\rangle D(e) + \langle D(b), e\rangle\varphi = \langle\varphi, b\rangle D(e) - \langle D(e), b\rangle\varphi = \delta_{-D(e)}(b).$$

Therefore D is inner, as required.

To prove that ${}_{\varphi}A$ is not $(2n)-$weakly amenable for any $n \in \mathbb{N}$, it is enough to show that ${}_{\varphi}A$ is not 2-weakly amenable, [3, Proposition 1.2]. To this end let $f \in ({}_{\varphi}A)^*$ be such that f and φ are linearly independent. It follows that $\langle f, a_0\rangle = \langle\varphi, b_0\rangle = 0$ and $\langle f, b_0\rangle = \langle\varphi, a_0\rangle = 1$, for some $a_0, b_0 \in {}_{\varphi}A$. Define $D: {}_{\varphi}A \to ({}_{\varphi}A)^{**}$ by $D(a) = \langle f - \varphi, a\rangle b_0$, then D is a derivation. If there exists $m \in ({}_{\varphi}A)^{**}$ with $D(a) = am - ma$ $(a \in {}_{\varphi}A)$, then by taking $a = b_0$, we obtain $b_0 = -\langle m, \varphi\rangle b_0$ which follows that $1 = -\langle m, \varphi\rangle$. Now if $a \in \ker\varphi$, then $\langle f, a\rangle b_0 = -\langle m, \varphi\rangle a = a$. It follows that $\dim(\ker\varphi) = 1$ that is a contradiction. □

As an immediate consequence of Theorem 2.2 we obtain the result of Zhang mentioned in the introduction.

Corollary 2.3 ([10, Page 507])**.** *For each* $n \in \mathbb{N}$, ${}_{\varphi_{s_0}}l^1(S)$ *is* $(2n-1)-$*weakly amenable, while it is not* $(2n)-$*weakly amenable for any* $n \in \mathbb{N}$.

Proposition 2.4. *A bounded linear map* $D: {}_{\varphi}A \to ({}_{\varphi}A)^{(2n)}, (n \in \mathbb{N})$, *is a derivation if and only if* $D({}_{\varphi}A) \subseteq \ker\varphi$.

Proof. A direct verification shows that $D: {}_{\varphi}A \to ({}_{\varphi}A)^{(2n)}$ is a derivation if and only if

$$\langle\varphi, a\rangle D(b) = D(ab) = D(a)b + aD(b) = \langle D(a), \varphi\rangle b + \langle\varphi, a\rangle D(b) \qquad (a, b \in {}_{\varphi}A).$$

And this is equivalent to $\langle D(a), \varphi\rangle = 0$, $(a \in {}_{\varphi}A)$; that is $D({}_{\varphi}A) \subseteq \ker\varphi$. Note that here φ is assumed to be an element of $({}_{\varphi}A)^{(2n+1)}$. □

The next results demonstrates that in contrast to Theorem 2.2, ${}_{\varphi}A$ is $(2n)-$weakly amenable in the case where $\dim(\ker\varphi) < 2$.

Proposition 2.5. *If* $\dim(\ker\varphi) < 2$ *then* ${}_{\varphi}A$ *is* $(2n)-$*weakly amenable for each* $n \in \mathbb{N}$.

Proof. The only reasonable case that we need to verify is $\dim(\ker\varphi) = 1$. In this case we have $\dim(A) = 2$. Therefore one may assume that A is generated by two elements $e, a \in A$ such that $\langle\varphi, e\rangle = 1$ and $\langle\varphi, a\rangle = 0$. Let $f \in ({}_{\varphi}A)^*$ satisfy $\langle f, e\rangle = 0$ and $\langle f, a\rangle = 1$. Then f and φ are linearly independent and generate A^*; indeed, every non-trivial element $g \in A^*$ has the form $g = \langle g, e\rangle\varphi + \langle g, a\rangle f$. Let $D: {}_{\varphi}A \to ({}_{\varphi}A)^{(2n)}$ be a derivation then as Proposition 2.5 demonstrates $D({}_{\varphi}A) \subseteq \ker\varphi$. Therefore $D(x) = \langle g, x\rangle a, (x \in {}_{\varphi}A)$, for some $g \in ({}_{\varphi}A)^*$. As for every $x \in {}_{\varphi}A$, $x = \langle\varphi, x\rangle e + \langle f, x\rangle a$, a direct calculation reveals that $D = \delta_{(\langle g,e\rangle a - \langle g,a\rangle e)}$; as required. □

Remark. (i) *If we go through the proof of Theorem 2.2 we see that the range of the derivation* $D(a) = \langle f - \varphi, a \rangle b_0$ *lies in* ${}_\varphi A$*, and the same argument may be applied to show that it is not inner as a derivation from* ${}_\varphi A$ *to* ${}_\varphi A$*. This shows that* ${}_\varphi A$ *is not* $0-$*weakly amenable, i.e.* $H^1({}_\varphi A, \ {}_\varphi A) \neq 0$; *see a remark just after* [3, Proposition 1.2]. *The same situation has occurred in the proof of Proposition 2.5.*

(ii) *As* ${}_\varphi A$ *has a left identity and it is a left ideal in* $({}_\varphi A)^{**}$*, it is worthwhile mentioning that, to prove the* $(2n-1)-$*weak amenability of* ${}_\varphi A$ *it suffices to show that* ${}_\varphi A$ *is weakly amenable;* [10, Theorem 3]*, and this has already done by Dales* et al. [6, Page 713].

We have seen in the first part of the proof of Theorem 2.2 that if $D : \ {}_\varphi A \to ({}_\varphi A)^{(2n-1)}$ is a derivation then $\langle D(a), b \rangle + \langle D(b), a \rangle = 0, \ (a, b \in \ {}_\varphi A)$; and the latter is known as a cyclic derivation for the case $n = 1$. Therefore as a consequence of Theorem 2.2 we get:

Corollary 2.6. *A bounded linear mapping* $D : \ {}_\varphi A \to ({}_\varphi A)^*$ *is a derivation if and only if it is a cyclic derivation. In particular,* ${}_\varphi A$ *is cyclicly amenable.*

We conclude with the following list consisting of some miscellaneous properties of ${}_\varphi A$ which can be verified straightforwardly.

• If $\varphi = \lambda\psi$ for some $\lambda \in \mathbb{C}$ then trivially ${}_\varphi A$ and ${}_\psi A$ are isomorphic; indeed, the mapping $a \to \lambda a$ defines an isomorphism. However, the converse is not valid, in general. For instance, let A be generated by two elements a, b. Choose $\varphi, \psi \in A^*$ such that $\langle \varphi, a \rangle = \langle \psi, b \rangle = 0$ and $\langle \varphi, b \rangle = \langle \psi, a \rangle = 1$, then ${}_\varphi A$ and ${}_\psi A$ are isomorphic (indeed, $\alpha a + \beta b \to \alpha b + \beta a$ defines an isomorphism), however φ and ψ are linearly independent.

• It can be readily verified that $\{0\} \cup \{a \in \ {}_\varphi A, \varphi(a) = 1\}$ is the set of all idempotents of ${}_\varphi A$. Moreover, this is actually the set of all minimal idempotents of ${}_\varphi A$.

• It is obvious that every subspace of ${}_\varphi A$ is a left ideal, while a subspace I is a right ideal if and only if either $I = \ {}_\varphi A$ or $I \subseteq \ker\varphi$. In particular, $\ker\varphi$ is the unique maximal ideal in ${}_\varphi A$.

• A subspace I of ${}_\varphi A$ is a modular left ideal if and only if either $I =_\varphi A$ or $I = ker\varphi$. In particular, $\ker\varphi$ is the unique primitive ideal in ${}_\varphi A$ and this implies that $rad({}_\varphi A) = \ker\varphi$ and so ${}_\varphi A$ is not semi-simple. Furthermore, for every non-zero proper closed ideal I, $rad(I) = I \cap rad({}_\varphi A) = I \cap \ker\varphi = I$.

• A direct verification reveals that $LM({}_\varphi A) = \mathbb{C}I$ and $RM({}_\varphi A) = B({}_\varphi A)$, where LM and RM stand for the left and right multipliers, respectively.

Acknowledgments. The authors would like to thank the referee for some comments that helped us improve this article.

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