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STRONG CONVERGENCE OF NOOR ITERATION FOR A GENERAL CLASS OF FUNCTIONS

(COMMUNICATED BY MARTIN HERMANN)

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Abstract. In this paper, we employ the notion of a general class of functions introduced by Bosede and Rhoades [6] to prove the strong convergence of Noor iteration considered in Banach spaces. We also establish the strong convergence of Ishikawa and Mann iterations as special cases. Our results generalize, improve and unify some of the known results in literature.

1. Introduction

Let (E,d) be a complete metric space, $T : E \longrightarrow E$ a selfmap of E and $F_T = \{p \in E : Tp = p\}$ the set of fixed points of T in E.
Let $\{x_n\}_{n=0}^{\infty} \subset E$ be a sequence generated by an iteration procedure involving the operator T, that is,

$$x_{n+1} = f(T, x_n),\ n = 0, 1, 2, ... \tag{1.1}$$

where $x_0 \in E$ is the initial approximation and f is some function.
Setting

$$f(T, x_n) = Tx_n,\ n = 0, 1, 2, ..., \tag{1.2}$$

in (1.1), we have the Picard iteration process.
Putting

$$f(T, x_n) = (1 - \alpha_n)x_n + \alpha_n Tx_n,\ n = 0, 1, 2, ..., \tag{1.3}$$

in (1.1), where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence of real numbers in $[0, 1]$, we have the Mann iteration process.
For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Tz_n \\ z_n &= (1 - \beta_n)x_n + \beta_n Tx_n \end{aligned} \tag{1.4}$$

where $\{\alpha_n\}_{n=o}^{\infty}$ and $\{\beta_n\}_{n=o}^{\infty}$ are sequences of real numbers in [0,1], is called the Ishikawa iteration process. [For Example, see Ishikawa [11]].

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For arbitrary $x_0 \in E$, let $\{x_n\}_{n=0}^{\infty}$ be the Noor iteration defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T z_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n T x_n, \end{aligned} \tag{1.5}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences of real numbers in [0,1].
The following result was due to Zamfirescu [27]:
Theorem 1.1. *Let* (E, d) *be a complete metric space and* $T : E \longrightarrow E$ *be a mapping for which there exist real numbers* α, β *and* γ *satisfying* $0 \leq \alpha < 1, 0 \leq \beta < 0.5$ *and* $0 \leq \gamma < 0.5$ *such that, for each* $x, y \in E$, *at least one of the following is true:*
(Z_1) $d(Tx, Ty) \leq \alpha d(x, y)$;
(Z_2) $d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)]$;
(Z_3) $d(Tx, Ty) \leq \gamma[d(x, Ty) + d(y, Tx)]$.
An operator T satisfying the contractive conditions $(Z_1), (Z_2)$ and (Z_3) in Theorem 1.1 above is called a *Zamfirescu operator*.

2. Preliminaries

Several authors including Rhoades [23, 24] employed the Zamfirescu condition to establish several interesting convergence results for Mann and Ishikawa iteration processes in a uniformly convex Banach space.
The results of Rhoades [23, 24] were also extended by Berinde [2] to an arbitrary Banach space for the same fixed point iteration processes. Several other researchers such as Bosede [3, 4] and Rafiq [21, 22] obtained some interesting convergence results for some iteration procedures using various contractive definitions.
Employing a new idea, Osilike [20] considered the following contractive definition: there exist $L \geq 0, a \in [0, 1)$ such that for each $x, y \in E$,

$$d(Tx, Ty) \leq Ld(x, Tx) + ad(x, y). \tag{2.1}$$

and established T-stability for such maps with respect to Picard, Kirk, Mann and Ishikawa iterations.
Imoru and Olatinwo [11] later extended the results of Osilike [20] and proved some stability results for Picard and Mann iteration processes using the following contractive condition: there exist $b \in [0, 1)$ and a monotone increasing function $\varphi : \Re^+ \longrightarrow \Re^+$ with $\varphi(0) = 0$ such that for each $x, y \in E$,

$$d(Tx, Ty) \leq \varphi(d(x, Tx)) + bd(x, y). \tag{2.2}$$

A lot of "generalizations" and contraction conditions similar to (2.2) were also employed by several authors especially Olatinwo [19] to establish strong convergence results for some iteration processes. [For Example, see Imoru and Olatinwo [11] and Olatinwo [19]].
In 2010, Bosede and Rhoades [6] observed that the process of "generalizing" (2.1) could continue ad infinitum. As a result of this observation, Bosede and Rhoades [6] introduced the notion of a general class of functions to prove the stability of Picard and Mann iterations. [For Example, See Bosede and Rhoades [6]].
Our aim in this paper is to prove the strong convergence of Noor iteration using the notion of a general class of functions considered in Banach spaces. We also establish the strong convergence of Ishikawa and Mann iterations as corollaries.
In the sequel, we shall employ the following contractive definition: Let $(E, \|.\|)$ be

a Banach space, $T : E \longrightarrow E$ a selfmap of E, with a fixed point p such that for each $y \in E$ and $0 \leq a < 1$, we have

$$\|p - Ty\| \leq a\,\|p - y\|\,. \tag{2.3}$$

Remark 2.1. The contractive condition (2.3) is more general than those considered by Imoru and Olatinwo [11], Osilike [20] and several others in the following sense:
By replacing L in (2.1) with more complicated expressions, the process of "generalizing" (2.1) could continue ad infinitum.
In this paper, we make an obvious assumption implied by (2.1), and one which renders all "generalizations" of the form (2.2) **unnecessary**.
Furthermore, the condition "$\varphi(0) = 0$" usually imposed by Imoru and Olatinwo [11] in the contractive definition (2.2) is **no longer necessary** in our contraction condition (2.3) and this is a further improvement to several known results in literature.

3. Main Results

Theorem 3.1. *Let $(E, \|.\|)$ be a Banach space, $T : E \longrightarrow E$ a selfmap of E with a fixed point p, satisfying the contractive condition (2.3). For $x_0 \in E$, let $\{x_n\}_{n=0}^{\infty}$ be the Noor iteration process defined by (1.5) converging to p, (that is, $Tp = p$), where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences of real numbers in [0,1] such that $\sum_{k=0}^{\infty}\alpha_k = \infty$. Then, the Noor iteration process converges strongly p.*

Proof. Using the Noor iteration (1.5), the contractive condition (2.3) and the triangle inequality, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\alpha_n)x_n - \alpha_n Ty_n - p\| \\
&= \|(1-\alpha_n)x_n + \alpha_n Ty_n - ((1-\alpha_n) + \alpha_n)p\| \\
&= \|(1-\alpha_n)(x_n - p) + \alpha_n(Ty_n - p)\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \|Ty_n - p\| \\
&= (1-\alpha_n)\|x_n - p\| + \alpha_n \|p - Ty_n\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n a \|p - y_n\| \\
&= (1-\alpha_n)\|x_n - p\| + \alpha_n a \|y_n - p\|\,.
\end{aligned} \tag{3.1}$$

For the estimate of $\|y_n - p\|$ in (3.1), we have

$$\begin{aligned}
\|y_n - p\| &= \|(1-\beta_n)x_n + \beta_n Tz_n - p\| \\
&= \|(1-\beta_n)x_n + \beta_n Tz_n - ((1-\beta_n) + \beta_n)p\| \\
&= \|(1-\beta_n)(x_n - p) + \beta_n(Tz_n - p)\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n \|Tz_n - p\| \\
&= (1-\beta_n)\|x_n - p\| + \beta_n \|p - Tz_n\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n a \|p - z_n\| \\
&= (1-\beta_n)\|x_n - p\| + \beta_n a \|z_n - p\|\,.
\end{aligned} \tag{3.2}$$

Substitute (3.2) into (3.1) gives

$$\|x_{n+1} - p\| \leq \Big(1 - (1-a)\alpha_n - (1-a)a\alpha_n\beta_n\Big)\|z_n - p\|\,. \tag{3.3}$$

Similarly, $\|z_n - p\|$ in (3.3) is estimated as follows:

$$\begin{aligned}
\|z_n - p\| &= \|(1-\gamma_n)x_n + \gamma_n T x_n - p\| \\
&= \|(1-\gamma_n)x_n + \gamma_n T x_n - ((1-\gamma_n)+\gamma_n)p\| \\
&= \|(1-\gamma_n)(x_n - p) + \gamma_n (T x_n - p)\| \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n \|T x_n - p\| \\
&= (1-\gamma_n)\|x_n - p\| + \gamma_n \|p - T x_n\| \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n a \|p - x_n\| \\
&= (1-\gamma_n + \gamma_n a)\|x_n - p\| .
\end{aligned} \tag{3.4}$$

Substitute (3.4) into (3.3) yields

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \Big(1-(1-a)\alpha_n - (1-a)a\alpha_n\beta_n\Big)(1-\gamma_n+\gamma_n a)\|x_n - p\| \\
&\leq [1-(1-a)\alpha_n]\|x_n - p\| \\
&\leq \prod_{k=0}^{n}[1-(1-a)\alpha_k]\|x_0 - p\| \\
&\leq \prod_{k=0}^{n} e^{-(1-a)\alpha_k}\|x_0 - p\| \\
&= e^{-(1-a)\sum_{k=0}^{n}\alpha_k}\|x_0 - p\| \longrightarrow 0,
\end{aligned} \tag{3.5}$$

as $n \longrightarrow \infty$. Since $\sum_{k=0}^{n}\alpha_k = \infty$, $a \in [0,1)$ and from (3.5), we have $\|x_n - p\| \longrightarrow 0$ as $n \longrightarrow \infty$, which implies that the Noor iteration process converges strongly to p. To prove the **uniqueness**, we take $p_1, p_2 \in F_T$, where F_T is the set of fixed points of T in E such that $p_1 = Tp_1$ and $p_2 = Tp_2$.
Suppose on the contrary that $p_1 \neq p_2$. Then, using the contractive condition (2.3) and since $0 \leq a < 1$, we have

$$\begin{aligned}
\|p_1 - p_2\| &= \|p_1 - Tp_2\| \\
&\leq a\|p_1 - p_2\| \\
&< \|p_1 - p_2\| ,
\end{aligned} \tag{3.6}$$

which is a contradiction. Therefore, $p_1 = p_2$.

This completes the proof.

Consequently, we have the following corollaries:

Corollary 3.2. *Let $(E, \|.\|)$ be a Banach space, $T : E \longrightarrow E$ a selfmap of E with a fixed point p, satisfying the contractive condition (2.3). For $x_0 \in E$, let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iteration process defined by (1.4) converging to p, (that is, $Tp = p$), where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of real numbers in [0,1] such that $\sum_{k=0}^{\infty}\alpha_k = \infty$. Then, Ishikawa iteration process converges strongly p.*

Corollary 3.3. *Let $(E, \|.\|)$ be a Banach space, $T : E \longrightarrow E$ a selfmap of E with a fixed point p, satisfying the contractive condition (2.3). For $x_0 \in E$, let $\{x_n\}_{n=0}^{\infty}$ be the Mann iteration process defined by (1.3) converging to p, (that is, $Tp = p$), where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence of real numbers in [0,1] such that $\sum_{k=0}^{\infty}\alpha_k = \infty$.*

Then, Mann iteration process converges strongly p.

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