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A GENERALIZED CLASS OF k-UNIFORMLY STARLIKE FUNCTIONS INVOLVING WGH OPERATORS

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ABSTRACT. In this paper, involving Wgh operators $\mathbf{W}_q^p([\alpha_1])$ and $\mathbf{W}_q^p([\alpha_1 + 1])$, a generalized class of k-uniformly starlike functions is defined. Some results on coefficient inequalities, inclusion and convolution properties for functions belonging to this class are derived. Our results generalize some of the previously obtained results as well as generate new ones.

1. INTRODUCTION

Let $\mathcal{S}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are univalent analytic in the open unit disk $\Delta = \{z \in \mathbb{C}; |z| < 1\}$. Let $\mathcal{S}^*$ and $\mathcal{CV}$ denote the subclasses of $\mathcal{S}$ whose members are, respectively, starlike and convex in Δ.

Subclasses k-$\mathcal{SP}$ and k-$\mathcal{UCV}$ of $\mathcal{S}^*$ and $\mathcal{CV}$, respectively, are studied by Kanas and Wiśniowska in [13], [14] (see [12], [16]) which are defined as follows:

Definition 1.1. *Let $f \in \mathcal{S}$, and $0 \le k < \infty$. Then $f \in k$-$\mathcal{SP}$ if and only if*

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|. \tag{1.2}$$

Definition 1.2. *Let $f \in \mathcal{S}$, and $0 \le k < \infty$. Then $f \in k$-$\mathcal{UCV}$ if and only if*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right|. \tag{1.3}$$

By Alexander property, we have $f \in k$-$\mathcal{UCV} \Leftrightarrow zf' \in k$-$\mathcal{SP}$. Note that these classes were introduced by Goodman [10] by giving a two-variable characterisation

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of 1-$\mathcal{UCV}$. Rønning [20] and independently Ma and Minda [17] have given a more applicable one-variable characterisation for this class.

Geometrically, the class k-$\mathcal{SP}$ (k-$\mathcal{UCV}$) is described as the family of functions f such that $p(z) = \frac{zf'(z)}{f(z)}$ $(1 + \frac{zf''(z)}{f'(z)})$ is subordinate to the univalent functions p_k such that $p_k(\Delta)$ describe a conic region:

$$\Omega_k = \left\{u + iv : u > k\sqrt{(u-1)^2 + v^2}\right\}, \tag{1.4}$$

with $1 \in \Omega_k$. Some explicit form of extremal functions p_k are given in [12].

For $\alpha_i \in \mathbb{C}$ $\left(\frac{\alpha_i}{A_i} \neq 0, -1, -2, ..., \ A_i > 0; i = 1, 2, ..., p\right)$ and $\beta_i \in \mathbb{C}$ $\left(\frac{\beta_i}{B_i} \neq 0, -1, -2, ..., \ B_i > 0; i = 1, 2, ..., q\right)$ such that $1 + \sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i \geq 0$, Wright's generalized hypergeometric (Wgh) function ${}_p\psi_q\ [z]$ [24] ([23]) is defined by

$$ {}_p\psi_q\ [z] = {}_p\psi_q \left[\begin{array}{c} (\alpha_i, A_i)_{1,p} \\ (\beta_i, B_i)_{1,q} \end{array} ; z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + nA_i)}{\prod_{i=1}^{q} \Gamma(\beta_i + nB_i)} \frac{z^n}{n!}, \tag{1.5}$$

which is analytic for bounded values of $|z|$. Involving Wgh function defined by (1.5) with $\alpha_i \neq 0, -1, -2, ..., \ i = 1, 2, ..., p$ and $\beta_i \neq 0, -1, -2, ..., \ i = 1, 2, ..., q$, a linear operator: $\mathbf{W}_q^p([\alpha_1]) = \mathbf{W}_q^p((\alpha_i, A_i)_{1,p}; (\beta_i, B_i)_{1,q}) : \mathcal{S} \to \mathcal{S}$ is defined with the use of convolution $*$ for f of the form (1.1) by

$$\begin{aligned} \mathbf{W}_q^p([\alpha_1])f(z) &= z \frac{\prod_{i=1}^{q} \Gamma(\beta_i)}{\prod_{i=1}^{p} \Gamma(\alpha_i)} \ {}_p\psi_q \left[\begin{array}{c} (\alpha_i, A_i)_{1,p} \\ (\beta_i, B_i)_{1,q} \end{array} ; z \right] * f(z) \qquad (1.6) \\ &= z + \sum_{n=2}^{\infty} a_n\ \theta_n\ z^n, z \in \Delta, \end{aligned}$$

where

$$\theta_n = \frac{\prod_{i=1}^{p} \frac{\Gamma(\alpha_i + (n-1)A_i)}{\Gamma(\alpha_i)}}{\prod_{i=1}^{q} \frac{\Gamma(\beta_i + (n-1)B_i)}{\Gamma(\beta_i)}} \frac{1}{(n-1)!}, n \geq 2. \tag{1.7}$$

Also, we get

$$\mathbf{W}_q^p([\alpha_1 + 1])f(z) := z + \sum_{n=2}^{\infty} \left(1 + \frac{(n-1)A_1}{\alpha_1}\right) a_n\ \theta_n\ z^n. \tag{1.8}$$

We call the operators $\mathbf{W}_q^p([\alpha_1])$, $\mathbf{W}_q^p([\alpha_1 + 1])$ as the Wgh operators. Note that the operator $\mathbf{W}_q^p([\alpha_1])$ was defined by Dziok and Raina in [6] and was used in several works, see [1], [2], [4], [5], [6], [7], [18], [19], [22].

Taking $A_i = 1\ (i = 1, 2, .., p)$ and $B_i = 1\ (i = 1, 2, .., q)$, Wgh operator $\mathbf{W}_q^p([\alpha_1])$ reduces to the Dziok-Srivastava operator $\mathbf{F}_q^p([\alpha_1])$ ([8]) which is defined for $f \in \mathcal{S}$ by

$$\mathbf{F}_q^p([\alpha_1])f(z) = z\ {}_pF_q\ [z] * f(z),$$

where ${}_pF_q\ [z]$ is the generalized hypergeometric function:

$$\begin{aligned} {}_pF_q\ [z] &= {}_pF_q\,(\alpha_1,\dots\alpha_p;\beta_1,\dots\beta_q;z) \\ &= \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{i=1}^{q}(\beta_i)_n}\frac{z^n}{n!},\ p\le 1+q. \end{aligned}$$

The symbol $(\lambda)_n$ is the Pochhammer symbol. Operator $\mathbf{F}_1^2([\alpha_1])$ is called the Hohlov operator [11], $\mathbf{F}_1^2(\alpha_1,1;\beta_1)$ is the Carlson and Shaffer operator [3] and $\mathbf{F}_1^2(1+\lambda,1;1)$ is the Ruscheweyh derivative operator [21] which is defined for $f\in\mathcal{S}$ by

$$\mathbf{D}^{\lambda}f(z)=\frac{z}{(1-z)^{1+\lambda}}*f(z),\lambda>-1.$$

We define here a new class $\mathcal{UC}(k,[\alpha_1])$ involving Wgh operators $\mathbf{W}_q^p([\alpha_1])$ and $\mathbf{W}_q^p([\alpha_1+1])$:

Definition 1.3. *Let Wgh operators $\mathbf{W}_q^p([\alpha_1])$ and $\mathbf{W}_q^p([\alpha_1+1])$ be defined by (1.6) and (1.8), respectively, then for $0\le k<\infty$, a function $f\in\mathcal{S}$ is said to be in the class $\mathcal{UC}(k,[\alpha_1])$ if it satisfies*

$$\Re\left(\frac{\mathbf{W}_q^p([\alpha_1+1])f(z)}{\mathbf{W}_q^p([\alpha_1])f(z)}\right)>k\left|\frac{\mathbf{W}_q^p([\alpha_1+1])f(z)}{\mathbf{W}_q^p([\alpha_1])f(z)}-1\right|. \tag{1.9}$$

On taking $p=q+1$ and $A_i=1\,(i=1,2,...,p)$, $\alpha_i=\beta_i=B_i=1\,(i=1,2,...,q)$ and if $\alpha_{q+1}=1+\lambda,\lambda>-1$, class $\mathcal{UC}(k,[\alpha_1])$ reduces to the class $\mathcal{UK}(\lambda,k)$ which involve the Ruscheweyh derivative operators $\mathbf{D}^{\lambda}$ and $\mathbf{D}^{\lambda+1}$ and is studied by Kanas and Yaguchi [12]. Clearly, if $p=q+1$ and $\alpha_i=A_i=1\,(i=1,2,...,p)$, $\beta_i=B_i=1\,(i=1,2,...,q)$, then the class $\mathcal{UC}(k,[\alpha_1])$ reduces to the class k-$\mathcal{SP}$ and also to the class k-$\mathcal{UCV}$ if we replace f by $zf^{'}$ in (1.9).

The purpose of this paper is to find some results for the class $\mathcal{UC}(k,[\alpha_1])$ which is defined by using its subordinate condition. Coefficient inequalities, inclusion and convolution properties for this class are derived with some of the consequent results.

2. Coefficient Inequalities

Theorem 2.1. *Let $\mathbf{W}_q^p([\alpha_1])$ be the Wgh operator defined by (1.6) with $\frac{\alpha_1}{A_1}\ge 1$, and if for $0\le k<\infty$, the function f of the form (1.1) belongs to the class $\mathcal{UC}(k,[\alpha_1])$, then there exists a convex univalent function:*

$$p_k(z)=\frac{\mathbf{W}_q^p([\alpha_1+1])f_k(z)}{\mathbf{W}_q^p([\alpha_1])f_k(z)},z\in\Delta,$$

$f_k(z)=z+d_2z^2+d_3z^3+...$ such that

$$|a_2|\le|d_2|\,,\ |a_3|\le|d_3|\ . \tag{2.1}$$

Proof. Let $f\in\mathcal{UC}(k,[\alpha_1])$, we get

$$p(z)\prec p_k(z),z\in\Delta, \tag{2.2}$$

where

$$p(z)=\frac{\mathbf{W}_q^p([\alpha_1+1])f(z)}{\mathbf{W}_q^p([\alpha_1])f(z)}=1+p_1z+p_2z^2+... \tag{2.3}$$

and

$$p_k(z)=\frac{\mathbf{W}_q^p([\alpha_1+1])f_k(z)}{\mathbf{W}_q^p([\alpha_1])f_k(z)}=1+P_1z+P_2z^2+...(P_j\geq 0,\ j=1,2,...).\tag{2.4}$$

On writing the series expansions of $\mathbf{W}_q^p([\alpha_1])f(z), \mathbf{W}_q^p([\alpha_1+1])f(z)$ and $\mathbf{W}_q^p([\alpha_1])f_k(z)$, $\mathbf{W}_q^p([\alpha_1+1])f_k(z)$, equations (2.3) and (2.4) provide

$$\left(1+p_1z+p_2z^2+..\right)\left(z+\sum_{n=2}^{\infty}a_n\ \theta_n\ z^n\right)=z+\sum_{n=2}^{\infty}\left(1+(n-1)\frac{A_1}{\alpha_1}\right)a_n\ \theta_n\ z^n\tag{2.5}$$

and

$$\left(1+P_1z+P_2z^2+..\right)\left(z+\sum_{n=2}^{\infty}d_n\ \theta_n\ z^n\right)=z+\sum_{n=2}^{\infty}\left(1+(n-1)\frac{A_1}{\alpha_1}\right)d_n\ \theta_n\ z^n.\tag{2.6}$$

Hence, from (2.5) and (2.6), we get the coefficient relations:

$$(m-1)\frac{A_1}{\alpha_1}a_m\ \theta_m=\sum_{j=1}^{m-1}p_j\ a_{m-j}\ \theta_{m-j}, m\geq 2\tag{2.7}$$

and

$$(m-1)\frac{A_1}{\alpha_1}d_m\ \theta_m=\sum_{j=1}^{m-1}P_j\ d_{m-j}\ \theta_{m-j}, m\geq 2,$$

$a_1=d_1=\theta_1=1$. Hence, we obtain

$$\frac{A_1}{\alpha_1}\theta_2a_2=p_1,\quad 2\frac{A_1}{\alpha_1}\theta_3a_3=\frac{\alpha_1}{A_1}p_1^2\quad +p_2\tag{2.8}$$

and

$$\frac{A_1}{\alpha_1}\theta_2d_2=P_1,\quad 2\frac{A_1}{\alpha_1}\theta_3d_3=\frac{\alpha_1}{A_1}P_1^2\quad +P_2.\tag{2.9}$$

As $P_j\geq 0,\ j=1,2,..$, from (2.9), it is clear that θ_2d_2 and θ_3d_3 are also non-negative real numbers. Further, with the use of subordination (2.2), we get $p(z)=p_k\left(w(z)\right)$ for some analytic function w with $w(0)=0$ and $|w(z)|<1, z\in\Delta$. If $w(z)=\frac{q(z)-1}{q(z)+1}$, where $q(z)=1+q_1z+q_2z^2+...$, with $\Re\left(q(z)\right)>0$, we write

$$p(z)=p_k\left(\frac{q(z)-1}{q(z)+1}\right).$$

On using their series expansions, we obtain

$$1+p_1z+p_2z^2+..=1+\frac{P_1q_1}{2}z+\left(\frac{P_1q_2}{2}-\frac{P_1q_1^2}{4}+\frac{P_2q_1^2}{4}\right)z^2+...\tag{2.10}$$

Thus, by (2.8), (2.9) and (2.10), we get

$$\left|\frac{A_1}{\alpha_1}\theta_2a_2\right|=|p_1|=\left|\frac{P_1q_1}{2}\right|\leq|P_1|=\left|\frac{A_1}{\alpha_1}\theta_2d_2\right|\tag{2.11}$$

and

$$\begin{aligned}\left|\frac{2A_1}{\alpha_1}\theta_3a_3\right| &= \left|\ \left(\frac{\alpha_1}{A_1}-1\right)\ p_1^2\ +\ p_2\ +\ p_1^2\right|\\ &\leq \left(\frac{\alpha_1}{A_1}-1\right)P_1^2\ +\ P_2\ +\ P_1^2=\left|\frac{2A_1}{\alpha_1}\theta_3d_3\right|,\end{aligned}\tag{2.12}$$

where we use the inequalities $|q_n| \leq 2, n \geq 1$ and $|p_2| + |p_1|^2 \leq P_2 + P_1^2$ ([15]). Thus, inequalities (2.11) and (2.12) imply the desired result (2.1). This proves Theorem 2.1. □

Theorem 2.2. *Let $\mathbf{W}_q^p([\alpha_1])$ be the Wgh operator defined by (1.6) with $\frac{\alpha_1}{A_1} \geq 1$, if the function f of the form (1.1) belongs to the class $\mathcal{UC}(k, [\alpha_1])$ for $0 \leq k < \infty$, then there exists a convex univalent function:*

$$p_k(z) = \frac{\mathbf{W}_q^p([\alpha_1 + 1]) f_k(z)}{\mathbf{W}_q^p([\alpha_1]) f_k(z)} = 1 + P_1 z + P_2 z^2 + ...(P_j \geq 0, j = 1, 2, ...),$$

such that

$$\left| \frac{A_1}{\alpha_1} \theta_n \ a_n \right| \leq \frac{P_1 \left(1 + \frac{\alpha_1}{A_1} P_1\right)_{n-2}}{(n-1)!} \ , n \geq 2,$$

where θ_n is given by (1.7).

Proof. By induction, it is shown by Theorem 2.1 that result holds for $n = 2$. Let the result be true for all j, $2 \leq j \leq n-1$. Thus, from coefficient relation (2.7) and by Rogosinski result $|p_j| \leq P_1, \ j = 1, 2, ..,$ we get

$$\begin{aligned}
(n-1) \left| \frac{A_1}{\alpha_1} a_n \ \theta_n \right| &= \left| \sum_{j=1}^{n-1} p_{n-j} \ a_j \ \theta_j \right|, n \geq 2 \\
&\leq P_1 + \sum_{j=2}^{n-1} |p_{n-j}| \ |a_j \ \theta_j| \\
&\leq P_1 \left[1 + \sum_{j=2}^{n-1} \frac{\frac{\alpha_1}{A_1} P_1 \left(1 + \frac{\alpha_1}{A_1} P_1\right)_{j-2}}{(j-1)!} \right]. \qquad (2.13)
\end{aligned}$$

We see that

$$\begin{aligned}
& 1 + \sum_{j=2}^{n-1} \frac{\frac{\alpha_1}{A_1} P_1 \left(1 + \frac{\alpha_1}{A_1} P_1\right)_{j-2}}{(j-1)!} \\
= \ & \frac{1}{1!} \left(1 + \frac{\alpha_1}{A_1} P_1\right) + \sum_{j=3}^{n-1} \frac{\frac{\alpha_1}{A_1} P_1 \left(1 + \frac{\alpha_1}{A_1} P_1\right)_{j-2}}{(j-1)!} \\
= \ & \frac{1}{2!} \left(1 + \frac{\alpha_1}{A_1} P_1\right) \left(2 + \frac{\alpha_1}{A_1} P_1\right) + \sum_{j=4}^{n-1} \frac{\frac{\alpha_1}{A_1} P_1 \left(1 + \frac{\alpha_1}{A_1} P_1\right)_{j-2}}{(j-1)!} \\
= \ & \frac{1}{(n-2)!} \left(1 + \frac{\alpha_1}{A_1} P_1\right)_{n-2}.
\end{aligned}$$

Hence, (2.13) proves that the result is true for n also. Thus the result holds for any $n \geq 2$. This proves Theorem 2.2. □

Replacing f by zf', we can obtain following result on the similar lines of the proofs of Theorems 2.1 and 2.2:

Theorem 2.3. *Let $\mathbf{W}_q^p([\alpha_1])$ be the Wgh operator defined by (1.6) with $\frac{\alpha_1}{A_1} \geq 1$, and if for $0 \leq k < \infty$, the function f of the form (1.1) satisfies*

$$\frac{\mathbf{W}_q^p([\alpha_1+1])\ zf^{'}(z)}{\mathbf{W}_q^p([\alpha_1])\ zf^{'}(z)} \prec \frac{\mathbf{W}_q^p([\alpha_1+1])\ zf_k^{'}(z)}{\mathbf{W}_q^p([\alpha_1])\ zf_k^{'}(z)}$$

then for $f_k(z) = z + d_2z^2 + d_3z^3 + ...$ and for

$$\frac{\mathbf{W}_q^p([\alpha_1+1])\ zf_k^{'}(z)}{\mathbf{W}_q^p([\alpha_1])\ zf_k^{'}(z)} = 1 + P_1z + P_2z^2 + ...(P_j \geq 0, j = 1, 2, ...),$$

$$|a_2| \leq |d_2|\,,\ |a_3| \leq |d_3|$$

and

$$\left|\frac{A_1}{\alpha_1}\theta_n\ a_n\right| \leq \frac{P_1\left(1+\frac{\alpha_1}{A_1}P_1\right)_{n-2}}{n!}, n \geq 2,\ . \tag{2.14}$$

where θ_n is given by (1.7).

Theorem 2.4. *If for the function f of the form (1.1) and for $0 \leq k < \infty$, θ_n given by (1.7), the inequality*

$$\sum_{n=2}^{\infty}\left\{(n-1)\ \ (k+1)\left|\frac{A_1}{\alpha_1}\right| + 1\right\}|a_n\ \theta_n| < 1 \tag{2.15}$$

holds, then $f \in \mathcal{UC}(k, [\alpha_1])$.

Proof. To prove $f \in \mathcal{UC}(k, [\alpha_1])$, we have to show from the condition (1.9) that

$$S_1 := k\left|\frac{\mathbf{W}_q^p([\alpha_1+1])f(z)}{\mathbf{W}_q^p([\alpha_1])f(z)} - 1\right| - \Re\left(\frac{\mathbf{W}_q^p([\alpha_1+1])f(z)}{\mathbf{W}_q^p([\alpha_1])f(z)} - 1\right) < 1.$$

From (1.6) and (1.8), we get

$$\begin{aligned} S_1 &\leq (k+1)\left|\frac{\mathbf{W}_q^p([\alpha_1+1])f(z)}{\mathbf{W}_q^p([\alpha_1])f(z)} - 1\right| \\ &\leq (k+1)\left[\frac{\sum\limits_{n=2}^{\infty}(n-1)\ \left|\frac{A_1}{\alpha_1}a_n\ \theta_n\right|}{1-\sum\limits_{n=2}^{\infty}|a_n\ \theta_n|}\right] < 1, \end{aligned}$$

if (2.15) holds. This proves Theorem 2.4. ∎

3. Inclusion Property

Theorem 3.1. *Let $\mathbf{W}_q^p([\alpha_1])$ be the Wgh operator defined by (1.6) with $0 < \frac{\alpha_1}{A_1} < (k+1)$, $0 \leq k < \infty$. Then $zf^{'}(z) \in \mathcal{UC}(k, [\alpha_1]) \Rightarrow f \in \mathcal{UC}(k, [\alpha_1])$.*

Proof. Let $zf^{'}(z) \in \mathcal{UC}(k, [\alpha_1])$, then there exists a univalent convex function $p_k(z), z \in \Delta$ describing the conic region Ω_k defined by (1.4) such that

$$\frac{\mathbf{W}_q^p([\alpha_1+1])\ zf^{'}(z)}{\mathbf{W}_q^p([\alpha_1])\ zf^{'}(z)} \prec p_k(z).$$

Set

$$p(z) = \frac{\mathbf{W}_q^p([\alpha_1+1])f(z)}{\mathbf{W}_q^p([\alpha_1])f(z)}. \tag{3.1}$$

Note that $z\left(\mathbf{W}_q^p([\alpha_1])f(z)\right)^{'} = \mathbf{W}_q^p([\alpha_1])\ zf^{'}(z)$. Hence, differentiation of (3.1) provides

$$\frac{\mathbf{W}_q^p([\alpha_1+1])\ zf^{'}(z)}{\mathbf{W}_q^p([\alpha_1+1])\ f(z)} = \frac{\mathbf{W}_q^p([\alpha_1])\ zf^{'}(z)}{\mathbf{W}_q^p([\alpha_1])\ f(z)} + \frac{zp^{'}(z)}{p(z)}$$

Using the identity:

$$\mathbf{W}_q^p([\alpha_1])\ zf^{'}(z) = \frac{\alpha_1}{A_1}\mathbf{W}_q^p([\alpha_1+1])\ f(z) - \left(\frac{\alpha_1}{A_1}-1\right)\mathbf{W}_q^p([\alpha_1])\ f(z),$$

we get

$$\frac{\mathbf{W}_q^p([\alpha_1+1])\ zf^{'}(z)}{\mathbf{W}_q^p([\alpha_1])\ zf^{'}(z)} = p(z) + \frac{zp^{'}(z)}{p(z)\frac{\alpha_1}{A_1}+\left(1-\frac{\alpha_1}{A_1}\right)} \prec p_k(z).$$

Therefore, by using a well known Lemma of Eenigenburg, Miller, Mocanu and Read [9], we get

$$p(z) \prec p_k(z),$$

provided that

$$\Re\left(p_k(z)\frac{\alpha_1}{A_1}+\left(1-\frac{\alpha_1}{A_1}\right)\right) > 0.$$

Since, from the definition of Ω_k, given by (1.4), we have $\Re\left(p_k(z)\right) > \frac{k}{k+1}$ and hence, by the hypothesis we get the result. $\square$

Remark. *Above result confirms that k-$\mathcal{UCV} \subset k$-$\mathcal{SP}$.*

4. Convolution Property

Theorem 4.1. *Let $\mathbf{W}_q^p([\alpha_1])$ be the Wgh operator defined by (1.6), then for $0 \leq k < \infty$, $f \in \mathcal{UC}(k,[\alpha_1])$ if and only if*

$$\frac{1}{z}\left(H_t * \mathbf{W}_q^p([\alpha_1])f\right)(z) \neq 0, z \in \Delta, \tag{4.1}$$

where

$$H_t(z) = \frac{1}{(1-C(t))}\frac{z}{(1-z)}\left(\frac{1-\left(1-\frac{A_1}{\alpha_1}\right)z}{(1-z)} - C(t)\right), z \in \Delta \tag{4.2}$$

and $C(t) = kt \pm i\sqrt{t^2-(kt-1)^2}$, $t \geq 0$, $t^2-(kt-1)^2 \geq 0$.

Proof. Let

$$p(z) = \frac{\mathbf{W}_q^p([\alpha_1+1])f(z)}{\mathbf{W}_q^p([\alpha_1])f(z)},\ z \in \Delta.$$

Since $p(0) = 1$, we have from the definition of conic region (1.4)

$$f \in \mathcal{UC}(k,[\alpha_1]) \Leftrightarrow p(z) \notin \partial\Omega_k\ ,\ z \in \Delta, \tag{4.3}$$

where

$$\partial\Omega_k = \left\{u+iv : u = k\sqrt{(u-1)^2+v^2}\right\}. \tag{4.4}$$

Note that $\partial\Omega_k = C(t) = kt \pm i\sqrt{t^2-(kt-1)^2}$, for $t \geq 0$, $t^2-(kt-1)^2 \geq 0$. Thus, we have

$$\frac{1}{z}\left(\frac{\mathbf{W}_q^p([\alpha_1+1])f(z) - C(t)\,\mathbf{W}_q^p([\alpha_1])\,f(z)}{(1-C(t))}\right) \neq 0, \tag{4.5}$$

By series expansions of $\mathbf{W}_q^p([\alpha_1+1])f(z)$ and $\mathbf{W}_q^p([\alpha_1])f(z)$, given in (1.8) and (1.6), we note that

$$\mathbf{W}_q^p([\alpha_1+1])f(z) = \frac{z\left(1-\left(1-\frac{A_1}{\alpha_1}\right)z\right)}{(1-z)^2} * \mathbf{W}_q^p\left([\alpha_1]\right)f(z)$$

and

$$\mathbf{W}_q^p\left([\alpha_1]\right)f(z) = \frac{z}{(1-z)} * \mathbf{W}_q^p\left([\alpha_1]\right)f(z).$$

Hence, by (4.5), we get

$$\begin{aligned}&\frac{1}{z}\left(H_t * \mathbf{W}_q^p([\alpha_1])f\right)(z)\\ = \ &\frac{1}{z}\left\{\frac{1}{(1-C(t))}\left(\frac{z\left(1-\left(1-\frac{A_1}{\alpha_1}\right)z\right)}{(1-z)^2} - C\left(t\right)\frac{z}{(1-z)}\right) * \mathbf{W}_q^p([\alpha_1])f(z)\right\} \neq 0.\end{aligned}$$

Thus,

$$\frac{1}{z}\left(H_t * \mathbf{W}_q^p([\alpha_1])f\right)(z) \neq 0 \Leftrightarrow p(z) \notin \partial\Omega_k \Leftrightarrow p(z) \in \Omega_k,\ z \in \Delta.$$

This proves the convolution property. $\square$

On taking $p = q+1$ and $\alpha_i = A_i = 1\,(i = 1,2,..,p)$, $\beta_i = B_i = 1\,(i = 1,2,..,q)$, in Theorem 4.1, we get following result for the class k-$\mathcal{SP}$

Corollary 4.2. *Let $0 \leq k < \infty$, then $f \in k$-$\mathcal{SP}$, if and only if*

$$\frac{1}{z}\left(G_t * f\right)(z) \neq 0, z \in \Delta,$$

where

$$G_t(z) = \frac{1}{(1-C(t))}\frac{z}{(1-z)}\left(\frac{1}{(1-z)} - C(t)\right), z \in \Delta,$$

$C\left(t\right) \ = kt \pm i\sqrt{t^2-(kt-1)^2}\ ,\ t \geq 0,\ t^2-(kt-1)^2 \geq 0.$

Note that for $f, g \in \mathcal{S}$

$$\left(g * zf^{'}\right)(z) = \left(zg^{'} * f\right)(z).$$

Hence, on replacing f by $zf^{'}$ in Corollary 4.2, we get following result of Kanas and Wiśniowska [[14], Theorem 3.5, p. 336] for the class k-$\mathcal{UCV}$.

Corollary 4.3. [14] *Let $0 \leq k < \infty$, then $f \in k$-$\mathcal{UCV}$, if and only if*

$$\frac{1}{z}\left[zG_t^{'} * f(z)\right] \neq 0,$$

where

$$zG_t^{'} = \frac{1}{(1-C(t))}\frac{z}{(1-z)^2}\left(\frac{1+z}{1-z} - C(t)\right), z \in \Delta,$$

$C\left(t\right) \ = kt \pm i\sqrt{t^2-(kt-1)^2}\ ,\ t \geq 0,\ t^2-(kt-1)^2 \geq 0.$

Further, Theorem 4.1 yields following result of Kanas and Yaguchi [12]:

Corollary 4.4. [12] *Let* $0 \leq k < \infty$ *and* $\lambda > -1$, *then* $f \in \mathcal{UK}(\lambda, k)$ *if and only if*

$$\frac{1}{z}\left(R_t * f\right)(z) \neq 0, \tag{4.6}$$

where

$$R_t(z) = \frac{z}{(1-z)^{\lambda+2}}\left(1 - \frac{C(t)\ z}{C(t)-1}\right)$$

and $C(t) \ = kt \pm i\sqrt{t^2 - (kt-1)^2}\ ,\ t \geq 0,\ t^2 - (kt-1)^2 \geq 0.$

Proof. Applying the argument similar to the argument applied in the proof of Theorem 4.1, we get

$$f \in \mathcal{UK}(\lambda, k) \Leftrightarrow \frac{1}{z}\left\{\frac{1}{(1-C(t))}\left(D^{\lambda+1}f(z) - C(t)\,D^{\lambda}f(z)\right)\right\} \neq 0.$$

Using the definition of Ruscheweyh derivative operators, we get

$$D^{\lambda+1}f(z) - C(t)\,D^{\lambda}f(z) = \left(\frac{z}{(1-z)^{\lambda+2}} - C(t)\frac{z}{(1-z)^{\lambda+1}}\right) * f(z),$$

which proves the condition (4.6). $\square$

Theorem 4.5. *Let* $\mathbf{W}_q^p([\alpha_1])$ *be the Wgh operator defined by (1.6) with* $\frac{\alpha_1}{A_1} \geq 1$, *then for* $0 < k < \infty$, $f \in \mathcal{UC}(k, [\alpha_1])$ *if and only if*

$$\frac{1}{z}\left(\mathbf{W}_q^p([\alpha_1])H_t * f\right)(z) \neq 0, z \in \Delta, \tag{4.7}$$

where for $H_t(z)$ *given by (4.2),*

$$\mathbf{W}_q^p([\alpha_1])H_t(z) = z + \sum_{n=2}^{\infty} h_n\ z^n, z \in \Delta,$$

$$|h_n| \leq \left\{1 + \frac{(n-1)A_1}{\alpha_1}(1+k)\right\}|\theta_n|\,, n \geq 2,$$

θ_n *is given by (1.7).*

Proof. Mentioning the proof of Theorem 4.1 and the convolution property: $\left(H_t * \mathbf{W}_q^p([\alpha_1])f\right)(z) = \left(\mathbf{W}_q^p([\alpha_1])H_t * f\right)(z)$, we get

$$f \in \mathcal{UC}(k, [\alpha_1]) \Leftrightarrow \frac{1}{z}\left(\mathbf{W}_q^p([\alpha_1])H_t * f\right)(z) \neq 0, z \in \Delta,$$

where the coefficient h_n in the series expansion of $\mathbf{W}_q^p([\alpha_1])H_t(z)$ is given by

$$h_n = \left\{\left(\frac{\alpha_1}{A_1} - 1\right) + \frac{C(t) - n}{(C(t)-1)}\right\}\ \frac{A_1}{\alpha_1}\theta_n.$$

Hence, maximum of $|h_n|$ for each $n \geq 2$, depends upon the maximum of $\left|\frac{C(t)-n}{(C(t)-1)}\right|$.
Since,

$$\begin{aligned}\left|\frac{C(t)-n}{(C(t)-1)}\right|^2 &= \left(\frac{\overline{C(t)} - n}{\left(\overline{C(t)} - 1\right)}\right)\left(\frac{C(t)-n}{(C(t)-1)}\right) \\ &= 1 - \frac{2k(n-1)}{t} + \frac{\left(n^2-1\right)}{t^2} := s(t),\end{aligned}$$

which decreases in the interval $\left[\frac{1}{k+1}, t_0\right)$ and increases in (t_0, ∞) with a minima at $t_0 = \frac{n+1}{k}$. But $s(\frac{1}{k+1}) = [n + k(n-1)]^2 > 1$. Thus,

$$|h_n| \leq \left[\left(\frac{\alpha_1}{A_1} - 1\right) + n + k(n-1)\right] \frac{A_1}{\alpha_1} |\theta_n|.$$

This proves the result of Theorem 4.5. □

Corollary 4.6. *The function $g(z) = z + Cz^n \in \mathcal{UC}(k, [\alpha_1])$ if and only if*

$$|C| \leq \frac{1}{\left\{1 + \frac{(n-1)A_1}{\alpha_1}(1+k)\right\} |\theta_n|}, n \geq 2, \tag{4.8}$$

θ_n is given by (1.7).

Proof. Let (4.8) holds. To prove the result by Theorem 4.5, we have to show

$$S_2 := \frac{1}{z}\left(\mathbf{W}_q^p([\alpha_1])H_t * g\right)(z) \neq 0, z \in \Delta.$$

Since,

$$|S_2| = \left|1 + h_n C\ z^{n-1}\right| > 1 - |h_n C\ z| \geq 1 - |z| > 0, z \in \Delta.$$

This proves $g \in \mathcal{UC}(k, [\alpha_1])$. Conversely, let $g \in \mathcal{UC}(k, [\alpha_1])$ and let

$$\mathbf{W}_q^p([\alpha_1])H(z) = z + \sum_{n=2}^{\infty} \left\{1 + \frac{(n-1)A_1}{\alpha_1}(1+k)\right\} |\theta_n|\ z^n.$$

Then

$$\frac{1}{z}\left(\mathbf{W}_q^p([\alpha_1])H * g\right)(z) = 1 + \left\{1 + \frac{(n-1)A_1}{\alpha_1}(1+k)\right\} |\theta_n|\ C\ z^{n-1}, z \in \Delta.$$

Thus, if

$$|C| > \frac{1}{\left\{1 + \frac{(n-1)A_1}{\alpha_1}(1+k)\right\} |\theta_n|},$$

then there exists a point $\zeta \in \Delta$ such that $\frac{1}{\zeta}\left(\mathbf{W}_q^p([\alpha_1])H * g\right)(\zeta) = 0$. This proves that the inequality (4.8) must hold. □

5. Concluding Remark

It is noted that taking $A_i = 1\,(i = 1, 2, .., p)$ and $B_i = 1\,(i = 1, 2, .., q)$, in our results (obtained in previous Sections 2-4), similar results can also be derived for Dziok-Srivastava operators $\mathbf{F}_q^p([\alpha_1])$ and $\mathbf{F}_q^p([\alpha_1 + 1])$ and for the special cases of these operators discussed in Introduction. In fact, involving Ruscheweyh derivative operators $\mathbf{D}^\lambda f(z)$ and $\mathbf{D}^{\lambda+1} f(z)$, some of the results have been obtained by Kanas and Yaguchi in [12]. Also, our results verify some of the results of Kanas and Wiśniowska [13] for the class k-$\mathcal{SP}$ and also for k-$\mathcal{UCV}$ [14] if f is replaced by zf'.

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References

[1] M.K. Aouf, J. Dziok, *Distortion and convolutional theorems for operators of generalized fractional calculus involving Wright function*, J. App. Anal. **14** (2) (2008), 183–192.

[2] M.K. Aouf, J. Dziok, *Certain class of analytic functions associated with the Wright generalized hypergeometric function*, J. Math. Appl. **30** (2008), 23–32.

[3] B.C. Carlson, D.B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal. **15** (4) (1984), 737–745.

[4] J. Dziok, G. Murugusundaramoorthy, K. Vijaya, *A generalized class of k-starlike functions with varying arguments of coefficients*, J. Inequal. Pure Appl. Math. **10** (3) Art. 66 (2009) 9 pp.

[5] J. Dziok, R.K. Raina, *Extremal problems in the class of analytic functions associated with Wright's function*, Advan. Pure Appld. Math. **1** (2) (2010), 247–258.

[6] J. Dziok, R.K. Raina, *Families of Analytic functions associated with the Wright generalized hypergeometric functions*, Demonstratio Math. **37** (3) (2004), 533–542.

[7] J. Dziok, R.K. Raina, *Some results based on first order differential subordination with the Wright's generalized hypergeometric function*, Comment. Math. Univ. St. Pauli. **58** (2) (2009), 87–94.

[8] J. Dziok, H.M. Srivastava, *Classes of analytic functions associated with the generalized hypergeometric functions*, Appl. Math. Comput. **103** (1999), 1–13.

[9] P. Eeingenburg, S.S. Miller, P.T. Mocanu, M.D. Reade, *On a Briot-Bouquet differential subordination*, General Inequalities, International Series of Numerical Mathematics, Birkhäuser Verlag, Basel, **64**, (3) (1983), 339–348.

[10] A.W. Goodman, *On uniformly convex functions*, Ann. Polon. Math. **56** (1991), 87–92.

[11] YU. E. Hohlov, *Operators and operations on the class of univalent functions*, Izv. Vyssh. Uchebn. Zaved. Mat. **10** (1978), 83–89.

[12] S. Kanas, T. Yaguchi, *Subclasses of k-uniformly convex and starlike functions defined by generalized derivative, II*, Publ. l'Inst. Math. **69** (83) (2001), 91–100.

[13] S. Kanas, A. Wiśniowska, *Conic regions and k-uniformly starlike functions*, Rev. Roumaine Math. Pures Appl. **45** (4) (2000), 647–657.

[14] S. Kanas, A. Wiśniowska, *Conic regions and k-uniform convexity*, J. Comput. Appl. Math. **105** (1999), 327–336.

[15] S. Kanas, A. Wiśniowska, *Conic regions and k-uniform convexity, II*, Folia Sci. Tech. Resov. **170** (1998), 65–78.

[16] S. Kanas, H.M. Srivastava, *Linear operators associated with k-uniformly convex functions*, Int. Transf. Spec. Func. **9** (2000), 121–132.

[17] W. Ma, D. Minda *Uniformly convex functions*, Ann. Polon. Math. **57** (1992), 165–175.

[18] R.K. Raina, *On generalized Wright's hypergeometric functions and fractional calculus operators*, East Asian Math. J. **21** (2) (2005), 191–203.

[19] R.K. Raina, P. Sharma, *Harmonic univalent functions associated with Wright's generalized hypergeometric functions*, Int. Transf. Spec. Func. (2011), To appear.

[20] F. Rønning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc. **118** (1993), 189–196.

[21] S. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. **77** (1979), 336–343.

[22] P. Sharma, *A class of multivalent analytic functions with fixed argument of coefficients involving Wright's generalized hypergeometric functions*, Bull. Math. Anal. Appl. **2** (1) (2010), 56–65.

[23] H.M. Srivastava, H.L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester) 1984.

[24] E.M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, Proc. London Math. Soc. **46** (1946), 389–408.

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