Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 3 Issue 3(2011), Pages 84-96.

SCHUR CONVEXITY WITH RESPECT TO A CLASS OF SYMMETRIC FUNCTIONS AND THEIR APPLICATIONS

(COMMUNICATED BY MUHAMMAD ASLAM NOOR)

WEIFENG XIA, YUMING CHU

Abstract. For $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+$, the symmetric function $\varphi_n(x, r)$ is defined by

$$\varphi_n(x,r) = \varphi_n(x_1, x_2, \cdots, x_n; r) = \prod_{1 \le i_1 < i_2 \cdots < i_r \le n} \left(\sum_{j=1}^{r} \frac{1 + x_{i_j}}{x_{i_j}} \right),$$

where $r = 1, 2, \cdots, n$, and $i_1, i_2, \cdots, i_n$ are positive integers.

In this article, the Schur convexity, Schur multiplicative convexity and Schur harmonic convexity of $\varphi_n(x, r)$ are discussed. As applications, some inequalities, including Weierstrass inequality, are established by use of the theory of majorization.

1. Introduction

In this paper, we shall adopt the notation and terminology as follows: $\mathbb{R}^n$ denotes the $n-$dimensional Euclidean space ($n \ge 2$), $\mathbb{R}^n_+ = \{(x_1, x_2, \cdots, x_n) : x_i > 0, i = 1, 2, \cdots, n\}$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = (0, +\infty)$ and $\mathbb{N} = \{1, 2, \cdots, n, \cdots\}$. For $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we denote by

$$\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n), \\ xy &= (x_1 y_1, x_2 y_2, \cdots, x_n y_n), \\ \alpha x &= (\alpha x_1, \alpha x_2, \cdots, \alpha x_n), \\ e^x &= (e^{x_1}, e^{x_2}, \cdots, e^{x_n}), \\ \alpha + x &= (\alpha + x_1, \alpha + x_2, \cdots, \alpha + x_n) \end{aligned}$$

and

$$\alpha - x = (\alpha - x_1, \alpha - x_2, \cdots, \alpha - x_n).$$

2000 *Mathematics Subject Classification.* 05E05, 26B25.

Key words and phrases. symmetric function; Schur convex; Schur multiplicatively convex; Schur harmonic convex.

Submitted April 7, 2011. Published June 20, 2011.

This work was supported by the Natural Science Foundation of China (11071069), the Natural Science Foundation of Zhejiang Province (Y7080185) and the Innovation Team Foundation of the Department of Education of Zhejiang Province (T200924).

Moreover, we denote by

$$x^{\alpha} = (x_1^{\alpha}, x_2^{\alpha}, \cdots, x_n^{\alpha}),$$

$$\log x = (\log x_1, \log x_2, \cdots, \log x_n)$$

and

$$\frac{1}{x} = \left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right)$$

for $x \in \mathbb{R}_+^n$.

For $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$, $r \in \mathbb{N}$ and $r \leq n$, the Hamy symmetric function $H_n(x, r)$ is defined by T. Hara, M. Uchiyama and S. Takahasi [1] as follows:

$$H_n(x, r) = H_n(x_1, x_2, \cdots, x_n; r) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} \left(\prod_{j=1}^{r} x_{i_j}\right)^{\frac{1}{r}},$$

where $i_1, i_2, \cdots, i_n \in \mathbb{N}$.

Corresponding to this is the $r-$th order Hamy mean

$$\sigma_n(x, r) = \sigma_n(x_1, x_2, \cdots, x_n; r) = \frac{1}{C_n^r} H_n(x, r),$$

where $C_n^r = \frac{n!}{(n-r)!r!}$. T. Hara, M. Uchiyama and S. Takahasi [1] established the following refinement of the classical arithmetic and geometric means inequalities:

$$G_n(x) = \sigma_n(x, n) \leq \sigma_n(x, n-1) \leq \cdots \leq \sigma_n(x, 2) \leq \sigma_n(x, 1) = A_n(x).$$

Here, $A_n(x) = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $G_n(x) = (\prod_{i=1}^{n} x_i)^{\frac{1}{n}}$ denote the classical arithmetic and geometric means of x, respectively. We also denote the harmonic mean of x by $H_n(x) = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}$.

The paper [2] by H. T. Ku, M. C. Ku and X. M. Zhang contains some interesting inequalities including the fact that $(\sigma_n(x, r))^{\frac{1}{r}}$ is log-concave. More results can be found in the book [3] by P. S. Bullen.

Recently, the Schur convexity of the Hamy symmetric function $H_n(x, r)$ was discussed and some analytic inequalities were established by K. Z. Guan [4].

In this article, we define the following new symmetric function:

$$\varphi_n(x, r) = \varphi_n(x_1, x_2, \cdots, x_n; r) = \prod_{1 \leq i_1 < i_2 \cdots < i_r \leq n} \left(\sum_{j=1}^{r} \frac{1 + x_{i_j}}{x_{i_j}}\right), \tag{1.1}$$

for $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$, $r \in \mathbb{N}$ and $r \leq n$. Here, $i_1, i_2, \cdots, i_n \in \mathbb{N}$.

The main purpose of this paper is to discuss the Schur convexity, Schur multiplicative convexity and Schur harmonic convexity for the symmetric function $\varphi_n(x, r)$. As applications, some inequalities are established by use of the theory of majorization.

Schur convex, Schur multiplicatively convex and Schur harmonic convex functions are defined as follows.

Definition 1.1. *Let $E \subseteq \mathbb{R}^n$ be a set, a real-valued function F on E is called a Schur convex function if*

$$F(x_1, x_2, \cdots, x_n) \leq F(y_1, y_2, \cdots, y_n)$$

for each pair of n-tuples $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ in E, such that $x \prec y$, i.e.

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, k = 1, 2, \cdots, n-1$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[i]}$ denotes the ith largest component in x. F is called Schur concave if $-F$ is Schur convex .

Definition 1.2. *Let $E \subseteq \mathbb{R}^n_+$ be a set, a real-valued function $F : E \to \mathbb{R}_+$ is called a Schur multiplicatively convex function on E if $F(x_1, x_2, \cdots, x_n) \leq F(y_1, y_2, \cdots, y_n)$ for each pair of $n-$tuples $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ in E, such that $\log x \prec \log y$. F is called Schur multiplicatively concave if $\frac{1}{F}$ is Schur multiplicatively convex.*

Definition 1.3. *Let $E \subseteq \mathbb{R}^n_+$ be a set, a real-valued function F on E is called a Schur harmonic convex function on E if*

$$F(x_1, x_2, \cdots, x_n) \leq F(y_1, y_2, \cdots, y_n) \tag{1.2}$$

for each pair of $n-$tuples $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ in E, such that $\frac{1}{x} \prec \frac{1}{y}$. F is called a Schur harmonic concave function on E if inequality (1.2) is reversed.

The Schur convexity was introduced by I. Schur [5] in 1923, G. H. Hardy, J. E. Littlewood and G. Pólya were also interested in some inequalities that are related to the Schur convexity [6]. It has many important applications in extended mean values [7], theory of statistical experiments [8], graphs and matrices [9], combinatorial optimization [10], reliability [11], gamma functions [12], information-theoretic topics [13], stochastic orderings [14] and other related fields. Recently, the Schur multiplicative and harmonic convexities were introduced and investigated in paper [15, 16, 22-24].

2. Lemmas

In this section, we introduce and establish some Lemmas, which are used in the proof of our main results.

Lemma 2.1. [5] *Let $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ be a continuous symmetric function. If f is differentiable in $\mathbb{R}^n_+$, then f is Schur convex in $\mathbb{R}^n_+$ if and only if*

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geq 0 \tag{2.1}$$

for all $i, j = 1, 2, \cdots, n$ and $x = (x_1, \cdots, x_n) \in \mathbb{R}^n_+$. And f is Schur concave in $\mathbb{R}^n_+$ if and only if inequality (2.1) is reversed for all $i, j = 1, 2, \cdots, n$ and

$x = (x_1, \cdots, x_n) \in \mathbb{R}_+^n$. *Here f is a symmetric function in $\mathbb{R}_+^n$ which means that $f(Px) = f(x)$ for any $x \in \mathbb{R}_+^n$ and any $n \times n$ permutation matrix P.*

Remark 2.1. *Since f is symmetric, the Schur's condition in Lemma 2.1, i.e. (2.1) can be reduced to*

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \ge 0.$$

Lemma 2.2. [15, 16] *Let $f : \mathbb{R}_+^n \to \mathbb{R}_+$ be a continuous symmetric function. If f is differentiable in $\mathbb{R}_+^n$, then f is Schur multiplicatively convex in $\mathbb{R}_+^n$ if and only if*

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial f}{\partial x_1} - x_2\frac{\partial f}{\partial x_2}\right) \ge 0 \tag{2.2}$$

for all $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$. And f is Schur multiplicatively concave in $\mathbb{R}_+^n$ if and only if inequality (2.2) is reversed.

Lemma 2.3. [22] *Let $f : \mathbb{R}_+^n \to \mathbb{R}_+$ be a continuous symmetric function. If f is differentiable in $\mathbb{R}_+^n$, then f is Schur harmonic convex in $\mathbb{R}_+^n$ if and only if*

$$(x_1 - x_2)\left(x_1^2\frac{\partial f}{\partial x_1} - x_2^2\frac{\partial f}{\partial x_2}\right) \ge 0 \tag{2.3}$$

for all $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$. And f is Schur harmonic concave in $\mathbb{R}_+^n$ if and only if inequality (2.3) is reversed.

Lemma 2.4. [4, 16, 17] *Let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$ and $\sum_{i=1}^n x_i = s$. If $c \ge s$, then*

$$\frac{c - x}{\frac{nc}{s} - 1} = \left(\frac{c - x_1}{\frac{nc}{s} - 1}, \frac{c - x_2}{\frac{nc}{s} - 1}, \cdots, \frac{c - x_n}{\frac{nc}{s} - 1}\right) \prec (x_1, x_2, \cdots, x_n) = x.$$

Lemma 2.5. [17] *Let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$ and $\sum_{i=1}^n x_i = s$. If $c \ge 0$, then*

$$\frac{c + x}{\frac{nc}{s} + 1} = \left(\frac{c + x_1}{\frac{nc}{s} + 1}, \frac{c + x_2}{\frac{nc}{s} + 1}, \cdots, \frac{c + x_n}{\frac{nc}{s} + 1}\right) \prec (x_1, x_2, \cdots, x_n) = x.$$

Lemma 2.6. [18] *Suppose that $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$ and $\sum_{i=1}^n x_i = s$. If $0 \le \lambda \le 1$, then*

$$\frac{s - \lambda x}{n - \lambda} = \left(\frac{s - \lambda x_1}{n - \lambda}, \frac{s - \lambda x_2}{n - \lambda}, \cdots, \frac{s - \lambda x_n}{n - \lambda}\right) \prec (x_1, x_2, \cdots, x_n) = x.$$

3. Main results

Theorem 3.1. *For $r \in \{1, 2, \cdots, n\}$, the symmetric function $\varphi_n(x, r)$ is Schur convex in $\mathbb{R}_+^n$.*

Proof. By Lemma 2.1 and Remark 2.1 we only need to prove that

$$(x_1 - x_2)\left(\frac{\partial \varphi_n(x, r)}{\partial x_1} - \frac{\partial \varphi_n(x, r)}{\partial x_2}\right) \ge 0. \tag{3.1}$$

for all $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$ and $r = 1, 2, \cdots, n$.

The proof is divided into four cases.

Case 1. If $r = 1$, then (1.1) leads to

$$\varphi_n(x,1)=\varphi_n(x_1,x_2,\cdots,x_n;1)=\prod_{i=1}^{n}\frac{1+x_i}{x_i}. \tag{3.2}$$

From (3.2) and simple computation we get

$$(x_1-x_2)\left(\frac{\partial\varphi_n(x,1)}{\partial x_1}-\frac{\partial\varphi_n(x,1)}{\partial x_2}\right)=\frac{(x_1-x_2)^2(1+x_1+x_2)}{x_1x_2(1+x_1)(1+x_2)}\varphi_n(x,1)\geq 0.$$

Case 2. If $n\geq 2$ and $r=n$, then (1.1) yields that

$$\varphi_n(x,n)=\varphi_n(x_1,x_2,\cdots,x_n;n)=\sum_{i=1}^{n}\frac{1+x_i}{x_i} \tag{3.3}$$

and

$$(x_1-x_2)\left(\frac{\partial\varphi_n(x,n)}{\partial x_1}-\frac{\partial\varphi_n(x,n)}{\partial x_2}\right)=\frac{(x_1-x_2)^2(x_1+x_2)}{x_1^2x_2^2}\geq 0.$$

Case 3. If $n\geq 3$ and $r=2$, then by (1.1) we have

$$\begin{aligned}
&\varphi_n(x,2)=\varphi_n(x_1,x_2,\cdots,x_n;2)\\
&=\left(\frac{1+x_1}{x_1}+\frac{1+x_2}{x_2}\right)\left[\prod_{j=3}^{n}\left(\frac{1+x_1}{x_1}+\frac{1+x_j}{x_j}\right)\right]\varphi_{n-1}(x_2,x_3,\cdots,x_n;2)\\
&=\left(\frac{1+x_2}{x_2}+\frac{1+x_1}{x_1}\right)\left[\prod_{j=3}^{n}\left(\frac{1+x_2}{x_2}+\frac{1+x_j}{x_j}\right)\right]\varphi_{n-1}(x_1,x_3,\cdots,x_n;2).
\end{aligned} \tag{3.4}$$

Simple computation and (3.4) lead to

$$\begin{aligned}
&(x_1-x_2)\left(\frac{\partial\varphi_n(x,2)}{\partial x_1}-\frac{\partial\varphi_n(x,2)}{\partial x_2}\right)\\
&=(x_1-x_2)^2\varphi_n(x,2)\left[\frac{x_1+x_2}{x_1x_2(x_1+2x_1x_2+x_2)}\right.\\
&\quad\left.+\sum_{j=3}^{n}\frac{x_j^2+(x_1+x_2)(1+2x_j)x_j}{x_1x_2(x_1+2x_1x_j+x_j)(x_2+2x_2x_j+x_j)}\right]\\
&\geq 0.
\end{aligned}$$

Case 4. If $n \geq 4$ and $3 \leq r \leq n-1$, then from (1.1) we have

$$\begin{aligned}
&\varphi_n(x,r) = \varphi_n(x_1, x_2, \cdots, x_n; r) \\
&= \varphi_{n-1}(x_2, x_3, \cdots, x_n; r) \prod_{3\leq i_1<i_2<\cdots<i_{r-1}\leq n} \left(\frac{1+x_1}{x_1} + \sum_{j=1}^{r-1} \frac{1+x_{i_j}}{x_{i_j}} \right) \\
&\times \prod_{3\leq i_1<i_2<\cdots<i_{r-2}\leq n} \left(\frac{1+x_1}{x_1} + \frac{1+x_2}{x_2} + \sum_{j=1}^{r-2} \frac{1+x_{i_j}}{x_{i_j}} \right) \\
&= \varphi_{n-1}(x_1, x_3, \cdots, x_n; r) \prod_{3\leq i_1<i_2<\cdots<i_{r-1}\leq n} \left(\frac{1+x_2}{x_2} + \sum_{j=1}^{r-1} \frac{1+x_{i_j}}{x_{i_j}} \right) \\
&\times \prod_{3\leq i_1<i_2<\cdots<i_{r-2}\leq n} \left(\frac{1+x_1}{x_1} + \frac{1+x_2}{x_2} + \sum_{j=1}^{r-2} \frac{1+x_{i_j}}{x_{i_j}} \right) \qquad (3.5)
\end{aligned}$$

and

$$\begin{aligned}
&(x_1 - x_2)\left(\frac{\partial \varphi_n(x,r)}{\partial x_1} - \frac{\partial \varphi_n(x,r)}{\partial x_2} \right) \\
&= \frac{(x_1-x_2)^2}{x_1^2 x_2^2} \varphi_n(x,r) \left[\sum_{3\leq i_1<i_2<\cdots<i_{r-2}\leq n} \frac{x_1+x_2}{\frac{1+x_1}{x_1} + \frac{1+x_2}{x_2} + \sum_{j=1}^{r-2} \frac{1+x_{i_j}}{x_{i_j}}} \right. \\
&+ \left. \sum_{3\leq i_1<i_2<\cdots<i_{r-1}\leq n} \frac{(1+x_1+x_2) + (x_1+x_2)\sum_{j=1}^{r-1} \frac{1+x_{i_j}}{x_{i_j}}}{(\frac{1+x_1}{x_1} + \sum_{j=1}^{r-1} \frac{1+x_{i_j}}{x_{i_j}})(\frac{1+x_2}{x_2} + \sum_{j=1}^{r-1} \frac{1+x_{i_j}}{x_{i_j}})} \right] \\
&\geq 0.
\end{aligned}$$

Therefore (3.1) follows from Cases 1-4, and the proof of Theorem 3.1 is completed. □

Theorem 3.2. *For $r \in \{1, 2, \cdots, n\}$, the symmetric function $\varphi_n(x,r)$ is Schur multiplicatively convex in $\mathbb{R}_+^n$.*

Proof. According to Lemma 2.2 we only need to prove that

$$(\log x_1 - \log x_2)\left(x_1 \frac{\partial \varphi_n(x,r)}{\partial x_1} - x_2 \frac{\partial \varphi_n(x,r)}{\partial x_2} \right) \geq 0 \qquad (3.6)$$

for all $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$ and $r = 1, 2, \cdots, n$.

The proof is divided into four cases.

Case I. If $r = 1$, then (3.2) leads to

$$\begin{aligned}
&(\log x_1 - \log x_2)\left(x_1 \frac{\partial \varphi_n(x,1)}{\partial x_1} - x_2 \frac{\partial \varphi_n(x,1)}{\partial x_2} \right) \\
&= \frac{(\log x_1 - \log x_2)(x_1 - x_2)}{(1+x_1)(1+x_2)} \varphi_n(x,1) \geq 0.
\end{aligned}$$

Case II. If $n \geq 2$ and $r = n$, then (3.3) implies that

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial\varphi_n(x,n)}{\partial x_1} - x_2\frac{\partial\varphi_n(x,n)}{\partial x_2}\right)$$
$$= \frac{(x_1-x_2)(\log x_1 - \log x_2)}{x_1x_2} \geq 0.$$

Case III. If $n \geq 3$ and $r = 2$, then (3.4) leads to

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial\varphi_n(x,2)}{\partial x_1} - x_2\frac{\partial\varphi_n(x,2)}{\partial x_2}\right)$$
$$= (x_1-x_2)(\log x_1 - \log x_2)\varphi_n(x,2)\left[\frac{1}{x_1+2x_1x_2+x_2}\right.$$
$$\left.+\sum_{j=3}^{n}\frac{(1+2x_j)x_j}{(x_1+2x_1x_j+x_j)(x_2+2x_2x_j+x_j)}\right] \geq 0.$$

Case IV. If $n \geq 4$ and $3 \leq r \leq n-1$, then (3.5) reveals

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial\varphi_n(x,r)}{\partial x_1} - x_2\frac{\partial\varphi_n(x,r)}{\partial x_2}\right)$$
$$= \frac{(x_1-x_2)(\log x_1 - \log x_2)}{x_1x_2}\left[\sum_{3\leq i_1<i_2<\cdots<i_{r-2}\leq n}\frac{1}{\frac{1+x_1}{x_1}+\frac{1+x_2}{x_2}+\sum_{j=1}^{r-2}\frac{1+x_{i_j}}{x_{i_j}}}\right.$$
$$\left.+\sum_{3\leq i_1<i_2<\cdots<i_{r-1}\leq n}\frac{1+\sum_{j=1}^{r-1}\frac{1+x_{i_j}}{x_{i_j}}}{(\frac{1+x_1}{x_1}+\sum_{j=1}^{r-1}\frac{1+x_{i_j}}{x_{i_j}})(\frac{1+x_2}{x_2}+\sum_{j=1}^{r-1}\frac{1+x_{i_j}}{x_{i_j}})}\right]\varphi_n(x,r) \geq 0.$$

Therefore (3.6) follows from Cases I-IV, and the proof of Theorem 3.2 is completed. □

Theorem 3.3. *For $r \in \{1,2,\cdots,n\}$, the symmetric function $\varphi_n(x,r)$ is Schur harmonic concave in $\mathbb{R}_+^n$.*

Proof. According to Lemma 2.3 we only need to prove that

$$(x_1-x_2)\left(x_1^2\frac{\partial\varphi_n(x,r)}{\partial x_1} - x_2^2\frac{\partial\varphi_n(x,r)}{\partial x_2}\right) \leq 0 \tag{3.7}$$

for all $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$ and $r = 1, 2, \cdots, n$.

The proof is divided into four cases.

Case A. If $r = 1$, then from (3.2) we have

$$(x_1-x_2)\left(x_1^2\frac{\partial\varphi_n(x,1)}{\partial x_1} - x_2^2\frac{\partial\varphi_n(x,1)}{\partial x_2}\right)$$
$$= -\frac{(x_1-x_2)^2}{(1+x_1)(1+x_2)}\varphi_n(x,1) \leq 0.$$

Case B. If $n \geq 2$ and $r = n$, then (3.3) leads to

$$(x_1-x_2)\left(x_1^2\frac{\partial\varphi_n(x,n)}{\partial x_1} - x_2^2\frac{\partial\varphi_n(x,n)}{\partial x_2}\right) = 0.$$

Case C. If $n \geq 3$ and $r = 2$, then (3.4) shows that

$$(x_1 - x_2)\left(x_1^2 \frac{\partial \varphi_n(x,2)}{\partial x_1} - x_2^2 \frac{\partial \varphi_n(x,2)}{\partial x_2}\right)$$
$$= -(x_1 - x_2)^2 \varphi_n(x,2) \sum_{j=3}^{n} \frac{x_j^2}{(x_1 + 2x_1x_j + x_j)(x_2 + 2x_2x_j + x_j)} \leq 0.$$

Case D. If $n \geq 4$ and $3 \leq r \leq n-1$, then (3.5) deduces that

$$(x_1 - x_2)\left(x_1^2 \frac{\partial \varphi_n(x,r)}{\partial x_1} - x_2^2 \frac{\partial \varphi_n(x,r)}{\partial x_2}\right)$$
$$= -\frac{(x_1 - x_2)^2}{x_1 x_2} \varphi_n(x,r)$$
$$\times \sum_{3 \leq i_1 < i_2 < \cdots < i_{r-1} \leq n} \frac{1}{\left(\frac{1+x_1}{x_1} + \sum_{j=1}^{r-1} \frac{1+x_{i_j}}{x_{i_j}}\right)\left(\frac{1+x_2}{x_2} + \sum_{j=1}^{r-1} \frac{1+x_{i_j}}{x_{i_j}}\right)} \leq 0.$$

Therefore (3.7) follows from Cases A-D, and the proof of Theorem 3.3 is completed. □

4. Applications

In this section, we establish some inequalities by use of Theorems 3.1-3.3 and the theory of majorization.

Theorem 4.1. *If $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$, $s = \sum_{i=1}^n x_i$ and $r \in \{1, 2, \cdots, n\}$, then*

(1) $\varphi_n(x,r) \geq \varphi_n\left(\frac{c-x}{\frac{nc}{s}-1}, r\right)$ for $c \geq s$;
(2) $\varphi_n(x,r) \leq \varphi_n\left(\frac{cH_n(x)-1}{cx-1}x, r\right)$ for $c \geq \sum_{i=1}^n \frac{1}{x_i}$;
(3) $\varphi_n(x,r) \geq \varphi_n\left(\frac{c+x}{\frac{nc}{s}+1}, r\right)$ for $c \geq 0$;
(4) $\varphi_n(x,r) \leq \varphi_n\left(\frac{cH_n(x)+1}{cx+1}x, r\right)$ for $c \geq 0$;
(5) $\varphi_n(x,r) \geq \varphi_n\left(\frac{s-\lambda x}{n-\lambda}, r\right)$ for $0 \leq \lambda \leq 1$;
(6) $\varphi_n(x,r) \leq \varphi_n\left(\frac{n-\lambda}{\sum_{i=1}^n \frac{1}{x_i} - \frac{\lambda}{x}}, r\right)$ for $0 \leq \lambda \leq 1$;
(7) $\varphi_n(x,r) \geq \varphi_n\left(\frac{s+\lambda x}{n+\lambda}, r\right)$ for $0 \leq \lambda \leq 1$;
(8) $\varphi_n(x,r) \leq \varphi_n\left(\frac{n+\lambda}{\sum_{i=1}^n \frac{1}{x_i} + \frac{\lambda}{x}}, r\right)$ for $0 \leq \lambda \leq 1$.

Proof. Theorem 4.1(1) and (2) follow from Lemma 2.4 , Theorem 3.1 and Theorem 3.3.

Theorem4.1(3) and (4) follow from Lemma 2.5 , Theorem 3.1 and Theorem 3.3.

Theorem 4.1(5) and (6) follow from Lemma 2.6 , Theorem 3.1 and Theorem 3.3.

Theorem 4.1(7) and (8) follow from Theorem 3.1 , Theorem 3.3 together with the face that

$$\frac{s+\lambda x}{n+\lambda} = \left(\frac{s+\lambda x_1}{n+\lambda}, \frac{s+\lambda x_2}{n+\lambda}, \cdots, \frac{s+\lambda x_n}{n+\lambda}\right) \prec (x_1, x_2, \cdots, x_n) = x.$$

□

Theorem 4.2. *If* $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+$ *and* $r \in \{1, 2, \cdots, n\}$, *then*

$$(i) \quad \prod_{1 \le i_1 < i_2 < \cdots < i_r \le n} \left(\sum_{j=1}^{r} \frac{1 + x_{i_j}}{x_{i_j}} \right) \ge \left[\frac{rA_n(1+x)}{A_n(x)} \right]^{\frac{n!}{r!(n-r)!}};$$

$$(ii) \quad \prod_{1 \le i_1 < i_2 < \cdots < i_r \le n} \left[\sum_{j=1}^{r} (1 + x_{i_j}) \right] \le [rA_n(1+x)]^{\frac{n!}{r!(n-r)!}}.$$

Proof. We clearly see that

$$(A_n(x), A_n(x), \cdots, A_n(x)) \prec (x_1, x_2, \cdots, x_n) = x. \tag{4.1}$$

Therefore, Theorem 4.2(i) follows from (4.1) and Theorem 3.1 together with (1.1), and Theorem 4.2 (ii) follows from (4.1) and Theorem 3.3 together with (1.1). □

If we take $r = 1$ in Theorem 4.2(i) and (ii), and $r = n$ in Theorem 4.2(i), respectively, then we have

Corollary 4.1. *If* $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+$, *then*

$$(i) \frac{G_n(1+x)}{G_n(x)} \ge \frac{A_n(1+x)}{A_n(x)};$$

$$(ii) G_n(1+x) \le A_n(1+x);$$

$$(iii) A_n\left(\frac{1+x}{x}\right) \ge \frac{A_n(1+x)}{A_n(x)}.$$

Remark 4.1. *If we take* $\sum_{i=1}^{n} x_i = 1$ *in Corollary 4.1(i), then we get the Weierstrass inequality (see [19, p. 260])*

$$\prod_{i=1}^{n} \left(\frac{1}{x_i} + 1 \right) \ge (n+1)^n.$$

Theorem 4.3. *If* $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+$ *and* $r \in \{1, 2, \cdots, n\}$, *then*

$$(i) \quad \prod_{1 \le i_1 < i_2 < \cdots < i_r \le n} \left[\sum_{j=1}^{r} (1 + x_{i_j}) \right] \ge [r(1 + H_n(x))]^{\frac{n!}{r!(n-r)!}};$$

$$(ii) \quad \prod_{1 \le i_1 < i_2 < \cdots < i_r \le n} \left(\sum_{j=1}^{r} \frac{1 + x_{i_j}}{x_{i_j}} \right) \le \left[\frac{r(1 + H_n(x))}{H_n(x)} \right]^{\frac{n!}{r!(n-r)!}}.$$

Proof. We clearly see that

$$\left(\frac{1}{H_n(x)}, \frac{1}{H_n(x)}, \cdots, \frac{1}{H_n(x)} \right) \prec \left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n} \right) = \frac{1}{x}. \tag{4.2}$$

Therefore, Theorem 4.3 (i) follows from (4.2), Theorem 3.1 and (1.1), and Theorem 4.3 (ii) follows from (4.2), Theorem 3.3 and (1.1).

If we take $r = 1$ and $r = n$ in Theorem 4.3, respectively, then we get the following Corollaries 4.2 and 4.3. □

Corollary 4.2. *If* $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$, *then*

$(i) G_n(1+x) \geq 1 + H_n(x)$;

$(ii) \dfrac{G_n(x)}{G_n(1+x)} \geq \dfrac{H_n(x)}{1+H_n(x)}$.

Corollary 4.3. *If* $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$, *then*

$(i) A_n(1+x) \geq 1 + H_n(x)$;

$(ii) A_n\left(\dfrac{1+x}{x}\right) \leq 1 + \dfrac{1}{H_n(x)}$.

Theorem 4.4. *If* $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$ *and* $r \in \{1, 2, \cdots, n\}$, *then*

$$\prod_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} \left(\sum_{j=1}^{r} \frac{1 + x_{i_j}}{x_{i_j}} \right) \geq \left[r \cdot \frac{1 + G_n(x)}{G_n(x)} \right]^{\frac{n!}{r!(n-r)!}}.$$

Proof. We clearly see that

$$\log\left(G_n(x), G_n(x), \cdots, G_n(x)\right) \prec \log(x_1, x_2, \cdots, x_n). \tag{4.3}$$

Therefore, Theorem 4.4 follows from (4.3), Theorem 3.2 and (1.1). □

If we take $r = 1$ and $r = n$ in Theorem 4.4, respectively, then we get

Corollary 4.4. *If* $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$, *then*

$(i) G_n(1+x) \geq 1 + G_n(x)$;

$(ii) A_n\left(\dfrac{1+x}{x}\right) \geq \dfrac{1+G_n(x)}{G_n(x)}$.

Theorem 4.5. *If* $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$, *then*

$$(i) \prod_{1 \leq i_1 < i_2 \cdots < i_r \leq n} \left(\sum_{j=1}^{r} \frac{2 + x_{i_j}}{1 + x_{i_j}} \right) \leq (2r)^{\frac{(n-1)!}{r!(n-r-1)!}} \left[\frac{2 + \sum_{i=1}^{n} x_i}{1 + \sum_{i=1}^{n} x_i} + 2(r-1) \right]^{\frac{(n-1)!}{(r-1)!(n-r)!}}$$

for $1 \leq r \leq n-1$;

$$(ii) \sum_{i=1}^{n} \frac{2 + x_i}{1 + x_i} \leq \frac{2 + \sum_{i=1}^{n} x_i}{1 + \sum_{i=1}^{n} x_i} + 2(n-1);$$

$$(iii) \prod_{1 \leq i_1 < i_2 \cdots < i_r \leq n} \left[\sum_{j=1}^{r} (2 + x_{i_j}) \right] \geq (2r)^{\frac{(n-1)!}{r!(n-r-1)!}} \left(2r + \sum_{i=1}^{n} x_i \right)^{\frac{(n-1)!}{(r-1)!(n-r)!}}$$

for $1 \leq r \leq n-1$.

Proof. Theorem 4.5 follows from Theorem 3.1, Theorem 3.3 and (1.1) together with the fact that

$$(1 + x_1, 1 + x_2, \cdots, 1 + x_n) \prec (1 + \sum_{i=1}^{n} x_i, 1, 1, \cdots, 1).$$

□

Theorem 4.6. *Let $\mathcal{A} = A_1A_2\cdots A_{n+1}$ be a $n-$dimensional simplex in $\mathbb{R}^n$ and P be an arbitrary point in the interior of $\mathcal{A}$. If B_i is the intersection point of straight line A_iP and hyperplane $\sum_i = A_1A_2\cdots A_{i-1}A_{i+1}\cdots A_{n+1}, i = 1, 2, \cdots, n+1$. Then for $r \in \{1, 2, \cdots, n+1\}$ we have*

$$(i) \quad \prod_{1\le i_1<i_2\cdots<i_r\le n+1} \left(\sum_{j=1}^{r} \frac{A_{i_j}B_{i_j} + PB_{i_j}}{PB_{i_j}} \right) \ge [r(n+2)]^{\frac{(n+1)!}{r!(n-r+1)!}};$$

$$(ii) \quad \prod_{1\le i_1<i_2\cdots<i_r\le n+1} \left(\sum_{j=1}^{r} \frac{A_{i_j}B_{i_j} + PB_{i_j}}{A_{i_j}B_{i_j}} \right) \le \left[\frac{r(n+2)}{n+1}\right]^{\frac{(n+1)!}{r!(n-r+1)!}};$$

$$(iii) \quad \prod_{1\le i_1<i_2\cdots<i_r\le n+1} \left(\sum_{j=1}^{r} \frac{A_{i_j}B_{i_j} + PA_{i_j}}{PA_{i_j}} \right) \ge \left[\frac{r(2n+1)}{n}\right]^{\frac{(n+1)!}{r!(n-r+1)!}};$$

$$(iv) \quad \prod_{1\le i_1<i_2\cdots<i_r\le n+1} \left(\sum_{j=1}^{r} \frac{A_{i_j}B_{i_j} + PA_{i_j}}{A_{i_j}B_{i_j}} \right) \le \left[\frac{r(2n+1)}{n+1}\right]^{\frac{(n+1)!}{r!(n-r+1)!}}.$$

Proof. It is easy to see that $\sum_{i=1}^{n+1} \frac{PB_i}{A_iB_i} = 1$ and $\sum_{i=1}^{n+1} \frac{PA_i}{A_iB_i} = n$, these imply that

$$\left(\frac{1}{n+1}, \frac{1}{n+1}, \cdots, \frac{1}{n+1}\right) \prec \left(\frac{PB_1}{A_1B_1}, \frac{PB_2}{A_2B_2}, \cdots, \frac{PB_{n+1}}{A_{n+1}B_{n+1}}\right) \tag{4.4}$$

and

$$\left(\frac{n}{n+1}, \frac{n}{n+1}, \cdots, \frac{n}{n+1}\right) \prec \left(\frac{PA_1}{A_1B_1}, \frac{PA_2}{A_2B_2}, \cdots, \frac{PA_{n+1}}{A_{n+1}B_{n+1}}\right). \tag{4.5}$$

Therefore, Theorem 4.6 follows from (4.4), (4.5), Theorem 3.1, Theorem 3.3 and (1.1). □

Remark 4.2. *D. S. Mitrinović, J. E. Pečairć and V. Volenec [20, p. 473-479] established a series of inequalities for $\frac{PA_i}{A_iB_i}$ and $\frac{PB_i}{A_iB_i}$, $i = 1, 2, \cdots, n+1$. Obvious, our inequalities in Theorem 4.6 are different from theirs.*

Theorem 4.7. *Suppose that $A \in M_n(C)(n \ge 2)$ is a complex matrix, $\lambda_1, \lambda_2, \cdots, \lambda_n$ and $\sigma_1, \sigma_2, \cdots, \sigma_n$ are the eigenvalues and singular values of A, respectively. If A*

is a positive definite Hermitian matrix and $r \in \{1, 2, \cdots, n\}$, *then*

$$(i) \quad \prod_{1\le i_1<i_2\cdots<i_r\le n} \left(\sum_{j=1}^{r} \frac{1+\lambda_{i_j}}{\lambda_{i_j}}\right) \ge \left[\frac{r(n+trA)}{trA}\right]^{\frac{n!}{r!(n-r)!}};$$

$$(ii) \quad \prod_{1\le i_1<i_2\cdots<i_r\le n} \left[\sum_{j=1}^{r} (1+\lambda_{i_j})\right] \le \left[\frac{r(n+trA)}{n}\right]^{\frac{n!}{r!(n-r)!}};$$

$$(iii) \quad \prod_{1\le i_1<i_2\cdots<i_r\le n} \left(\sum_{j=1}^{r} \frac{2+\lambda_{i_j}}{1+\lambda_{i_j}}\right) \ge \left[\frac{r(1+\sqrt[n]{det(I+A)})}{\sqrt[n]{det(I+A)}}\right]^{\frac{n!}{r!(n-r)!}};$$

$$(iv) \quad \prod_{1\le i_1<i_2\cdots<i_r\le n} \left(\sum_{j=1}^{r} \frac{trA+\lambda_{i_j}}{\lambda_{i_j}}\right) \ge \left[\frac{r(trA+\sqrt[n]{detA})}{\sqrt[n]{detA}}\right]^{\frac{n!}{r!(n-r)!}};$$

$$(v) \quad \prod_{1\le i_1<i_2\cdots<i_r\le n} \left(\sum_{j=1}^{r} \frac{1+\lambda_{i_j}}{\lambda_{i_j}}\right) \le \prod_{1\le i_1<i_2\cdots<i_r\le n} \left(\sum_{j=1}^{r} \frac{1+\sigma_{i_j}}{\sigma_{i_j}}\right).$$

Proof. $(i)-(ii)$ We clearly see that $\lambda_i > 0 (i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} \lambda_i = trA$, these lead to

$$\left(\frac{trA}{n}, \frac{trA}{n}, \cdots, \frac{trA}{n}\right) \prec (\lambda_1, \lambda_2, \cdots, \lambda_n). \tag{4.6}$$

Therefore, Theorem 4.7 (i) and (ii) follows from (4.6), Theorem 3.1, Theorem 3.3 and (1.1).

(iii) It is easy to see that $1+\lambda_1, 1+\lambda_2, \cdots, 1+\lambda_n$ are the eigenvalues of matrix $I+A$ and $\prod_{i=1}^{n}(1+\lambda_i) = det(I+A)$, these yield that

$$\begin{aligned} &\log\left(\sqrt[n]{det(I+A)}, \sqrt[n]{det(I+A)}, \cdots, \sqrt[n]{det(I+A)}\right) \\ &\prec \log(1+\lambda_1, 1+\lambda_2, \cdots, 1+\lambda_n). \end{aligned} \tag{4.7}$$

Therefore, Theorem 4.7 (iii) follows from (4.7), Theorem 3.2 and (1.1).

(iv) Theorem 4.7(iv) follows from (1.1), Theorem 3.2 and the fact that

$$\log\left(\frac{\sqrt[n]{detA}}{trA}, \frac{\sqrt[n]{detA}}{trA}, \cdots, \frac{\sqrt[n]{detA}}{trA}\right) \prec \log\left(\frac{\lambda_1}{trA}, \frac{\lambda_2}{trA}, \cdots, \frac{\lambda_n}{trA}\right).$$

(v) A result due to H. Weyl [21] (see also [5, p. 231]) gives

$$\log(\lambda_1, \lambda_2, \cdots, \lambda_n) \prec \log(\sigma_1, \sigma_2, \cdots, \sigma_n). \tag{4.8}$$

Therefore, Theorem 4.7 (v) follows from (4.8), Theorem 3.2 and (1.1). □

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

References

[1] T. Hara, M. Uchiyama, S. Takahasi, *A refinement of various mean inequalities*, J. Inequal. Appl. **2**(1998)387-395.

[2] H. T. Ku, M. C. Ku, X. M. Zhang, *Inequalities for symmetric means, symmetric harmonic means, and their applications*, Bull. Austral. Math. Soc. **56**(1997) 409-420.

[3] P. S. Bullen, *Handbook of Means and Their Inequalities*, Kluwer Academic Publishers Group, Dordrecht, 2003.

[4] K. Z. Guan, *The Hamy symmetric function and its generalization*, Math. Inequal. Appl. **9**(2006)797-805.

[5] A. W. Marshall, I. Olkin, *Inequalities: Theorey of Majorization and Its Applications*, Academic Press, New York, 1979.

[6] G. H. Hardy, J. E. Littlewood, G. Pólya, *Some simple inequalities satisfied by convex functions*, Messenger Math. **58**(1929)145-152.

[7] Y. M. Chu, X. M. Zhang, *Necessary and sufficient conditions such that extended mean values are Schur-convex or Schur-concave*, J. Math. Kyoto Univ. **48**(2008)229-238.

[8] C. Stepniak, *Stochastic ordering and Schur-convex functions in comparison of linear experiments*, Metrika, **36**(1989)291-298.

[9] G. M. Constantine, *Schur-convex functions on the spetra of graphs*, Discrete Math. **45**(1983)181-188.

[10] F. K. Hwang, U. G. Rothblum, *Partition-optimization with Schur convex sum objective functions*, SIAM J. Discrete Math. **18**(2004/2005)512-524.

[11] F. K. Hwang, U. G. Rothblum, L. Shepp, *Monotone optimal multipartitions using Schur convexity with respect to partial orders*, SIAM J. Discrete Math. **6**(1993)533-547.

[12] M. Merkle, *Convexity, Schur-convexity and bounds for the gamma function involving the digamma function*, Rocky Mountain J. Math. **28**(1998)1053-1066.

[13] A. Forcina, A. Giovagnoli, *Homogeneity indices and Schur-convex functions*, Statistica **42** (1982)529-542.

[14] M. Shaked, J. G. Shanthikumar, Y. L. Tong, *Parametric Schur convexity and arrangement monotonicity properties of partial sums*, J. Multivariate Anal. **53**(1995)293-310.

[15] Y. M. Chu, X. M. Zhang, G. D. Wang, *The Schur geometrical convexity of the extended mean values*, J. Convex Anal. **4**(2008)707-718.

[16] K. Z. Guan, *A class of symmetric functions for multiplicatively convex function*, Math. Inequal. Appl. **10**(2007)745-753.

[17] K. Z. Guan, *Schur-convexity of the complete symmetric function*, Math. Inequal. Appl. **9**(2006)567-576.

[18] S. H. Wu, *Generalizaiton and sharpness of the power means inequality and their applications*, J. Math. Anal. Appl. **312**(2005)637-652.

[19] P. S. Bullen, *A Dictionary of Inequalities*, Longman, Harlow, 1998.

[20] D. S. Mitrinović, J. E. Pečarić, V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers Group, Dordrecht, 1989.

[21] H. Weyl, *Inequalities between two kinds of eigenvalues of a linear transformation*, Proc. Nat. Acad. Sci. U. S. A. **35**(1949)408-411.

[22] Y. M. Chu, Y. P. Lv, *The Schur harmonic convexity of the Hamy symmetric function and its applications*, J. Inequal. Appl. **2009**, Article ID 838529, 10 pages.

[23] W. F. Xia, Y. M. Chu, *On Schur convexity of some symmetric functions*, J. Inequal. Appl. **2010**, Article ID 543250, 12 pages.

[24] Y. M. Chu, G. D. Wang, X. H. Zhang, *The Schur multiplicative and harmonic convexities of the complete symmetric function*, Math. Nachr. **5-6**(2011)653-663.

WEIFENG XIA

School of Teacher Education, Huzhou Teachers College, Zhejiang 313000, China

E-mail address: `xwf212@hutc.zj.cn`

YUMING CHU

Department of Mathematics, Huzhou Teachers College, Huzhou 313000, China

E-mail address: `chuyuming@hutc.zj.cn`