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INCLUSION PROPERTIES OF CERTAIN CLASSES OF MEROMORPHIC SPIRAL-LIKE FUNCTIONS OF COMPLEX ORDER ASSOICATED WITH THE GENERALIZED HYPERGEOMETRIC FUNCTION

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Abstract. The purpose of the present paper is to introduce new classes of meromorphic spiral-like functions defined by using a meromorphic analogue of the Choi-Saigo-Srivastava operator for the generalized hypergeometric function and investigate a number of inclusion relationships of these classes.

1. Introduction

Let M denote the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}a_k\ z^k, \tag{1.1}$$

which are analytic in the punctured unit disk $E^*=\{z:0<|z|<1\}=E\backslash\{0\}$.

If f and g are analytic in $E=E^*\cup\{0\}$, we say that f is subordinate to g, written $f\prec g$ or $f(z)\prec g(z)$, if there exists a Schwarz function w in E such that $f(z)=g(w(z))$.

Let P be the class of all functions ϕ which are analytic and univalent in E and for which $\phi(E)$ is convex with $\phi(0)=1$ and $\operatorname{Re}\{\phi(z\}>0$ $(z\in E)$.

For a complex parameters $\alpha_1,...\alpha_q$ and $\beta_1,...\beta_s$ $\quad(\beta_j\in\mathbb{C}\backslash\mathbb{Z}_0^-=\{0,-1,-2,...\};$ $j=1,...s)$, we now define the generalized hypergeometric function [16, 17] as follows:

$$_qF_s(\alpha_1,...\alpha_q;\beta_1,...\beta_s)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k...(\alpha_q)_k}{(\beta_1)_k...(\beta_s)_k\quad k!}z^k, \tag{1.2}$$

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where ($q \leq s+1; s \in \mathbb{N} \cup \{0\}; \mathbb{N} = \{1, 2, ...\}$) and $(v)_k$ is the Pochhammer symbol (or shifted factorial) defined in (terms of the Gamma function) by

$$(v)_k = \frac{\Gamma(v+k)}{\Gamma(v)} = \begin{cases} 1 \ \text{ if } k = 0 \text{ and } v \in \mathbb{C}\backslash\{0\} \\ v(v+1)...(v+k-1) \ \text{ if } k \in \mathbb{N} \text{ and } v \in \mathbb{C}. \end{cases}$$

Corresponding to a function

$$\mathcal{F}(\alpha_1, ...\alpha_q; \beta_1, ...\beta_s; z) = z^{-1} \ {}_qF_s(\alpha_1, ...\alpha_q; \beta_1, ...\beta_s; z). \tag{1.3}$$

Liu and Srivastava [11] consider a linear operator $H(\alpha_1, ...\alpha_q; \beta_1, ...\beta_s) : M \longrightarrow M$ defined by the following Hadamard product(or convolution):

$$H(\alpha_1, ..\alpha_q; \beta_1, ..\beta_s)f(z) = h(\alpha_1, ...\alpha_q; \beta_1, ...\beta_s; z) * f(z).$$

We note that the linear operator $H(\alpha_1, ..\alpha_q; \beta_1, ..\beta_s)$ was motivated essentially by Dzoik and Srivastava [4]. Some interesting developments with the generalized hypergeometric function were considered recently by Dzoik and Srivastava [5, 6] and Liu and Srivastava [9, 10]. Corresponding to the function $h(\alpha_1, ...\alpha_q; \beta_1, ...\beta_s; z)$ defined by (1.3), we introduce a function $h_\lambda(\alpha_1, ...\alpha_q; \beta_1, ...\beta_s; z)$ given by

$$h(\alpha_1, ...\alpha_q; \beta_1, ...\beta_s; z) * h_\lambda(\alpha_1, ...\alpha_q; \beta_1, ...\beta_s; z) = \frac{1}{z(1-z)^\lambda} \quad (\lambda > 0). \tag{1.5}$$

Analogous to $H(\alpha_1, ..\alpha_q; \beta_1, ..\beta_s)$ defined by (1.4), we now define the linear operator $H_\lambda(\alpha_1, ..\alpha_q; \beta_1, ..\beta_s)$ on M as follows:

$$H_\lambda(\alpha_1, ..\alpha_q; \beta_1, ..\beta_s)f(z) = h_\lambda(\alpha_1, ...\alpha_q; \beta_1, ...\beta_s; z) * f(z), \tag{1.6}$$

where $\alpha_i,\ \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ i = 1, ..q;\ j = 1, ..s;\ \lambda > 0;\ z \in E^*;\ f \in M.$

For convenience, we write

$$H_{\lambda,q,s}(\alpha_1) = H_\lambda(\alpha_1, ..\alpha_q; \beta_1, ..\beta_s).$$

It is easily verified from the definition (1.5) and (1.6) that

$$z(H_{\lambda,q,s}(\alpha_1+1)f(z))' = \alpha_1 H_{\lambda,q,s}(\alpha_1)f(z) - (\alpha_1+1)H_{\lambda,q,s}(\alpha_1+1)f(z), \tag{1.7}$$

and

$$z(H_{\lambda,q,s}(\alpha_1)f(z))' = \lambda H_{\lambda+1,q,s}(\alpha_1)f(z) - (\lambda+1)H_{\lambda,q,s}(\alpha_1)f(z). \tag{1.8}$$

We note that the operator $H_{\lambda,q,s}(\alpha_1)$ is closely related to the Choi-Saigo-Srivastava operator [3] for analytic functions, which includes the integral operator studied by Liu [8] and Noor et al [13, 15]. The interested readers are refered to the work done by the authors [1, 2, 14].

Definition 1.1. Using the subordination principle between two analytic functions, we introduce the subclasses $MS_b^\alpha(\phi(z))$, $MC_b^\alpha(\phi(z))$ and $MK_{b,c}^{\alpha,\beta}(\phi(z), \psi(z))$ of the class M as follows:

$$MS_b^\alpha(\phi(z)) = \left\{ f(z) \in M : \ 1 + \frac{e^{i\alpha}}{b\cos\alpha}\left(-\frac{zf'(z)}{f(z)} - 1\right) \prec \phi(z) \ \text{ in } E \right\}$$

$$MC_b^\alpha(\phi(z)) = \left\{ f(z) \in M : \ 1 + \frac{e^{i\alpha}}{b\cos\alpha}\left(-\frac{(zf'(z))'}{f'(z)} - 1\right) \prec \phi(z) \ \text{ in } E \right\}$$

$$MK_{b,c}^{\alpha,\beta}(\phi(z), \psi(z)) = \left\{ f(z) \in M : \ 1 + \frac{e^{i\beta}}{b\cos\beta}\left(-\frac{zf'(z)}{g(z)} - 1\right) \prec \psi(z), \right\},$$

with $g(z) \in MS_b^\alpha(\phi(z))$ in E and $\alpha, \beta \in \mathbb{R} : |\alpha| < \frac{\pi}{2},\ |\beta| < \frac{\pi}{2}, b, c \neq 0$ with $b.c \in \mathbb{C}$ and $\phi(z), \psi(z) \in P, z \in E$.

Now by using the operator $(H_{\lambda,q,s}(\alpha_1)$,we introduce a new subclasses of meromorphic functions.

$$MS^{\alpha}_{b,\lambda,q,s,\alpha_1}(\phi(z)) = \{f(z)\in M:\ H_{\lambda,q,s}(\alpha_1)f(z)\in MS^{\alpha}_{b}(\phi(z))\ \} \quad (1.9)$$

$$MC^{\alpha}_{b,\lambda,q,s,\alpha_1}(\phi(z)) = \{f(z)\in M:\ H_{\lambda,q,s}(\alpha_1)f(z)\in MC^{\alpha}_{b}(\phi(z))\ \} \quad (1.10)$$

$$MK^{\alpha,\beta}_{b,c,\lambda,q,s,,\alpha_1}(\phi(z),\psi(z)) = \left\{f(z)\in M: H_{\lambda,q,s}(\alpha_1)f(z)\in MK^{\alpha,\beta}_{b,c}(\phi(z),\psi(z))\right\}, \quad (1.11)$$

where $\alpha,\ \beta\in\mathbb{R}: |\alpha|<\frac{\pi}{2},\ |\beta|<\frac{\pi}{2},\ b,\ c\neq 0$ with $b.c\in\mathbb{C}$ and $\phi(z),\ \psi(z)\in P$. From (1.9) and (1.10), it is clear that

$$f(z)\in MC^{\alpha}_{b,\lambda,q,s,\alpha_1}(\phi(z)) \Longleftrightarrow -zf'(z)\in MS^{\alpha}_{b,\lambda,q,s,\alpha_1}(\phi(z)). \quad (1.12)$$

2. Preliminary Results

To establish our main results we need the following Lemmas.

Lemma 2.1 [7]. Let ϕ be convex univalent in E with $\phi(0)=1$ and $\mathrm{Re}\{\gamma\phi(z)+t)>0$ $(\gamma,t\in\mathbb{C})$. If p is analytic in E with $p(0)=1$, then

$$p(z)+\frac{z\ p'(z)}{\gamma p(z)+t}\prec\phi(z)\ \ (z\in E)\,,\Rightarrow p(z)\prec\phi(z).$$

Lemma 2.2 [12]. Let $\phi(z)\in P$ be convex univalent in E and $\omega(z)$ be analytic in E with $\mathrm{Re}\{\omega(z)\}\geq 0$. If p is analytic in E with $p(0)=\phi(0)$, then

$$p(z)+\omega(z)zp'(z)\prec\phi(z)\ \ (z\in E)\Longrightarrow p(z)\prec\phi(z).$$

3. Main Results

Theorem 3.1. Let $\alpha\in\mathbb{R}$, where $|\alpha|<\frac{\pi}{2}$ and let $b=b_1+ib_2\neq 0$, $\tan\nu=\frac{b_2}{b_1}$, $\phi(z)\in P$ for $z\in E$ $(\lambda,\ \alpha_1>0)$. Then

$$MS^{\alpha}_{b,\lambda+1,q,s,\alpha_1}(\phi(z))\subset MS^{\alpha}_{b,\lambda,q,s,\alpha_1}(\phi(z))\subset MS^{\alpha}_{b,\lambda,q,s,\alpha_1+1}(\phi(z)),$$

for $\mathrm{Im}\,\phi(z)<(\mathrm{Re}\,\phi(z)-1)\cot(\alpha-v)$.

Proof. To prove the first part of Theorem 3.1, let $f\in MS^{\alpha}_{b,\lambda+1,q,s,\alpha_1}(\phi(z))$ and set

$$p(z)=\frac{1}{b\cos\alpha}\left(-e^{i\alpha}\frac{z(H_{\lambda,q,s}(\alpha_1)f(z))'}{H_{\lambda,q,s}(\alpha_1)f(z)}-(1-b)\cos\alpha-i\sin\alpha\right). \quad (3.1)$$

Then $p(z)$ is analytic in E with $p(0)=1$. Applying (1.8) in (3.1) and with a simple computations, we have for $\lambda>0$

$$\left\{1+\frac{e^{i\alpha}}{b\cos\alpha}\left(-\frac{z(H_{\lambda_{+1},q,s}(\alpha_1)f(z))'}{H_{\lambda_{+1},q,s}(\alpha_1)f(z)}-1\right)\right\}=p(z)+\frac{zp'(z)}{-e^{-i\alpha}b\cos\alpha(p(z)-1)+\lambda+1}\prec\phi(z). \quad (3.2)$$

Since $\mathrm{Re}\{-e^{-i\alpha}b\cos\alpha(\phi(z)-1)+\lambda+1)>0$ for $\mathrm{Im}\,\phi(z)<(\mathrm{Re}\,\phi(z)-1)\cot(\alpha-v)$ and where $\tan\nu=\frac{b_2}{b_1}$, so by Lemma 2.1 and (3.2), we have $p(z)\prec\phi(z)$. This proves that

$$MS^{\alpha}_{b,\lambda+1,q,s,\alpha_1}(\phi(z))\subset MS^{\alpha}_{b,\lambda,q,s,\alpha_1}(\phi(z)). \quad (3.3)$$

To prove the second part of Theorem 3.1, we consider

$$p(z)=\frac{1}{b\cos\alpha}\left(-e^{i\alpha}\frac{z(H_{\lambda,q,s}(\alpha_1+1)f(z))'}{H_{\lambda,q,s}(\alpha_1+1)f(z)}-(1-b)\cos\alpha-i\sin\alpha\right). \quad (3.4)$$

Then $p(z)$ is analytic in E with $p(0)=1$. Applying (1.7) in (3.1) and with a simple computation, we have for $\alpha_1>0$

$$\left\{1+\frac{e^{i\alpha}}{b\cos\alpha}\left(-\frac{z(H_{\lambda,q,s}(\alpha_1)f(z))'}{H_{\lambda,q,s}(\alpha_1)f(z)}-1\right)\right\}=p(z)+\frac{zp'(z)}{-e^{-i\alpha}b\cos\alpha(p(z)-1)+\alpha_1+1}\prec\phi(z). \tag{3.5}$$

Since $\operatorname{Re}\{-e^{-i\alpha}b\cos\alpha(\phi(z)-1)+\alpha_1+1)>0$ for $\operatorname{Im}\phi(z)<(\operatorname{Re}\phi(z)-1)\cot(\alpha-v)$ and where $\tan\nu=\frac{b_2}{b_1}$, so by Lemma 2.1 and (3.5), we have $p(z)\prec\phi(z)$. This complete the proof of second inclusion.

Theorem 3.2. Let $\alpha\in\mathbb{R}$, where $|\alpha|<\frac{\pi}{2}$ and let $b=b_1+ib_2\neq 0$, $\tan\nu=\frac{b_2}{b_1}$, $\phi(z)\in P$ for $z\in E$ $(\lambda,\alpha_1>0)$. Then

$$MC^{\alpha}_{b,\lambda+1,q,s,\alpha_1}(\phi(z))\subset MC^{\alpha}_{b,\lambda,q,s,\alpha_1}(\phi(z))\subset MC^{\alpha}_{b,\lambda,q,s,\alpha_1+1}(\phi(z)).$$

for $\operatorname{Im}\phi(z)<(\operatorname{Re}\phi(z)-1)\cot(\alpha-v)$, $z\in E$.

Proof. The proof follows from Theorem 3.1 and (1.12).

Taking

$$\phi(z)=\frac{1+Az}{1+Bz}\ (-1<B<A\leq 1; z\in E)$$

Corollary 3.3. Let $\alpha\in\mathbb{R}$, where $|\alpha|<\frac{\pi}{2}$ and let $b=b_1+ib_2\neq 0$, $\tan\nu=\frac{b_2}{b_1}$, $\frac{1+A}{1+B}<\min\{\lambda+1/e^{-i\alpha}b\cos\alpha,\ \alpha_1+1/e^{-i\alpha}b\cos\alpha\}$, $-1<B<A\leq 1$. Then

$$MS^{\alpha}_{b,\lambda+1,q,s,\alpha_1}(A,B))\subset MS^{\alpha}_{b,\lambda,q,s,\alpha_1}(A,B)\subset MS^{\alpha}_{b,\lambda,q,s,\alpha_1+1}(A,B),$$

and

$$MC^{\alpha}_{b,\lambda+1,q,s,\alpha_1}(A,B)\subset MC^{\alpha}_{b,\lambda,q,s,\alpha_1}(A,B)\subset MC^{\alpha}_{b,\lambda,q,s,\alpha_1+1}(A,B).$$

Next, by using Lemma 2.2, we obtain the following Inclusion relation for the class of meromorphically close to convex functions.

Theorem 3.4. Let $\alpha,\beta\in\mathbb{R}$, where $|\alpha|<\frac{\pi}{2}$, $|\beta|<\frac{\pi}{2}$ and let $b=b_1+ib_2\neq 0$, $\tan\nu=\frac{b_2}{b_1}$, $\phi(z)$, $\psi(z)\in P$ for $z\in E$. Then

$$MK^{\alpha,\beta}_{b,c,\lambda+1,q,s,,\alpha_1}(\phi(z),\psi(z))\subset MK^{\alpha,\beta}_{b,c,\lambda,q,s,,\alpha_1}(\phi(z),\psi(z))\subset MK^{\alpha,\beta}_{b,c,\lambda,q,s,,\alpha_1+1}(\phi(z),\psi(z)),$$

for $\operatorname{Im}\phi(z)<\operatorname{Re}(\operatorname{Re}\phi(z)-1)\cot(\alpha-v)$, $\operatorname{Im}(q(z)<(\operatorname{Re}q(z)-1)\cot(\alpha-v)(z\in E)$, $(\lambda,\alpha_1>0)$.

Proof. To prove the first inclusion of Theorem 3.4, let $f\in MK^{\alpha,\beta}_{b,c,\lambda+1,q,s,,\alpha_1}(\phi(z),\psi(z))$. Then from the definition of $MK^{\alpha,\beta}_{b,c,\lambda+1,q,s,,\alpha_1}(\phi(z),\psi(z))$, there existsa function $g\in MS^{\alpha}_{b,\lambda+1,q,s,\alpha_1}(\psi(z)))$ such that

$$\frac{1}{c\cos\beta}\left(-e^{i\beta}\frac{z(H_{\lambda+1,q,s}(\alpha_1)f(z))'}{H_{\lambda+1,q,s}(\alpha_1)g(z)}-(1-c)\cos\beta-i\sin\beta\right)\prec\psi(z).(z\in E). \tag{3.6}$$

Now let

$$p(z)=\frac{1}{c\cos\beta}\left(-e^{i\beta}\frac{z(H_{\lambda,q,s}(\alpha_1)f(z))'}{H_{\lambda,q,s}(\alpha_1)g(z)}-(1-c)\cos\beta-i\sin\beta\right), \tag{3.7}$$

where $p(z)$ is analytic in E with $p(0) = 1$. Using (1.8), we obtain that

$$\begin{aligned}&\frac{1}{c\cos\beta}\left(-e^{i\beta}\frac{z(H_{\lambda+1,q,s}(\alpha_1)f(z))'}{H_{\lambda+1,q,s}(\alpha_1)g(z)}-(1-c)\cos\beta-i\sin\beta\right)\\ =\;&\frac{1}{c\cos\beta}\left(e^{i\beta}\frac{\frac{z(H_{\lambda,q,s}(\alpha_1)(-zf(z))'}{H_{\lambda,q,s}(\alpha_1)g(z)}+(\lambda+1)\frac{z(H_{\lambda,q,s}(\alpha_1)(-zf(z))}{H_{\lambda,q,s}(\alpha_1)g(z)}}{\frac{(H_{\lambda,q,s}(\alpha_1)g(z))'}{H_{\lambda,q,s}(\alpha_1)g(z)}+\lambda+1}-(1-c)\cos\beta-i\sin\beta\right).\end{aligned} \tag{3.8}$$

Since $g(z) \in MS^{\alpha}_{b,\lambda+1,q,s,\alpha_1}(\phi(z)) \subset MS^{\alpha}_{b,\lambda,q,s,\alpha_1}(\phi(z))$, by Theorem 3.1, we set

$$q(z)=\frac{1}{b\cos\alpha}\left(-e^{i\alpha}\frac{z(H_{\lambda,q,s}(\alpha_1)f(z))'}{H_{\lambda,q,s}(\alpha_1)g(z)}-(1-b)\cos\alpha-i\sin\alpha\right), \tag{3.9}$$

where $q \prec \phi$ in E with assumption $\phi \in P$. Then , by virtue of (3.7), (3.8) and (3.9), we obtain that

$$\begin{aligned}&\frac{1}{c\cos\beta}\left(-e^{i\beta}\frac{z(H_{\lambda+1,q,s}(\alpha_1)f(z))'}{H_{\lambda+1,q,s}(\alpha_1)g(z)}-(1-c)\cos\beta-i\sin\beta\right)\\ =\;&p(z)+\frac{zp'(z)}{-e^{-i\alpha}(q(z)-1)+\lambda+1}\prec\psi(z)\quad(z\in E).\end{aligned} \tag{3.10}$$

Since $q \prec \phi$ and $\lambda > 0$ in E with $\mathrm{Re}(-e^{-i\alpha}(q(z)-1)+\lambda+1) > 0$ for $\mathrm{Im}\,\phi(z) < \mathrm{Re}(\mathrm{Re}\,\phi(z)-1)\cot(\alpha-v)$, $\mathrm{Im}(q(z) < (\mathrm{Re}\,q(z)-1)\cot(\alpha-v)$. Hence, by taking

$$\omega(z)=\frac{1}{-e^{-i\alpha}(q(z)-1)+\lambda+1},$$

in (3.10), and applying Lemma2.2, we can show that $p(z) \prec \psi(z)$ in E, so that $f(z) \in MK^{\alpha,\beta}_{b,c,\lambda,q,s,,\alpha_1}(\phi(z),\psi(z))$. Moreover, we have the second inclusion by using the similar arguments to those detailed above with (1.7). Therefore we complete the proof of the Theorem 3.4.

Inclusion properties involving the Integral operaton F_μ

Consider the operator F_μ, defined by

$$F_\mu(f)(z)=\frac{\mu}{z^{\mu+1}}\int_0^z t^\mu f(t)dt\quad(f\in M;\mu>0). \tag{3.11}$$

From the definition of F_μ defined by (3.11), we observe that

$$z(H_{\lambda,q,s}(\alpha_1)F_\mu f(z))'=\mu H_{\lambda,q,s}(\alpha_1)f(z)-(\mu+1)H_{\lambda,q,s}(\alpha_1)F_\mu f(z). \tag{3.12}$$

Theorem 3.5. Let $\alpha \in \mathbb{R}$, where $|\alpha| < \frac{\pi}{2}$ and let $b = b_1 + ib_2 \neq 0$, $\phi(z) \in P$ for $z \in E$ $(\lambda, \alpha_1 > 0)$. Then for $f(z) \in MS^{\alpha}_{b,\lambda,q,s,\alpha_1}(\phi(z))$, then $F_\mu(f)(z) \in MS^{\alpha}_{b,\lambda,q,s,\alpha_1}(\phi(z))$, for $\mathrm{Im}\,\phi(z) < (\mathrm{Re}\,\phi(z)-1)\cot(\alpha-v)$, where $\tan v = \frac{b_2}{b_1}$, $z \in E$.

Proof. Consider

$$p(z)=\frac{1}{b\cos\alpha}\left(-e^{i\alpha}\frac{z(H_{\lambda,q,s}(\alpha_1)F_\mu(f)(z))'}{H_{\lambda,q,s}(\alpha_1)F_\mu(f)(z)}-(1-b)\cos\alpha-i\sin\alpha\right), \tag{3.13}$$

where $p(z)$ is analytic in E with $p(0) = 1$. Using (3.12) in (3.13) and after simple computation we have

$$p(z)+\frac{zp'(z)}{-e^{-i\alpha}b\cos\alpha(p(z)-1)+\mu}\prec\phi(z).$$

For $\operatorname{Im}\phi(z) < (\operatorname{Re}\phi(z)-1)\cot(\alpha - v)$, where $\tan v = \frac{b_2}{b_1}$, we have

$$\operatorname{Re}(-e^{-i\alpha} b\cos\alpha(p(z)-1)+\mu) > 0.$$

Thus, by Lemma 2.1 yeilds $p(z) \prec \phi(z)$. Hence we have the desired proof.

Next, we derive an inclusion property involving F_μ which is obtaind by applying (1.8) and Theorem 3.5.

Theorem 3.6. Let $\alpha \in \mathbb{R}$, where $|\alpha| < \frac{\pi}{2}$ and let $b = b_1 + ib_2 \neq 0$, $\phi(z) \in P$ for $z \in E$ $(\lambda, \alpha_1 > 0)$. Then for $f(z) \in MC^{\alpha}_{b,\lambda,q,s,\alpha_1}(\phi(z))$, then $F_\mu(f)(z) \in MC^{\alpha}_{b,\lambda,q,s,\alpha_1}(\phi(z))$, for $\operatorname{Im}\phi(z) < (\operatorname{Re}\phi(z)-1)\cot(\alpha - v)$, where $\tan v = \frac{b_2}{b_1}$, $z \in E$.

Finally, we obtain Theorem 3.7 below by using the same lines of proof as we used in the proof of Theorem3.4.

Theorem 3.7. Let $\alpha, \beta \in \mathbb{R}$,where $|\alpha| < \frac{\pi}{2}$, $|\beta| < \frac{\pi}{2}$ and let $b = b_1 + ib_2 \neq 0$, $\tan\nu = \frac{b_2}{b_1}$, $\phi(z), \psi(z) \in P$ for $z \in E$ $(\lambda, \alpha_1 > 0)$. If $f \in MK^{\alpha,\beta}_{b,c,\lambda+1,q,s,,\alpha_1}(\phi(z), \psi(z))$, Then $F_\mu(f)(z) \in MK^{\alpha,\beta}_{b,c,\lambda+1,q,s,,\alpha_1}(\phi(z), \psi(z))$ $(\mu > 0)$ for

$$\operatorname{Im}\phi(z) < (\operatorname{Re}\phi(z)-1)\cot(\alpha - v),\ \ \operatorname{Im}q(z) < (\operatorname{Re}q(z)-1)\cot(\alpha - v) \text{ and } q(z) \prec \phi(z), z \in E.$$

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References

[1] Ali Muhammad, On certain class of meromorphic functions defined by means of a linear operator, J. Acta. Univesitatis. Apulensis, no 23 (2010), 251-262.

[2] N. E. Cho and In. Hwa. Kim, Inclusion properties of certain of certain classes of meromophic functions associated with the gerneralized hepergeometric functions. Appl. Math. Comput. 187 (2007), 115-121.

[3] J. H. Choi, M. Saigo and H. M. Srivastava, Some inclusion properties of a certain family of integral operators, J. Math. Anal. Appl. 276 (2002), 432 -445.

[4] J. Dziok, and H. M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, Appl. Math. Comput, 103 (1999), 1-13.

[5] J. Dziok, and H. M. Srivastava, Some subclasses of analytic functions with fixed argument of coefficients associated with the generalized hypergeometric functions, Adv. Stud. Contemp. Math. 5 (2002), 115 -125.

[6] J. Dziok, and H. M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, Integral Trans. Spec. Funct. 14 (2003), 7-18.

[7] P. Eenigenberg, S. S. Miller, P.T. Mocanu and M. O. Reade, On a Briot-Bouquet differential subordination, General Inequalities 3 (1983),339-348.

[8] J. L. Liu, The Noor integral and strongly starlike functions, J. Math. Anal. Appl. 261 (2001), 441- 447.

[9] J. L. Liu and H. M. Srivastava, A linear operator and associated families of meromorphically multivalent functions, J. Math. Anal. Appl.259 (2001), 566-581.

[10] J. L. Liu and H. M. Srivastava, Certain properties of the Dzoik Srivastava operator, Appl. Math. Comput. 159 (2004), 485-493.

[11] J. L. Liu and H. M. Srivastava, Classes of meromorphically multivalent functions associated with the generalized hypergeormetric function, Math. Comut. Modell. 39 (2004), 21- 34.

[12] S. S. Miller and P. T. Mocanu, Differential subordinations and univalent functions, Michigan Math. J. 28 (1981), 157-171.

[13] K. I. Noor, On new classes of integral operators, J. Natur. Geom. 16 (1999), 71- 80.

[14] K. I. Noor and Ali Muhammad, On certain subclasses of meromorphic univalent functions, Bull. Institute. Maths. Academia. Sinica, Vol 5 (2010), 83-94.

[15] K. I. Noor and M. A. Noor, On integral operators, J. Math. Anal. Appl. 238 (1999), 341- 352.

[16] S. Owa and H. M. Srivastava, univalent and starlike generalized hypergeometric functions, canad. J. Math. 39 (1987), 1057-1077.

[17] H. M. Srivastava and S. Owa, Some characterization and distortion theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators, and certain subclasses of analytic functions, Nagoya Math. J. 106 (1987), 1-28.

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