EXISTENCE RESULTS FOR THREE-POINT BOUNDARY VALUE PROBLEMS FOR SECOND ORDER IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. In this paper, under weak conditions on the impulse functions, we prove existence results for three-point boundary value problems for second order impulsive differential equations. The approach is based on fixed point theorems.

1. Introduction

The purpose of this paper is to establish existence of solutions for the following problem

$$x''(t) + f(t, x(t), x'(t)) = 0, \ \text{a.e. } t \in J := [0, 1], \ t \neq t_1, \tag{1.1}$$

$$\Delta x(t_1) = I_1(x(t_1), x'(t_1)), \quad \Delta x'(t_1) = I_2(x(t_1), x'(t_1)), \tag{1.2}$$

$$x(0) = 0, \quad x(1) = \alpha x(\eta), \tag{1.3}$$

where the function $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and the impulse functions $I_1, I_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are given, the impulsive moment t_1 is such that $0 < t_1 < 1$, $\Delta x(t_1) = x(t_1^+) - x(t_1^-)$, $\Delta x'(t_1) = x'(t_1^+) - x'(t_1^-)$ and $\alpha \in \mathbb{R}$, $0 < \eta < 1$ are such that $\alpha\eta \neq 1$.

We note that we could consider three-point boundary value problems for second order impulsive differential equations with an arbitrary finite number of impulses. However, for clarity and brevity, we restrict our attention to the case of one impulse. In addition, the difference between the theory of one or an arbitrary finite number of impulses is quite minimal.

Impulsive differential equations are important mathematical tools for providing a better understanding of many real-world models. Relative to the theory of impulsive differential equations and its applications, we refer the interested reader to [14] and references therein, and the monographs [1, 12, 16].

In the literature, many existence results for impulsive differential equations are proved under restrictive conditions on the impulse functions (see for instance [2, 3, 4, 5, 7, 8, 9, 13, 18]). In references [10] and [11], the first author proved some existence results under weak conditions on the impulse functions. Here, we give some existence results of solutions for the initial value problem (1.1)-(1.3) using some fixed points theorems. In our main results (Theorem 3.1 and Theorem 3.2), only the continuity of the impulse functions, I_1 and I_2, is required. The proofs are based on the following well-known fixed points theorems.

Theorem 1.1 (Schaefer Fixed Point Theorem [17]). *Let E be a normed space and let $\Gamma : E \to E$ be a completely continuous map, that is, it is a continuous mapping which is relatively compact on each bounded subset of E. If the set $\mathcal{E} = \{x \in E : \lambda x = \Gamma x \text{ for some } \lambda > 1\}$ is bounded, then Γ has a fixed point.*

Theorem 1.2 (Sadovskii Fixed Point Theorem [15]). *Let E be a Banach space and let $\Gamma : E \to E$ be a completely continuous map. If $\Gamma(B) \subset B$ for a nonempty closed, convex and bounded set B of E, then Γ has a fixed point in B.*

Theorem 1.3 (Banach Fixed Point Theorem [6]). *Let E be a Banach space and let $\Gamma : E \to E$ be a contraction map, that is, for all $x, y \in E$, $|\Gamma(x) - \Gamma(y)| \leq L|x - y|$, with $L < 1$, then Γ has a unique fixed point.*

The paper is formulated as follows. In Section 2, some definitions and lemmas are given and hypotheses on data f, I_1 and I_2 are stated. In Section 3, we establish our existence theorems for the initial value problem (1.1)-(1.3).

2. Preliminaries

Let $L^1(J, \mathbb{R}^n)$ be the Banach space of measurable functions $x : J \to \mathbb{R}^n$ which are Lebesgue integrable, normed by

$$\|x\|_{L^1} = \int_0^1 |x(t)|dt, \quad x \in L^1(J, \mathbb{R}^n).$$

By $PC(J, \mathbb{R}^n)$ we denote the Banach space of functions $x : J \to \mathbb{R}^n$ which are continuous at $t \neq t_1$, left continuous at $t = t_1$ and right-hand limit at $t = t_1$ exists, equipped with the norm

$$\|x\| = \sup\{|x(t)| : t \in J\}, \quad x \in PC(J, \mathbb{R}^n).$$

In a similar fashion to above, we define $PC^1(J, \mathbb{R}^n)$ as the Banach space of functions $x : J \to \mathbb{R}^n$, $x \in PC(J, \mathbb{R}^n)$, which are continuously differentiable at $t \neq t_1$, with x' left continuous at $t = t_1$ and right-hand limit at $t = t_1$ exists, equipped with the norm

$$\|x\|_0 = \max\{\|x\|, \|x'\|\}, \quad x \in PC^1(J, \mathbb{R}^n).$$

A function $x \in PC^1(J, \mathbb{R}^n)$ is called a solution of (1.1)-(1.3) if x satisfies the differential equation (1.1) almost everywhere on $J \setminus \{t_1\}$ and the conditions (1.2)-(1.3).

To establish our main results, we need the following assumptions for the initial value problem (1.1)-(1.3).

(H1) The function $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is Carathéodory, that is,
    (i) for every $x, y \in \mathbb{R}^n$, the function $f(\cdot, x, y) : J \to \mathbb{R}^n$ is measurable,
    (ii) for almost every $t \in J$, the function $f(t, \cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

(H2) There exist functions $p, q, r \in L^1(J, \mathbb{R}_+)$ such that

$$|f(t, x, y)| \le p(t)|x| + q(t)|y| + r(t) \text{ for almost every } t \in J \text{ and all } x, y \in \mathbb{R}^n.$$

(H2)$_*$ There exist a function $q \in L^1(J, \mathbb{R}_+)$ and a continuous nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t, x, y)| \le q(t)\psi(\max\{|x|, |y|\}) \text{ for almost every } t \in J \text{ and all } x, y \in \mathbb{R}^n.$$

(H3) The impulse functions $I_1, I_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous.

Let us now give some lemmas which are needed in the sequel.

Lemma 2.1. *Let $a, b, \alpha \in \mathbb{R}$. Suppose that $0 < t_1 < \eta < 1$, $\alpha\eta \ne 1$ and $f : J \to \mathbb{R}^n$ is in $L^1(J, \mathbb{R}^n)$. Then the boundary value problem*

$$x''(t) + f(t) = 0, \ a.e. \ t \in J := [0, 1], \ t \ne t_1, \tag{2.1}$$

$$\Delta x(t_1) = a, \quad \Delta x'(t_1) = b, \tag{2.2}$$

$$x(0) = 0, \quad x(1) = \alpha x(\eta) \tag{2.3}$$

has a unique solution

$$x(t) = \int_0^1 G(t, s)f(s)ds + aV(t) + bW(t), \quad t \in J, \tag{2.4}$$

where

$$G(t, s) = \begin{cases} s\dfrac{(1-t) + \alpha(t-\eta)}{1-\alpha\eta}, & s \le \min\{t, \eta\} \\ t\dfrac{(1-s) + \alpha(s-\eta)}{1-\alpha\eta}, & t \le s \le \eta, \\ \dfrac{s(1-t) + \alpha\eta(t-s)}{1-\alpha\eta}, & \eta \le s \le t, \\ t\dfrac{(1-s)}{1-\alpha\eta}, & s \ge \max\{t, \eta\}, \end{cases} \quad (t, s) \in J \times J, \tag{2.5}$$

$$V(t) = \frac{(H(t-t_1) - t) + \alpha(t - \eta H(t-t_1))}{1 - \alpha\eta}, \quad t \in J, \tag{2.6}$$

and

$$W(t) = -(1 - H(t-t_1))t - \frac{(H(t-t_1) - t) + \alpha(t - \eta H(t-t_1))}{1 - \alpha\eta}t_1, \quad t \in J. \tag{2.7}$$

Remark. *The function $H : \mathbb{R} \to \{0, 1\}$ in (2.6) and (2.7) is the Heaviside function which is defined by $H(s) = 0$ if $s \le 0$ and $H(s) = 1$ if $s > 0$.*

Proof. Without boundary value conditions (2.3), problem (2.1)-(2.2) has solutions in the form

$$x(t) = A + Bt - \int_0^t (t-s)f(s)ds + H(t-t_1)b(t-t_1) + H(t-t_1)a.$$

By the boundary conditions (2.3) and standard calculation we get $A = 0$ and

$$B = \int_0^1 \frac{1-s}{1-\alpha\eta}f(s)ds - \int_0^\eta \frac{\alpha(\eta-s)}{1-\alpha\eta}f(s)ds - \frac{(1-t_1) + \alpha(t_1-\eta)}{1-\alpha\eta}b - \frac{1-\alpha}{1-\alpha\eta}a$$

so that

$$
\begin{aligned}
x(t) = \quad & -\int_0^t (t-s)f(s)ds + t\int_0^1 \frac{(1-s)}{1-\alpha\eta}f(s)ds - t\int_0^\eta \frac{\alpha(\eta-s)}{1-\alpha\eta}f(s)ds \\
& +\frac{(H(t-t_1)-t)+\alpha(t-\eta H(t-t_1))}{1-\alpha\eta}a \\
& -\left\{(1-H(t-t_1))t + \frac{(H(t-t_1)-t)+\alpha(t-\eta H(t-t_1))}{1-\alpha\eta}t_1\right\}b.
\end{aligned}
\tag{2.8}
$$

Now, if $t \le \eta$, (2.8) can be rewritten as

$$
\begin{aligned}
x(t) = \quad & -\frac{t\alpha}{1-\alpha\eta}\left\{\int_0^t (\eta-s)f(s)ds + \int_t^\eta (\eta-s)f(s)ds\right\} \\
& +\frac{t}{1-\alpha\eta}\left\{\int_0^t (1-s)f(s)ds + \int_t^\eta (1-s)f(s)ds + \int_\eta^1 (1-s)f(s)ds\right\} \\
& -\int_0^t (t-s)f(s)ds + \frac{(H(t-t_1)-t)+\alpha(t-\eta H(t-t_1))}{1-\alpha\eta}a \\
& -\left\{(1-H(t-t_1))t + \frac{(H(t-t_1)-t)+\alpha(t-\eta H(t-t_1))}{1-\alpha\eta}t_1\right\}b \\
= \quad & \int_\eta^1 t\frac{1-s}{1-\alpha\eta}f(s)ds + \int_t^\eta t\frac{(1-s)+\alpha(s-\eta)}{1-\alpha\eta}f(s)ds \\
& +\int_0^t s\frac{(1-t)+\alpha(t-\eta)}{1-\alpha\eta}f(s)ds + \frac{(H(t-t_1)-t)+\alpha(t-\eta H(t-t_1))}{1-\alpha\eta}a \\
& -\left\{(1-H(t-t_1))t + \frac{(H(t-t_1)-t)+\alpha(t-\eta H(t-t_1))}{1-\alpha\eta}t_1\right\}b.
\end{aligned}
$$

Similarly, if $\eta \le t$, (2.8) can be expressed

$$
\begin{aligned}
x(t) = \quad & -\int_0^\eta (t-s)f(s)ds - \int_\eta^t (t-s)f(s)ds - \frac{t\alpha}{1-\alpha\eta}\int_0^\eta (\eta-s)f(s)ds \\
& +\frac{t}{1-\alpha\eta}\left\{\int_0^\eta (1-s)f(s)ds + \int_\eta^t (1-s)f(s)ds + \int_t^1 (1-s)f(s)ds\right\} \\
& +\frac{(H(t-t_1)-t)+\alpha(t-\eta H(t-t_1))}{1-\alpha\eta}a \\
& -\left\{(1-H(t-t_1))t + \frac{(H(t-t_1)-t)+\alpha(t-\eta H(t-t_1))}{1-\alpha\eta}t_1\right\}b \\
= \quad & \int_0^\eta s\frac{(1-t)+\alpha(t-\eta)}{1-\alpha\eta}f(s)ds + \int_\eta^t \frac{s(1-t)+\alpha\eta(t-s)}{1-\alpha\eta}f(s)ds \\
& +\int_t^1 \frac{t(1-s)}{1-\alpha\eta}f(s)ds + \frac{(H(t-t_1)-t)+\alpha(t-\eta H(t-t_1))}{1-\alpha\eta}a \\
& -\left\{(1-H(t-t_1))t + \frac{(H(t-t_1)-t)+\alpha(t-\eta H(t-t_1))}{1-\alpha\eta}t_1\right\}b.
\end{aligned}
$$

The lemma is proved. $\qquad\square$

In view of Lemma 2.1 a useful operator will now be introduced so that fixed points of the operator will be solutions of the initial value problem (1.1)-(1.3).

Lemma 2.2. *Consider the initial value problem (1.1)-(1.3). Suppose that the function $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies conditions (H1) and (H2) or (H1) and (H2)$_*$, and the impulse functions $I_1, I_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy condition (H3). Consider the operator $\Gamma : PC^1(J, \mathbb{R}^n) \to PC^1(J, \mathbb{R}^n)$ defined, for $x \in PC^1(J, \mathbb{R})$ and $t \in J$, by*

$$
\begin{aligned}
(\Gamma x)(t) = \ & \int_0^1 G(t,s) f(s, x(s), x'(s)) ds \\
& + V(t) I_1(x(t_1), x'(t_1)) + W(t) I_2(x(t_1), x'(t_1)),
\end{aligned}
\tag{2.9}
$$

where G, V and W are given in (2.5)-(2.7), with $0 < t_1 < \eta < 1$ and $\alpha\eta \neq 1$.
If x is a fixed point of Γ, then x is a solution of problem (1.1)-(1.3).

Proof. Let $x \in PC^1(J, \mathbb{R}^n)$ be a fixed point of Γ. To prove that x satisfies the problem (1.1)-(1.3), just differentiate

$$
x(t) = \int_0^1 G(t,s) f(s, x(s), x'(s)) ds + V(t) I_1(x(t_1), x'(t_1)) + W(t) I_2(x(t_1), x'(t_1))
$$

to obtain (1.1) and also show that (1.2)-(1.3) hold by direct computation. $\square$

The result below is needed to prove our main theorems (Theorem 3.1 and Theorem 3.2).

Lemma 2.3. *Suppose that the function $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies conditions (H1) and (H2) or (H1) and (H2)$_*$, and the impulse functions $I_1, I_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy condition (H3). Then the operator $\Gamma : PC^1(J, \mathbb{R}^n) \to PC^1(J, \mathbb{R}^n)$ given by (2.9) is completely continuous.*

Proof. This follows in a standard step-by-step process and so omitted. $\square$

Remark. *In the rest of this paper, M_0, M_1, V_0 and W_0 are the constants defined by*

$$
\begin{aligned}
M_0 := \sup_{(t,s) \in J \times J} |G(t,s)|, \ M_1 := \sup_{(t,s) \in J \times J} |\frac{\partial G}{\partial t}(t,s)|, \\
V_0 := \sup_{t \in J} |V(t)|, \ V_1 := \sup_{t \in J} |V'(t)|, \\
W_0 := \sup_{t \in J} |W(t)| \ and \ W_1 := \sup_{t \in J} |W'(t)|,
\end{aligned}
$$

where G, V and W are given in (2.5)-(2.7), with $0 < t_1 < \eta < 1$ and $\alpha\eta \neq 1$.

3. Existence results

In this section we state and prove our existence results for problem (1.1)-(1.3).

Theorem 3.1. *Assume that (H1), (H2) and (H3) hold. Further if*

$$
M_0 \|q\|_{L^1} < 1, \quad M_1 \|q\|_{L^1} < 1 \quad and \quad M_0 \|p\|_{L^1} \left[1 + \frac{M_1 \|q\|_{L^1}}{1 - M_1 \|q\|_{L^1}} \right] < 1, \tag{3.1}
$$

then the initial value problem (1.1)-(1.3) has a solution on J.

Proof. We will apply Theorem 1.1 to obtain the existence of a solution for $x = \Gamma x$, where $\Gamma : PC^1(J, \mathbb{R}^n) \to PC^1(J, \mathbb{R}^n)$ is given by (2.9).

Note that, by Lemma 2.3, Γ is a completely continuous operator. Then it suffices to verify that all possible solutions of the family of problems

$$\lambda x = \Gamma x, \qquad \lambda > 1 \tag{3.2}$$

are bounded a priori in $PC^1(J, \mathbb{R}^n)$ by a constant independent of λ.

Let x be a solution to (3.2) and let $\lambda > 1$ be such that $\lambda x = \Gamma x$. Then $x|_{[0,t_1]}$ satisfies, for each $t \in [0, t_1]$,

$$x(t) = \lambda^{-1} \left(\int_0^1 G(t, s) f(s, x(s), x'(s)) ds \right).$$

It is straightforward to verify that

$$|x(t)| \leq M_0 \left(\int_0^1 [p(s) \sup_{u \in [0,t_1]} |x(u)| + q(s) \sup_{u \in [0,t_1]} |x'(u)| + r(s)] ds \right). \tag{3.3}$$

Introduce the constants $\xi = \sup\{|x(s)| : s \in [0, t_1]\}$ and $\overline{\xi} = \sup\{|x'(s)| : s \in [0, t_1]\}$ in (3.3) to obtain

$$|x(t)| \leq M_0(\|p\|_{L^1} \xi + \|q\|_{L^1} \overline{\xi} + \|r\|_{L^1})$$

from which we deduce that

$$\xi \leq M_0(\|p\|_{L^1} \xi + \|q\|_{L^1} \overline{\xi} + \|r\|_{L^1}). \tag{3.4}$$

In the other hand, $x|_{[0,t_1]}$ is such that, for each $t \in [0, t_1]$,

$$x'(t) = \lambda^{-1} \left(\int_0^1 \frac{\partial G}{\partial t}(t, s) f(s, x(s), x'(s)) ds \right).$$

It is easy to verify that

$$\begin{aligned}
|x'(t)| &\leq M_1 \left(\int_0^1 [p(s) \sup_{u \in [0,t_1]} |x(u)| + q(s) \sup_{u \in [0,t_1]} |x'(u)| + r(s)] ds \right) \\
&\leq M_1 \left(\|p\|_{L^1} \xi + \|q\|_{L^1} \overline{\xi} + \|r\|_{L^1} \right)
\end{aligned}$$

from which we get

$$\overline{\xi} \leq M_1 \left(\|p\|_{L^1} \xi + \|q\|_{L^1} \overline{\xi} + \|r\|_{L^1} \right)$$

and this gives

$$\overline{\xi} \leq \frac{M_1 \left(\|p\|_{L^1} \xi + \|r\|_{L^1} \right)}{(1 - M_1 \|q\|_{L^1})}. \tag{3.5}$$

Replace (3.5) into (3.4) to obtain after some calculation

$$\xi \leq \frac{M_0 \left(1 + \frac{M_1 \|q\|_{L^1}}{1 - M_1 \|q\|_{L^1}} \right) \|r\|_{L^1}}{1 - M_0 \|p\|_{L^1} \left(1 + \frac{M_1 \|q\|_{L^1}}{1 - M_1 \|q\|_{L^1}} \right)} := \xi_1$$

then, from (3.5), we get

$$\overline{\xi} \leq \frac{M_1 \left(\|p\|_{L^1} \xi_1 + \|r\|_{L^1} \right)}{(1 - M_1 \|q\|_{L^1})} := \overline{\xi}_1. \tag{3.6}$$

Now, consider $x|_J$. It satisfies, for each $t \in J$,

$$x(t) = \lambda^{-1}\Big(\int_0^1 G(t,s)f(s,x(s),x'(s))ds$$
$$+V(t)I_1(x(t_1),x'(t_1)) + W(t)I_2(x(t_1),x'(t_1))\Big).$$

Therefore,

$$|x(t)| \leq M_0\left(\int_0^1 [p(s)\sup_{u\in[0,1]}|x(u)| + q(s)\sup_{u\in[0,1]}|x'(u)| + r(s)]ds\right)$$
$$+V_0\sup_{|u|\leq\xi_1,|v|\leq\bar{\xi}_1}|I_1(u,v)| + W_0\sup_{|u|\leq\xi_1,|v|\leq\bar{\xi}_1}|I_2(u,v)|. \tag{3.7}$$

Denote $\rho = \sup\{|x(t)| : t \in J\}$, $\bar{\rho} = \sup\{|x'(t)| : t \in J\}$, $\bar{I}_1 = \sup\{|I_1(u,v)| : |u| \leq \xi_1, |v| \leq \bar{\xi}_1\}$ and $\bar{I}_2 = \sup\{|I_2(u,v)| : |u| \leq \xi_1, |v| \leq \bar{\xi}_1\}$. From (3.7) we obtain

$$|x(t)| \leq M_0(\|p\|_{L^1}\rho + \|q\|_{L^1}\bar{\rho} + \|r\|_{L^1}) + V_0\bar{I}_1 + W_0\bar{I}_2$$

from which we get

$$\rho \leq M_0(\|p\|_{L^1}\rho + \|q\|_{L^1}\bar{\rho} + \|r\|_{L^1}) + V_0\bar{I}_1 + W_0\bar{I}_2. \tag{3.8}$$

In the other hand, $x|_J$ is such that, for each $t \in J$,

$$x'(t) = \lambda^{-1}\Big(\int_0^1 \frac{\partial G}{\partial t}(t,s)f(s,x(s),x'(s))ds$$
$$+V'(t)I_1(x(t_1),x'(t_1)) + W'(t)I_2(x(t_1),x'(t_1))\Big)$$

from which one can get

$$\bar{\rho} \leq M_1(\|p\|_{L^1}\rho + \|q\|_{L^1}\bar{\rho} + \|r\|_{L^1}) + V_1\bar{I}_1 + W_1\bar{I}_2$$

and then

$$\bar{\rho} \leq \frac{M_1(\|p\|_{L^1}\rho + \|r\|_{L^1}) + V_1\bar{I}_1 + W_1\bar{I}_2}{1 - M_1\|q\|_{L^1}}. \tag{3.9}$$

Replace (3.9) into (3.8) and compute to obtain

$$\rho \leq \frac{M_0\left(1 + \frac{M_1\|q\|_{L^1}}{1-M_1\|q\|_{L^1}}\right)\|r\|_{L^1} + M_0\frac{V_1\bar{I}_1+W_1\bar{I}_2}{1-M_1\|q\|_{L^1}} + V_0\bar{I}_1 + W_0\bar{I}_2}{1 - M_0\|p\|_{L^1}\left(1 + \frac{M_1\|q\|_{L^1}}{1-M_1\|q\|_{L^1}}\right)} := \rho_1 \tag{3.10}$$

and then, from (3.9), we get

$$\bar{\rho} \leq \frac{M_1(\|p\|_{L^1}\rho_1 + \|r\|_{L^1}) + V_1\bar{I}_1 + W_1\bar{I}_2}{1 - M_1\|q\|_{L^1}} := \bar{\rho}_1. \tag{3.11}$$

Hence, there exists a constant $\varrho = \max\{\rho_1, \bar{\rho}_1\}$ such that

$$\|x\|_0 = \max\{\sup_{t\in J}|x(t)|, \sup_{t\in J}|x'(t)|\} \leq \varrho.$$

This finish to show that all possible solutions of (3.2) are bounded in $PC^1(J,\mathbb{R}^n)$ by the constant ϱ.

As a result the conclusion of Theorem 1.1 holds and consequently the initial value problem (1.1)-(1.3) has a solution x on J. This completes the proof. □

If hypothesis (H2) and condition (3.1) in Theorem 3.5 are replaced by hypothesis $(H2)_*$ and condition (3.12) below, we obtain a new existence result. This result is not a consequence of Theorem 3.5.

Theorem 3.2. *Under conditions (H1), $(H2)_*$ and (H3), the initial value problem (1.1)-(1.3) has a solution on J, provided that the function ψ in $(H2)_*$ satisfies*

$$M_0\|q\|_{L^1}\liminf_{r\to+\infty}\frac{\psi(r)}{r}$$
$$+V_0\liminf_{r\to+\infty}\frac{\sup_{|u|,|v|\le r}|I_1(u,v)|}{r}+W_0\liminf_{r\to+\infty}\frac{\sup_{|u|,|v|\le r}|I_2(u,v)|}{r}<1. \tag{3.12}$$

Proof. We claim that there exists $r>0$ such that $\Gamma(B_r)\subseteq B_r$ where the operator Γ is defined by (2.9) and B_r is the closed ball in $PC^1(J,\mathbb{R}^n)$ with center 0 and radius r. If this property is false, then for each $r>0$ there exist $x^r\in B_r$ and $t^r\in J$ such that $|(\Gamma x^r)(t^r)|>r$. From this it follows that

$$r < |(\Gamma x^r)(t^r)|$$
$$= \left|\int_0^1 G(t^r,s)f(s,x^r(s),x^{r\prime}(s))ds\right.$$
$$\left. + V(t)I_1(x^r(t_1),x^{r\prime}(t_1))+W(t)I_2(x^r(t_1),x^{r\prime}(t_1))\right|$$
$$\le M_0\int_0^1 q(s)\psi(\max\{|x^r(s)|,|x^{r\prime}(s)|\})ds$$
$$+ V_0 I_1(x^r(t_1),x^{r\prime}(t_1))+W_0 I_2(x^r(t_1),x^{r\prime}(t_1))$$
$$\le M_0\|q\|_{L^1}\psi(r)+V_0\sup_{|u|,|v|\le r}|I_1(u,v)|+W_0\sup_{|u|,|v|\le r}|I_2(u,v)|.$$

Hence, we obtain

$$1 \le M_0\|q\|_{L^1}\liminf_{r\to+\infty}\frac{\psi(r)}{r}$$
$$+ V_0\liminf_{r\to+\infty}\frac{\sup_{|u|,|v|\le r}|I_1(u,v)|}{r}+W_0\liminf_{r\to+\infty}\frac{\sup_{|u|,|v|\le r}|I_2(u,v)|}{r},$$

which contradicts (3.12).

Let $r>0$ be such that $\Gamma:B_r\to B_r$. By Lemma 2.3, Γ is completely continuous, and from Theorem 1.2 we conclude that the initial value problem (1.1)-(1.3) has a solution x on J. The proof is finished. $\square$

From the proof of Theorem 3.2, we immediately obtain the following corollaries.

Corollary 3.3. *Assume that (H1) and $(H2)_*$ hold. In addition, assume that the following condition is satisfied.*

$(H3)_*$ *The impulse functions $I_1,I_2:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ are continuous and there exist constants $a_i,b_i\in\mathbb{R}_+$, $i=1,2,3$, such that $|I_1(x,y)|\le a_1|x|+a_2|y|+a_3$ and $|I_2(x,y)|\le b_1|x|+b_2|y|+b_3|$, for all $x,y\in\mathbb{R}^n$.*

Then the initial value problem (1.1)-(1.3) has a solution on J, provided that

$$M_0\|q\|_{L^1}\liminf_{r\to+\infty}\frac{\psi(r)}{r}+V_0(a_1+a_2)+W_0(b_1+b_2)<1.$$

Corollary 3.4. *Assume that (H1), (H2)$_*$ and the following condition hold.*

(H3)$_{**}$ *The impulse functions $I_1, I_2 : \mathbb{R}^n \to \mathbb{R}^n$ are continuous and there exist constants $a_i, b_i \in \mathbb{R}_+$, $i = 1,2,3$, $\alpha_i, \beta_i \in [0,1)$, $i = 1,2$, such that $|I_1(x,y)| \leq a_1|x|^{\alpha_1} + a_2|y|^{\alpha_2} + a_3$ and $|I_2(x,y)| \leq b_1|x|^{\beta_1} + b_2|y|^{\beta_2} + b_3$, for all $x, y \in \mathbb{R}^n$.*

Then the initial value problem (1.1)-(1.3) has a solution on J, provided that

$$M_0\|q\|_{L^1} \liminf_{r \to +\infty} \frac{\psi(r)}{r} < 1.$$

We finish this paper with the following uniqueness result for solutions of problem (1.1)-(1.3) which involves Lipschitz condition on the functions f, I_1 and I_2.

Theorem 3.5. *Assume that, for every $x, y \in \mathbb{R}^n$, the function $f(\cdot, x, y) : J \to \mathbb{R}^n$ is measurable and the following conditions hold.*

(H2)$_{**}$ *There exist functions $p, q \in L^1(J, \mathbb{R}_+)$ such that, for almost every $t \in J$ and all $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$,*

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq p(t)|x_1 - x_2| + q(t)|y_1 - y_2|.$$

(H3)$_{***}$ *There exist constants $\alpha_i, \beta_i \in \mathbb{R}_+$, $i = 1,2$, such that, for almost every $t \in J$ and all $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$,*

$$|I_1(x_1, y_1) - I_1(x_2, y_2)| \leq \alpha_1|x_1 - x_2| + \alpha_2|y_1 - y_2|,$$
$$|I_2(x_1, y_1) - I_2(x_2, y_2)| \leq \beta_1|x_1 - x_2| + \beta_2|y_1 - y_2|.$$

Further if
$$\overline{M}[\|p\|_{L^1} + \|q\|_{L^1}] + \overline{V}[\alpha_1 + \alpha_2] + \overline{W}[\beta_1 + \beta_2] < 1,$$
where $\overline{M} = \max\{M_0, M_1\}$, $\overline{V} = \max\{V_0, V_1\}$ and $\overline{V} = \max\{W_0, W_1\}$, then the initial value problem (1.1)-(1.3) has a unique solution on J.

Proof. Let $x, y \in PC^1(J, \mathbb{R}^n)$. We have, for each $t \in J$,

$$\begin{aligned}
|(\Gamma x)(t) - (\Gamma y)(t)| \leq {} & \int_0^1 |G(t,s)||f(s, x(s), x'(s)) - f(s, y(s), y'(s))|ds \\
& + |V(t)||I_1(x(t_1), x'(t_1)) - I_1(y(t_1), y'(t_1))| \\
& + |W(t)||I_2(x(t_1), x'(t_1)) - I_2(y(t_1), y'(t_1))|
\end{aligned}$$

from which it is easy to deduce the following inequality

$$|(\Gamma x)(t) - (\Gamma y)(t)| \leq \left(\overline{M}(\|p\|_{L^1} + \|q\|_{L^1}) + \overline{V}(\alpha_1 + \alpha_2) + \overline{W}(\beta_1 + \beta_2)\right)\|x - y\|_0.$$

Moreover, we have on the other hand

$$\begin{aligned}
|(\Gamma x)'(t) - (\Gamma y)'(t)| \leq {} & \int_0^1 \left|\frac{\partial G}{\partial t}(t,s)\right| |f(s, x(s), x'(s)) - f(s, y(s), y'(s))|ds \\
& + |V'(t)||I_1(x(t_1), x'(t_1)) - I_1(y(t_1), y'(t_1))| \\
& + |W'(t)||I_2(x(t_1), x'(t_1)) - I_2(y(t_1), y'(t_1))| \\
\leq {} & \left(\overline{M}(\|p\|_{L^1} + \|q\|_{L^1}) + \overline{V}(\alpha_1 + \alpha_2) \right. \\
& \left. + \overline{W}(\beta_1 + \beta_2)\right)\|x - y\|_0.
\end{aligned}$$

Hence,

$$\|\Gamma x - \Gamma y\|_0 \leq \left(\overline{M}[\|p\|_{L^1} + \|q\|_{L^1}] + \overline{V}[\alpha_1 + \alpha_2] + \overline{W}[\beta_1 + \beta_2]\right)\|x - y\|_0,$$

that is, Γ is a contraction operator.

Thus, Theorem 1.3 applies yielding the existence of the unique fixed point of Γ and thus problem (1.1)-(1.3) has a unique solution on J. $\qquad\qquad\square$

References

[1] M. Benchohra, J. Henderson and S. Ntouyas, *Inpulsive Differential Equations and Inclusions*, Series Contemporary Mathematics and Its Applications, vol. 2, Hindawi Publ. Corp., 2006.

[2] M. Benchohra, J. Henderson, S. K. Ntouyas and A. Ouahab, Multiple solutions for impulsive semilinear functional and neutral functional differential equations in Hilbert space, *Journal of Inequalities and Applications*, vol. 2005 (2005), No. 2, 189-205.

[3] G. Cai, Z. Du and W. Ge, On a class of second order impulsive boundary value problem at resonance, *International Journal of Mathematics and Mathematical Sciences*, vol. 2006 (2006), Article ID 62512, 11 pages.

[4] G. Cai and W. Ge, Positive solutions for second order impulsive differential equations with dependence on first order derivative, *Journal of Mathematical Research And Exposition*, Vol.26 (2006), No.4, 725-734.

[5] B. Chuanzhi and Y. Dandan, Existence of solutions for second order nonlinear impulsive differential equations with periodic boundary value conditions, *Boundary Value Problems*, vol. 2007 (2007), Article ID 41589, 13 pages.

[6] C. Corduneanu, *Integral equations and applications*, Combridge University Press, 1991.

[7] M. Eduardo Hernandez, A second order impulsive Cauchy problems, *International Journal of Mathematics and Mathematical Sciences*, vol. 31 (2002), No. 8, 451-461.

[8] J. Henderson and A. Ouahab, Local and global existence and uniqueness results for second and higher order impulsive functional differential equations with infinte delay, *The Australian Journal of Mathematical Analysis and Applications*, vol. 4, issue 2 (2007), 1-32.

[9] L. Jianli and S. Jianhua, Existence of positive solutions for second order impulsive boundary value problems on infinity intervals, *Boundary Value Problems*, vol. 2006 (2006), Article ID 14594, 11 pages.

[10] M. Lakrib, Existence of solutions for impulsive neutral functional differential equations with multiple delays. *Electron. J. Differential Equations*, 2008, No. 36, 7 pp.

[11] M. Lakrib, A. Oumansour and K. Yadi, Existence results for second order impulsive functional differential equations with infinite delay. *The Australian Journal of Mathematical Analysis and Applications*. In press.

[12] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific Pub. Co., Singapore, 1989.

[13] I. Rachunkova and M. Tvrdy, Second order periodic problem with phi-Laplacian and impulses, *Nonlinear Analysis, T.M.A.* 63 (2005), 257-266.

[14] Y. V. Rogovchenko, Impulsive evolution systems: Main results and new trends, *Dynamics of Continuous, Discrete and Impulsive Systems* 3 (1997), 57-88.

[15] B. N. Sadovskii, On a fixed point principle, *Functional Analysis and Its Applications*, 1 (1967), No. 2, 74-76.

[16] A. Samoilenko and N. Peresyuk, *Differential Equations with Impusive Effectes*, World Scientific Pub. Co., Singapore, 1995.

[17] D. R. Smart, *Fixed Point Theorems*, Cambridge Tracts in Mathematics, vol. 66, Cambridge University Press, Cambridge, 1980.

[18] J. Tomecek, Nonlinear boundary value problem for nonlinear second order differential equations with impulses, *Electronic Journal of Qualitative Theory of Differential Equations* 2005 (2005), No. 10, 1-22.

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