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A PATHOLOGICAL EXAMPLE OF A DOMINANT TERRACED MATRIX

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Dedicated to Nancy Lee Shell

Abstract. A hyponormal terraced matrix is modified to produce an example of a non-hyponormal dominant terraced matrix.

1. Introduction

This brief paper addresses a question left open at the end of [5] – Does there exist a terraced matrix, acting as a bounded linear operator on ℓ^2, that is dominant but not hyponormal? The answer will be provided by modifying one particular entry of a known hyponormal terraced matrix.

A *terraced* matrix M is a lower triangular infinite matrix with constant row segments. The matrix M is *dominant* [6] if $Ran(M-\lambda) \subset Ran(M-\lambda)^*$ for all λ in the spectrum of M, and M is *hyponormal* if it satisfies $\langle (M^*M - MM^*)f, f\rangle \geq 0$ for all f in ℓ^2. Hyponormal operators are necessarily dominant. From [3] we know that M is dominant if and only if for each complex number λ there exists an operator $T = T(\lambda)$ on ℓ^2 such that $(M-\lambda) = (M-\lambda)^*T$.

2. Main Results

Our first theorem involves the terraced matrix $M :\equiv M(a)$ associated with a sequence $a = \{a_n : n = 0, 1, 2, 3, ...\}$ of real numbers. Throughout this section we assume that M acts through matrix multiplication to give a bounded linear operator on ℓ^2.

Theorem 2.1. *Suppose that $M(a)$ is the terraced matrix associated with a sequence $a = \{a_n\}$ satisfying the following conditions:*
(1) $\{a_n\}$ is a strictly decreasing sequence that converges to 0;
(2) $\{(n+1)a_n\}$ is a strictly increasing sequence that converges to $L < +\infty$; and
(3) $\frac{1}{a_{n+1}} \geq \frac{1}{2}(\frac{1}{a_n} + \frac{1}{a_{n+2}})$ for all n.

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If the sequence $b = \{b_n\}$ satisfies $0 < b_0 < 2L$ and $b_n = a_n$ for all $n \geq 1$, then $M(b)$ is dominant.

Proof. First we show that

$$Ran(M(b) - \lambda) \subset Ran(M(b) - \lambda)^*$$

for all $\lambda \neq b_0$. Since our hypothesis guarantees that $M :\equiv M(a)$ is hyponormal (see [4, Theorem 2.2]) and therefore also dominant, for each complex number λ there must exist an operator $T = [t_{ij}]$ on ℓ^2 such that $(M - \lambda) = (M - \lambda)^* T$. For $\lambda \neq b_0$, replace the first row of T by

$$< \frac{a_0 - \overline{\lambda}}{b_0 - \overline{\lambda}} t_{00} - \frac{a_0 - b_0}{b_0 - \overline{\lambda}}, \frac{a_0 - \overline{\lambda}}{b_0 - \overline{\lambda}} t_{01}, \frac{a_0 - \overline{\lambda}}{b_0 - \overline{\lambda}} t_{02}, \frac{a_0 - \overline{\lambda}}{b_0 - \overline{\lambda}} t_{03}, \frac{a_0 - \overline{\lambda}}{b_0 - \overline{\lambda}} t_{04}, ... >$$

and call the new matrix T'. Clearly T' is bounded on ℓ^2 since T is, and it is routine to verify that $(M(b) - \lambda) = (M(b) - \lambda)^* T'$ for $\lambda \neq b_0$.

We now consider the case $\lambda = b_0$. If $x :\equiv < x_0, x_1, x_2, ... >^T \in \ell^2$, it must be shown that $(M(b) - b_0)x \in Ran(M(b) - b_0)^*$. Since $M(a)$ is dominant, we know that

$$(M(a) - b_0)x = (M(a) - b_0)^* y$$

for some $y :\equiv < y_0, y_1, y_2, ... >^T \in \ell^2$. It can be verified that

$$(M(b) - b_0)^* y = (a_0 - b_0)(x_0 - y_0)e_0 + (M(b) - b_0)x,$$

where $\{e_n : n \geq 0\}$ is the standard orthonormal basis for ℓ^2. We want to find $z = < z_0, z_1, z_2, z_3, ... >^T \in \ell^2$ satisfying

$$(M(b) - b_0)^* z = (b_0 - a_0)(x_0 - y_0)e_0.$$

Computations reveal that z_0 can be chosen arbitrarily, but we must have

$$z_n = \frac{\prod_{j=0}^{n-1}(b_0 - a_j)}{b_0^n}(x_0 - y_0)$$

for $n \geq 1$. Raabe's Test [2, p. 396] can then be used to verify that $z \in \ell^2$ when $2L > b_0$. This means we have $(M(b) - b_0)^*(y + z) = (M(b) - b_0)x$, and this completes our proof that $M(b)$ is dominant for $0 < b_0 < 2L$. □

We note the following corollary to the proof of the theorem.

Corollary 2.2. *Suppose that $M(a)$ is the terraced matrix associated with a decreasing sequence $a = \{a_n\}$ of positive numbers converging to 0 and that $\{(n+1)a_n\}$ converges to a finite number $L > 0$. If $M(a)$ is a hyponormal operator on ℓ^2 and the sequence $b = \{b_n\}$ satisfies $0 < b_0 < 2L$ and $b_n = a_n$ for all $n \geq 1$, then $M(b)$ is dominant.*

We observe that the preceding theorem and corollary have made no assertion regarding hyponormality for $M(b)$. In the following, we let S_n denote the n-by-n section in the northwest corner of the matrix of the self-commutator $M(b)^* M(b) - M(b)M(b)^*$.

Example 2.1. *(Modified Cesàro Matrix). Start with $M(a)$ given by $a_n = \frac{1}{n+1}$ for all n. Take $b_0 \in (0, 2)$ and $b_n = \frac{1}{n+1}$ for all $n \geq 1$. We observe that this example satisfies the hypothesis of Corollary 2.2 with $L = 1$ since the Cesàro operator $M(a)$*

on ℓ^2 is known to be hyponormal (see [1]), so $M(b)$ is dominant when $0 < b_0 < 2$. If z_1, z_2 denote the zeroes of

$$y = -[\frac{1}{36}(\frac{\pi^2}{6} - \frac{19}{12}) + \frac{1}{108}]x^2 + [\frac{5}{36}(\frac{\pi^2}{6} - \frac{19}{12}) + \frac{1}{54}]x - [\frac{5}{72}(\frac{\pi^2}{6} - \frac{19}{12}) + \frac{1}{108}],$$

then $det(S_3) < 0$ when

$$b_0 \in (0, z_1 \approx 0.69665) \cup (z_2 \approx 1.77128, 2),$$

so $M(b)$ will not be hyponormal for those values of b_0.

We note that with a little more effort it can be demonstrated that $M(b)$ is not hyponormal for any $b_0 \in (0,2) \setminus \{1\}$. This can be accomplished by applying an obvious sequence of elementary row and column operations to reduce S_n to arrowhead form and then to upper triangular form, in which the first diagonal element is negative for $n = n(b_0)$ sufficiently large and all of the rest of the diagonal elements are positive, so $det(S_n) < 0$ when n is sufficiently large.

We will now present a result that applies to terraced matrices associated with sequences of complex numbers.

Theorem 2.3. *Assume that $M :\equiv M(\alpha)$ is a terraced matrix associated with an injective sequence $\alpha = \{\alpha_n : n = 0, 1, 2, 3, ...\}$ of nonzero complex numbers, and let $M(\beta)$ denote the terraced matrix associated with the sequence $\beta = \{\beta_n\}$ given by $\beta_0 = \alpha_1$ and $\beta_n = \alpha_n$ for all $n \geq 1$. If $M(\alpha)$ is hyponormal, then $M(\beta)$ is dominant but not hyponormal.*

Proof. The proof that $Ran(M(\beta) - \lambda) \subset Ran(M(\beta) - \lambda)^*$ for all $\lambda \neq \beta_0$ requires only a minor adjustment of the argument used in the proof of Theorem 2.1, so we leave that to the reader. We now show that if $\lambda = \beta_0$, then

$$Ran(M(\beta) - \lambda) \subset Ran(M(\beta) - \lambda)^*.$$

Recall that $\beta_0 = \alpha_1$. If $x :\equiv < x_0, x_1, x_2, ... >^T \in \ell^2$, it must be shown that

$$(M(\beta) - \alpha_1)x \in Ran(M(\beta) - \alpha_1)^*.$$

Since $M :\equiv M(\alpha)$ is hyponormal and therefore also dominant, we know that

$$(M - \alpha_1)x = (M - \alpha_1)^* y$$

for some $y :\equiv < y_0, y_1, y_2, ... >^T \in \ell^2$. It can be verified that

$$(M(\beta) - \alpha_1)^* y = [(\alpha_0 - \alpha_1)x_0 - (\overline{\alpha_0} - \overline{\alpha_1})y_0]e_0 + (M(\beta) - \alpha_1)x.$$

If

$$z :\equiv \frac{1}{\overline{\alpha_1}}[(\overline{\alpha_0} - \overline{\alpha_1})y_0 - (\alpha_0 - \alpha_1)x_0]e_1,$$

then $(M(\beta) - \alpha_1)^* z = [(\overline{\alpha_0} - \overline{\alpha_1})y_0 - (\alpha_0 - \alpha_1)x_0]e_0$. It follows that

$$(M(\beta) - \alpha_1)^*(y + z) = (M(\beta) - \alpha_1)x,$$

and now the proof that $M(\beta)$ is dominant is complete. Finally, since $det(S_2) = -|\alpha_1|^4 < 0$, $M(\beta)$ cannot be hyponormal. □

Example 2.2. *Recall that for fixed $k > 0$, the generalized Cesàro matrices of order one are the terraced matrices $C_k :\equiv M(a)$ that occur when $a_n = \frac{1}{k+n}$ for all n. C_k is hyponormal for $k \geq 1$. If $M_k :\equiv M(b)$ is the terraced matrix associated with the sequence defined by $b_0 = \frac{1}{k+1}$ and $b_n = \frac{1}{k+n}$ for all $n \geq 1$, then we know from Theorem 2.3 that M_k is dominant but not hyponormal for $k \geq 1$.*

In closing, we are reminded of another question left open at the conclusion of [5] – Is C_k dominant for $\frac{1}{2} \leq k < 1$? The next result provides a partial answer to that question.

Proposition 2.4. *C_k is not dominant when $k = \frac{1}{2}$.*

Proof. It can easily be verified that

$$(C_k - \frac{1}{k})[k(e_0 - e_1)] = e_1,$$

so $e_1 \in Ran(C_k - \frac{1}{k})$ for all $k > 0$. For C_k to be dominant, then it must also be true that $e_1 \in Ran(C_k - \frac{1}{k})^*$. A straightforward calculation reveals that in order to have $e_1 = (C_k - \frac{1}{k})^* z$ for some $z =< z_0, z_1, z_2, z_3, ... >^T$, it is necessary that $z_1 = -k$ and $z_n = \frac{(n-1)!k^2}{\prod_{j=1}^{n-1}(k+j)}$ for all $n \geq 2$. However, it then follows from a refinement (see [2, Theorem III, p. 396]) of Raabe's test that $z \notin \ell^2$ for $k = \frac{1}{2}$ and hence $e_1 \notin Ran(C_k - \frac{1}{k})^*$ for that value of k. □

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