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EXISTENCE OF WEAK SOLUTIONS FOR A SEMILINEAR PROBLEM WITH A NONLINEAR BOUNDARY CONDITION

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Abstract. This paper shows conditions for the existence of weak solutions of the problem

$$\begin{cases} -\Delta u = \lambda_1 u + f(x,u) - h(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = g(x,u) & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary, $\frac{\partial u}{\partial n}$ denotes the derivative of u with respect to the outer normal n, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are bounded Carathéodory functions, $h \in L^2(\Omega)$ and $\lambda_1 > 0$ is the principal eigenvalue of $-\Delta$ on Ω with zero Dirichlet boundary conditions. Our method is based on the minimum principle.

1. Preliminaries

The aim of this paper is to investigate the following semilinear problem

$$\begin{cases} -\Delta u = \lambda_1 u + f(x,u) - h(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary, $\frac{\partial u}{\partial n}$ denotes the derivative of u with respect to the outer normal n, $f : \Omega\times\mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega\times\mathbb{R} \to \mathbb{R}$ are bounded Carathéodory functions and $h \in L^2(\Omega)$.

Boundary value problems for partial differential equations play a fundamental role both in theory and applications. To establish the existence of solutions to nonlinear differential problems is very important as well as the application of such results in the physical reality. In fact, it is well known that the mathematical modelling of important questions in different fields of research, such as mechanical engineering, control systems, economics, computer science and many others, leads naturally to the consideration of nonlinear differential equations.
Elliptic problems involving the Laplacian have been studied by several authors; see, e.g., [2, 5, 6, 7, 9] and their references. We follow the proof of main results from Arcoya and Orsina [4].

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It is well known that the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a principal eigenvalue (i.e., the least one) $\lambda_1 > 0$ which is simple and characterized variationally by

$$\lambda_1 = \inf_{u \in W^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u(x)|^2\, dx}{\int_\Omega |u(x)|^2\, dx}.$$

Let us denote by φ_1 the positive (in Ω) eigenfunction associated with λ_1.
We will suppose that f and g satisfy the following conditions:

(F) $\lim_{s\to\pm\infty} f(x,s) = f_{\pm\infty}(x)$, for a.a. $x \in \Omega$,

(G) $\lim_{\tau\to\pm\infty} g(x,\tau) = g_{\pm\infty}(x)$, for a.a. $x \in \partial\Omega$.

By a (weak) solution of (1.1), we mean any $u \in W^{1,2}(\Omega)$ such that

$$\int_\Omega \nabla u(x)\nabla v(x)dx - \lambda_1 \int_\Omega u(x)v(x)dx - \int_\Omega f(x,u(x))v(x)dx + \int_\Omega h(x)v(x)dx - \int_{\partial\Omega} g(x,u(x))v(x)dS = 0 \quad (1.2)$$

for all test function $v \in W^{1,2}(\Omega)$, where dS is the measure on the boundary.

2. Existence results

Our main result is the following.

Theorem 2.1. *Let $f(x,\cdot)$ and $g(x,\cdot)$ be strictly decreasing and let the conditions (F) and (G) be satisfied. Then the problem (1.1) has at least one weak solution if and only if*

$$\int_\Omega f_{+\infty}(x)\varphi_1(x)dx + \int_{\partial\Omega} g_{+\infty}(x)\varphi_1(x)dS < \int_\Omega h(x)\varphi_1(x)dx < \int_\Omega f_{-\infty}(x)\varphi_1(x)dx + \int_{\partial\Omega} g_{-\infty}(x)\varphi_1(x)dS. \quad (2.1)$$

The associated energy functional to the problem (1.1), $E : W^{1,2}(\Omega) \to \mathbb{R}$ is defined by

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \frac{\lambda_1}{2}\int_\Omega |u(x)|^2 dx - \int_\Omega F(x,u(x))dx + \int_\Omega h(x)u(x)dx - \int_{\partial\Omega} G(x,u(x))dS, \quad (2.2)$$

where

$$F(x,s) := \int_0^s f(x,t)dt \quad \text{for a.a. } x \in \Omega \quad \text{and} \quad s \in \mathbb{R}$$

and

$$G(x,\tau) := \int_0^\tau g(x,t)dt \quad \text{for a.a. } x \in \partial\Omega \quad \text{and} \quad \tau \in \mathbb{R}.$$

By the hypotheses on f and g, E is well defined and $E \in C^1(W^{1,2}(\Omega),\mathbb{R})$. Also, the weak solutions of (1.1) are exactly the critical points of the functional E.

Definition 2.2. We say that a functional $E : W^{1,2}(\Omega) \to \mathbb{R}$ satisfies the (PS) condition, if every sequence $\{u_n\}_{n=1}^{\infty} \subset W^{1,2}(\Omega)$ satisfying

$$d := \sup_n E(u_n) < \infty, \qquad \nabla E(u_n) \to 0,$$

contains a convergent subsequence.

Lemma 2.3. *Let E be the energy functional associated with (1.1) and the Landesman-Lazer type condition (2.1) be satisfied. Then each (PS)-sequence for E is bounded.*

Proof. Let $\{u_n\}_{n=1}^{\infty} \subset W^{1,2}(\Omega)$ be such that there exists $c > 0$ such that

$$|E(u_n)| \le c \quad \forall n \in \mathbb{N}, \tag{2.3}$$

and there exists a strictly decreasing sequence $\{\epsilon_n\}_{n=1}^{\infty}$, $\lim_{n\to\infty} \epsilon_n = 0$, such that

$$|\langle E'(u_n), v\rangle| \le \epsilon_n \|v\| \quad \forall n \in \mathbb{N}, \quad \forall v \in W^{1,2}(\Omega). \tag{2.4}$$

Suppose by contradiction that $\|u_n\| \to \infty$, and define $v_n := \frac{u_n}{\|u_n\|}$. Thus $\{v_n\}_{n=1}^{\infty}$ is bounded in $W^{1,2}(\Omega)$ and hence, at least its subsequence, converges to a function v_0 weakly in $W^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$ and $L^2(\partial\Omega)$ (see [8, Theorem A.8]).

Dividing (2.2) with $u = u_n$ by $\|u_n\|^2$, we get due to (2.3),

$$\limsup_{n\to\infty} \left[\frac{1}{2} - \frac{\lambda_1}{2}\int_\Omega |v_n(x)|^2 dx - \int_\Omega \frac{F(x,u_n(x))}{\|u_n\|^2} dx + \int_\Omega h(x)\frac{u_n(x)}{\|u_n\|^2} dx - \int_{\partial\Omega} \frac{G(x,u_n(x))}{\|u_n\|^2} dS\right] \le 0.$$

Since

$$\lim_{n\to\infty} \left[\int_\Omega \frac{F(x,u_n(x))}{\|u_n\|^2} dx + \int_\Omega h(x)\frac{u_n(x)}{\|u_n\|^2} dx - \int_{\partial\Omega} \frac{G(x,u_n(x))}{\|u_n\|^2} dS\right] = 0,$$

by the hypotheses on f, h, g and $\{u_n\}_{n=1}^{\infty}$ while

$$\lim_{n\to\infty} \int_\Omega |v_n(x)|^2 dx = \int_\Omega |v_0(x)|^2 dx,$$

we have

$$\lambda_1 \int_\Omega |v_0(x)|^2 dx \ge 1.$$

Using the weak lower semicontinuity of the norm and the variational characterization of λ_1, we get

$$1 \le \lambda_1 \int_\Omega |v_0(x)|^2 dx \le \int_\Omega |\nabla v_0(x)|^2 dx \le \liminf_{n\to\infty} \int_\Omega |\nabla v_n(x)|^2 dx = 1.$$

Thus

$$\|v_0\| = 1 \quad \text{and} \quad \int_\Omega |\nabla v_0(x)|^2 dx = \lambda_1 \int_\Omega |v_0(x)|^2 dx.$$

This implies, by the definition of φ_1, that $v_0 = \pm\varphi_1$. Choosing $v = v_n - \varphi_1$ in (2.4), we obtain

$$\Big| \int_\Omega \nabla v_n(x) \nabla\big(v_n(x) - \varphi_1(x)\big) dx - \lambda_1 \int_\Omega v_n(x)\big(v_n(x) - \varphi_1(x)\big) dx - \int_\Omega f(x, v_n(x))\big(v_n(x) - \varphi_1(x)\big) dx + \int_\Omega h(x)\big(v_n(x) - \varphi_1(x)\big) dx - \int_{\partial\Omega} g(x, v_n(x))\big(v_n(x) - \varphi_1(x)\big) dS \Big| \le \epsilon_n \|v_n - \varphi_1\|.$$

Since $v_n \to \varphi_1$ in $L^2(\Omega)$ and in $L^2(\partial\Omega)$, by the hypotheses on f, g and h,

$$\lim_{n\to\infty} \int_\Omega v_n(x)\big(v_n(x) - \varphi_1(x)\big)dx = 0,$$

$$\lim_{n\to\infty} \int_\Omega f(x, v_n(x))\big(v_n(x) - \varphi_1(x)\big)dx = 0,$$

$$\lim_{n\to\infty} \int_{\partial\Omega} g(x, v_n(x))\big(v_n(x) - \varphi_1(x)\big)dS = 0,$$

$$\lim_{n\to\infty} \int_\Omega h(x)\big(v_n(x) - \varphi_1(x)\big)dx = 0,$$

we have

$$\lim_{n\to\infty} \int_\Omega \nabla v_n(x)\nabla\big(v_n(x) - \varphi_1(x)\big)dx = 0.$$

Subtracting

$$\lim_{n\to\infty} \int_\Omega \nabla\varphi_1(x)\big(\nabla v_n(x) - \nabla\varphi_1(x)\big)dx,$$

we conclude that

$$0 = \lim_{n\to\infty} \int_\Omega \big(\nabla v_n(x) - \nabla\varphi_1(x)\big)\big(\nabla v_n(x) - \nabla\varphi_1(x)\big)dx \geq \lim_{n\to\infty} \big(\|v_n\| - \|\varphi_1\|\big)^2 \geq 0,$$

which implies $\|v_n\| \to \|\varphi_1\|$. The uniform convexity of $W^{1,2}(\Omega)$ yields that v_n converges strongly to φ_1 in $W^{1,2}(\Omega)$.
Now we write (2.3) and (2.4) with $v = u_n$ in the equivalent forms

$$\begin{aligned} -2c \quad &\leq \quad \int_\Omega |\nabla u_n(x)|^2 dx - \lambda_1 \int_\Omega |u_n(x)|^2 dx - 2\int_\Omega F(x, u_n(x))dx \\ &\quad + 2\int_\Omega h(x)u_n(x)dx - 2\int_{\partial\Omega} G(x, u_n(x))dS \quad \leq \quad 2c \end{aligned}$$

and

$$\begin{aligned} -\epsilon_n\|u_n\| \quad &\leq \quad -\int_\Omega |\nabla u_n(x)|^2 dx + \lambda_1 \int_\Omega |u_n(x)|^2 dx + \int_\Omega f(x, u_n(x))u_n(x)dx \\ &\quad - \int_\Omega h(x)u_n(x)dx + \int_{\partial\Omega} g(x, u_n(x))u_n(x)dS \quad \leq \quad \epsilon_n\|u_n\|. \end{aligned}$$

Summing up and dividing by $\|u_n\|$, we have

$$\begin{aligned} &\bigg| \int_\Omega f(x, u_n(x))v_n(x)dx - 2\int_\Omega \psi(x, u_n(x))v_n(x)dx + \int_\Omega h(x)v_n(x)dx \\ &+ \int_{\partial\Omega} g(x, u_n(x))v_n(x)dS - 2\int_{\partial\Omega} \phi(x, u_n(x))v_n(x)dS \bigg| \leq \frac{2c}{\|u_n\|} + \epsilon_n, \end{aligned}$$

where

$$\psi(x, s) = \begin{cases} \frac{F(x,s)}{s} & \text{if } s \neq 0, \\ f(x, 0) & \text{if } s = 0 \end{cases}$$

and

$$\phi(x, s) = \begin{cases} \frac{G(x,s)}{s} & \text{if } s \neq 0, \\ g(x, 0) & \text{if } s = 0. \end{cases}$$

Letting n to infinity and supposing for example $v_n \to \varphi_1$, we obtain

$$\lim_{n\to\infty}\Big[\int_\Omega f(x,u_n(x))v_n(x)dx - 2\int_\Omega \psi(x,u_n(x))v_n(x)dx + \int_{\partial\Omega} g(x,u_n(x))v_n(x)dS$$
$$-2\int_{\partial\Omega}\phi(x,u_n(x))v_n(x)dS\Big] = -\int_\Omega h(x)\varphi_1(x)dx.$$

Since v_n converges to φ_1, we have $\lim_{n\to\infty} u_n(x) = \infty$ for a.a. $x\in\Omega$ and so

$$f(x,u_n(x)) \to f_{+\infty}(x) \quad \text{for a.a.} \quad x\in\Omega,$$
$$\psi(x,u_n(x)) \to f_{+\infty}(x) \quad \text{for a.a.} \quad x\in\Omega,$$
$$g(x,u_n(x)) \to g_{+\infty}(x) \quad \text{for a.a.} \quad x\in\partial\Omega,$$
$$\phi(x,u_n(x)) \to g_{+\infty}(x) \quad \text{for a.a.} \quad x\in\partial\Omega.$$

The properties of f, F, g and G and the Lebesgue Dominated Convergence Theorem then imply

$$\lim_{n\to\infty}\Big[\int_\Omega f(x,u_n(x))v_n(x)dx - 2\int_\Omega \psi(x,u_n(x))v_n(x)dx + \int_{\partial\Omega} g(x,u_n(x))v_n(x)dS$$
$$-2\int_{\partial\Omega}\phi(x,u_n(x))v_n(x)dS\Big] = -\int_\Omega f_{+\infty}(x)\varphi_1(x)dx - \int_{\partial\Omega} g_{+\infty}(x)\varphi_1(x)dS,$$

and so

$$\int_\Omega f_{+\infty}(x)\varphi_1(x)dx + \int_{\partial\Omega} g_{+\infty}(x)\varphi_1(x)ds = \int_\Omega h(x)\varphi_1(x)dx,$$

which contradicts (2.1) and the Lemma is proved. □

Lemma 2.4. *The functional E given by (2.2) is weakly coercive in $W^{1,2}(\Omega)$.*

Proof. We proceed by contradiction. It is possible to choose a sequence $\{u_n\}_{n=1}^\infty$ such that

$$\|u_n\| \to \infty, \qquad E(u_n) \le c \qquad \text{and} \qquad v_n \to +\varphi_1 \qquad \text{in } W^{1,2}(\Omega).$$

We get

$$\begin{aligned}\int_\Omega h(x)\varphi_1(x)dx \quad &- \quad \int_\Omega f_{+\infty}(x)\varphi_1(x)dx + \int_{\partial\Omega} g_{+\infty}(x)\varphi_1(x)dS \\ &= \lim_{n\to\infty}\Big[\int_\Omega h(x)v_n(x) - \int_\Omega \frac{F(x,u_n(x))}{\|u_n\|}dx + \int_{\partial\Omega}\frac{G(x,u_n(x))}{\|u_n\|}dS\Big] \\ &\le \limsup_{n\to\infty}\frac{E(u_n)}{\|u_n\|} \le \lim_{n\to\infty}\frac{c}{\|u_n\|} = 0,\end{aligned}$$

which contradicts (2.1). This proves the Lemma. □

3. Proof of Theorem 2.1

By lemma 2.4 and weak lower semicontinuity of E, applying the Minimum principle (see [8, p. 4, Theorem 1.2]), the functional E has a global minimum and the problem (1.1) admits a weak solution.

Next, we show that (2.1) is a necessary condition. Let $u \in W^{1,2}(\Omega)$ be a weak solution of (1.1). Then taking $v = \varphi_1$ as a test function in (1.2), we obtain

$$\int_\Omega f(x,u(x))\varphi_1(x)dx + \int_{\partial\Omega} g(x,u(x))\varphi_1(x)dS = \int_\Omega h(x)\varphi_1(x)dx \tag{3.1}$$

due to

$$\int_\Omega \nabla u(x)\nabla\varphi_1(x)dx = \lambda_1 \int_\Omega u(x)\varphi_1(x)dx.$$

Since $f(x,\cdot)$ and $g(x,\cdot)$ are strictly decreasing functions, we obtain

$$\int_\Omega f_{+\infty}(x)\varphi_1(x)dx < \int_\Omega f(x,u(x))\varphi_1(x)dx < \int_\Omega f_{-\infty}(x)\varphi_1(x)dx$$

for a.a. $x \in \Omega$, and

$$\int_{\partial\Omega} g_{+\infty}(x)\varphi_1(x)dS < \int_{\partial\Omega} g(x,u(x))\varphi_1(x)dS < \int_{\partial\Omega} g_{-\infty}(x)\varphi_1(x)dS$$

for a.a. $x \in \partial\Omega$. Summing up and using (3.1), then the proof is complete.

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