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ON LOCAL PROPERTY OF ABSOLUTE WEIGHTED MEAN SUMMABILITY OF FOURIER SERIES

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Abstract. We improve and generalize a result on a local property of $|T|_k$ summability of factored Fourier series due to Sarıgöl [6].

1. Introduction

Let T be a lower triangular matrix, (s_n) a sequence of the partial sums of the series $\sum a_n$, then

$$T_n := \sum_{v=0}^{n} t_{nv} s_v. \tag{1}$$

A series $\sum a_n$ is said to be summable $|T|_k$, $k \geq 1$, if (see [6])

$$\sum_{n=1}^{\infty} |t_{nn}|^{1-k} |\Delta T_{n-1}|^k < \infty. \tag{2}$$

Given any lower triangular matrix T one can associate the matrices $\overline{T}$ and $\hat{T}$, with entries defined by

$$\bar{t}_{nv} = \sum_{i=v}^{n} t_{ni}, \quad n, i = 0, 1, 2, \ldots, \qquad \widehat{t}_{nv} = \bar{t}_{nv} - \bar{t}_{n-1,v}$$

respectively. With $s_n = \sum\limits_{i=0}^{n} a_i \lambda_i$,

$$t_n = \sum_{v=0}^{n} t_{nv} s_v = \sum_{v=0}^{n} t_{nv} \sum_{i=0}^{v} a_i \lambda_i = \sum_{i=0}^{n} a_i \lambda_i \sum_{v=i}^{n} t_{nv} = \sum_{i=0}^{n} \bar{t}_{ni} a_i \lambda_i \tag{3}$$

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$$\begin{aligned} Y_n \quad : \quad &= t_n - t_{n-1} = \sum_{i=0}^{n} \bar{t}_{ni} a_i \lambda_i - \sum_{i=0}^{n-1} \bar{t}_{n-1,i} a_i \lambda_i \\ &= \sum_{i=0}^{n} \hat{t}_{ni} a_i \lambda_i \text{ as } \bar{t}_{n-1,n} = 0. \end{aligned} \tag{4}$$

Recall that $\hat{t}_{nn} = t_{nn}$. (p_n) is assumed to be positive sequences of numbers such that

$$P_n = p_0 + p_1 + \ldots + p_n \to \infty, \text{ as } n \to \infty \ ,$$

The series $\sum a_n$ is said to be summable $|R, p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta z_{n-1}|^k < \infty \quad ,$$

where

$$z_n = \sum_{i=0}^{n} p_i s_i .$$

The series $\sum a_n$ is said to be summable $\left|\overline{N}, p_n\right|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta z_{n-1}|^k < \infty \quad .$$

Let f be 2π-periodic function, Lebesgue integrable over $[-\pi, \pi]$. Without loss of generality, we may assume that the constant term in the Fourier series representation for f is zero and that

$$f(t) \approx \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} A_n(t). \tag{5}$$

It is known that the convergence of the Fourier series of f at $t = x$ is a local property of f, that is the convergence depends only on the behavior of f in an arbitrary small neighborhood of x. Therefore it follows that the summability of the Fourier series at $t = x$ by any regular summability method is also a local property of f.

A sequence (λ_n) is said to be convex if $\Delta^2 \lambda_n \geq 0$, $(\Delta^2 \lambda_n = \Delta(\Delta\lambda_n))$.

Mohanty [5] proved that the $|R, \log n, 1|$ summability of factored Fourier series

$$\sum \frac{A(t)}{\log(n+1)} \tag{6}$$

at $t = x$ is a local property of f. Matsumoto [3] improved the previous result by replacing the series in (6) with

$$\sum \frac{A_n(t)}{(\log\log(n+1))^{\delta}}, \qquad \delta > 1. \tag{7}$$

Bhatt [1], in turn, generalized the result of Matsumoto by giving the following result

Theorem 1.1. *If (λ_n) is convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then the summability $|R, \log n, 1|$ of the series $\sum A_n(t)\lambda_n \log n$ at a point can be ensured by a local property.*

Bor [2] introduced the following theorem on the local property of the summability $\left|\overline{N}, p_n\right|_k$ of the factored Fourier series, which generalizes most of the above results under more appropriate conditions than those given in them.

Theorem 1.2. *Let the positive sequence (p_n) and a sequence (λ_n) be such that*

$$\Delta X_n = O(1/n), \tag{8}$$

$$\sum_{n=1}^{\infty} n^{-1}\left(|\lambda_n|^k + |\lambda_{n+1}|^k\right) X_n^{k-1} < \infty, \tag{9}$$

$$\sum_{n=1}^{\infty} \left(X_n^k + 1\right)|\Delta\lambda_n| < \infty, \tag{10}$$

where $X_n = (np_n)^{-1}P_n$. Then the summability $\left|\overline{N}, p_n\right|_k$, $k \geq 1$, of the series $\sum \lambda_n X_n A_n(t)$ at any point is a local property of f.

Finally, Sarıgöl [6] generalized Bor's result by giving the following

Theorem 1.3. *Suppose that $T = (t_{nv})$ is a normal matrix such that*

$$t_{n-1,v} \geq t_{nv} \quad \text{for } n \geq v+1, \tag{11}$$

$$\bar{t}_{n0} = 1, \quad n = 0, 1, \ldots, \tag{12}$$

$$\sum_{v=1}^{n-1} t_{vv}\hat{t}_{n,v-1} = O(a_{nn}), \tag{13}$$

$$\Delta X_n = O(1/n), \tag{14}$$

where $X_n = (nt_{nn})^{-1}$. If a sequence (λ_n) holds for $k \geq 1$ and the conditions (9), (10) are satisfied, then the summability $|T|_k$ of the series $\sum\limits_{n=1}^{\infty} \lambda_n X_n A_n(t)$ at any point is a local property of f.

The object of the present paper is to improve and generalize Sarıgöl's result. In fact we do the following

(1) The matrix we use is not positive in general.
(2) the condition (10) is replaced by following weaker one

$$\sum_{n=1}^{\infty} X_n |\Delta\lambda_n| < \infty, \quad X_n \geq 1. \tag{15}$$

2. Results

We prove the following

Theorem 2.1. *Suppose that $T = (t_{nv})$ is a normal matrix satisfying*

$$\left|\hat{t}_{n,v+1}\right| \leq |t_{nn}|, \tag{16}$$

$$\sum_{n=v+1}^{\infty} \left|\hat{t}_{n,v+1}\right| < \infty, \tag{17}$$

$$\sum_{v=1}^{n-1} |t_{vv}|\left|\hat{t}_{n,v+1}\right| = O(|t_{nn}|), \tag{18}$$

$$\sum_{v=1}^{n-1}\left|\Delta_v \hat{t}_{nv}\right| = O(|t_{nn}|). \tag{19}$$

Suppose that $X_n = (n|t_{nn}|)^{-1}$ *and satisfied (14). If a sequence* (λ_n) *holds for* $k \geq 1$ *and the conditions (9), (15) are also satisfied, then the summability* $|T|_k$ *of the series* $\sum_{n=1}^{\infty}\lambda_n X_n A_n(t)$ *at any point is a local property of* f.

The following lemma is needed for the proof of the theorem

Lemma 2.2. *Suppose that the matrix* T *and the sequence* (λ_n) *satisfying the conditions of the theorem, and that* (s_n) *is bounded. Then the series* $\sum_{n=1}^{\infty}\lambda_n X_n A_n(t)$ *is summable* $|T|_k$, $k \geq 1$.

Proof. Let (T_n) be a T-transform of the series $\sum_{n=1}^{\infty}\lambda_n X_n A_n(t)$. By (4)

$$\Delta T_n = \sum_{v=1}^{n}\hat{t}_{nv}\lambda_v X_v, \quad X_0 = 0.$$

Via Abel's transformation, we have

$$\begin{aligned}\Delta T_n &= \sum_{v=1}^{n-1}\hat{t}_{n,v+1}X_v\Delta\lambda_v s_v + \sum_{v=1}^{n-1}\hat{t}_{n,v+1}\lambda_v\Delta X_v s_v + \sum_{v=1}^{n-1}\Delta\hat{t}_{nv}\lambda_v X_v s_v + t_{nn}\lambda_n X_n s_n\\ &= T_{n1}+T_{n2}+T_{n3}+T_{n4}.\end{aligned}$$

In order the prove the lemma, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}|t_{nn}|^{1-k}|T_{nj}|^k < \infty, \qquad j = 1,2,3,4.$$

By Hölder's inequality,

$$\begin{aligned}\sum_{n=2}^{\infty}|t_{nn}|^{1-k}|T_{n1}|^k &= \sum_{n=2}^{\infty}|t_{nn}|^{1-k}\left|\sum_{v=1}^{n-1}\hat{t}_{n,v+1}X_v\Delta\lambda_v s_v\right|^k\\ &\leq \sum_{n=2}^{\infty}|t_{nn}|^{1-k}\sum_{v=1}^{n-1}\left|\hat{t}_{n,v+1}\right|^k X_v|\Delta\lambda_v||s_v|^k\left(\sum_{v=1}^{n-1}X_v|\Delta\lambda_v|\right)^{k-1}\\ &= O(1)\sum_{n=2}^{\infty}|t_{nn}|^{1-k}\sum_{v=1}^{n-1}\left|\hat{t}_{n,v+1}\right|^k X_v|\Delta\lambda_v||s_v|^k\\ &= O(1)\sum_{v=1}^{\infty}X_v|\Delta\lambda_v|\sum_{n=v+1}^{\infty}|t_{nn}|^{1-k}\left|\hat{t}_{n,v+1}\right|^k\\ &= O(1)\sum_{v=1}^{\infty}X_v|\Delta\lambda_v|\sum_{n=v+1}^{\infty}|t_{nn}|^{1-k}\left|\hat{t}_{n,v+1}\right|^{k-1}\left|\hat{t}_{n,v+1}\right|\\ &= O(1)\sum_{v=1}^{\infty}X_v|\Delta\lambda_v|\sum_{n=v+1}^{\infty}\left|\hat{t}_{n,v+1}\right|\\ &= O(1)\sum_{v=1}^{\infty}X_v|\Delta\lambda_v| = O(1).\end{aligned}$$

$$\begin{aligned}\sum_{n=2}^{\infty}|t_{nn}|^{1-k}|T_{n2}|^{k} &= \sum_{n=2}^{\infty}|t_{nn}|^{1-k}\left|\sum_{v=1}^{n-1}s_v\hat{t}_{n,v+1}\lambda_{v+1}\Delta X_v\right|^{k}\\ &\le \sum_{n=2}^{\infty}|t_{nn}|^{1-k}\sum_{v=1}^{n-1}|s_v|^{k}\left|\hat{t}_{n,v+1}\right||t_{vv}|^{1-k}|\lambda_{v+1}|^{k}|\Delta X_v|^{k}\left(\sum_{v=1}^{n-1}|t_{vv}|\left|\hat{t}_{n,v+1}\right|\right)^{k-1}\\ &= O(1)\sum_{n=2}^{\infty}\sum_{v=1}^{n-1}|s_v|^{k}\left|\hat{t}_{n,v+1}\right||t_{vv}|^{1-k}|\lambda_{v+1}|^{k}|\Delta X_v|^{k}\\ &= O(1)\sum_{v=1}^{\infty}|t_{vv}|^{1-k}|\Delta X_v|^{k}|\lambda_{v+1}|^{k}\sum_{n=v+1}^{\infty}\left|\hat{t}_{n,v+1}\right|\\ &= O(1)\sum_{v=1}^{\infty}|\lambda_{v+1}|^{k}|t_{vv}|^{1-k}n^{-k}\\ &= O(1)\sum_{v=1}^{\infty}v^{-1}|\lambda_{v+1}|^{k}X_v^{k-1}=O(1).\end{aligned}$$

$$\begin{aligned}\sum_{n=2}^{\infty}|t_{nn}|^{1-k}|T_{n3}|^{k} &= \sum_{n=2}^{\infty}|t_{nn}|^{1-k}\left|\sum_{v=1}^{n-1}s_v\Delta\hat{t}_{nv}\lambda_v X_v\right|^{k}\\ &\le \sum_{n=2}^{\infty}|t_{nn}|^{1-k}\sum_{v=1}^{n-1}|s_v|^{k}\left|\Delta_v\hat{t}_{nv}\right||\lambda_v|^{k}X_v^{k}\left(\sum_{v=1}^{n-1}\left|\Delta_v\hat{t}_{nv}\right|\right)^{k-1}\\ &= O(1)\sum_{n=2}^{\infty}|t_{nn}|^{1-k}|t_{nn}|^{k-1}\sum_{v=1}^{n-1}|s_v|^{k}\left|\Delta_v\hat{t}_{nv}\right||\lambda_v|^{k}X_v^{k}\\ &= O(1)\sum_{v=1}^{\infty}|s_v|^{k}|\lambda_v|^{k}X_v^{k}\sum_{n=v+1}^{\infty}\left|\Delta_v\hat{t}_{nv}\right|\\ &= O(1)\sum_{v=1}^{\infty}|\lambda_v|^{k}X_v^{k}\left|\hat{t}_{vv}\right|\\ &= O(1)\sum_{v=1}^{\infty}v^{-1}|\lambda_v|^{k}X_v^{k-1}=O(1).\end{aligned}$$

$$\begin{aligned}\sum_{n=2}^{\infty}|t_{nn}|^{1-k}|T_{n4}|^{k} &= \sum_{n=2}^{\infty}|t_{nn}|^{1-k}\left|s_n\hat{t}_{nn}\lambda_n X_n\right|^{k}\\ &= O(1)\sum_{n=1}^{\infty}|t_{nn}|^{1-k}|t_{nn}|^{k}|\lambda_n|^{k}X_n^{k}\\ &= O(1)\sum_{n=1}^{\infty}n^{-1}|\lambda_n|^{k}X_n^{k-1}=O(1).\end{aligned}$$

This completes the proof of the lemma. □

Remark 2.3. It may be mentioned that whenever T is positive , then conditions (16)-(19) are replaced by conditions (11)-(13).

Proof of Theorem 2.1. Since the convergence of the Fourier series at a point is a local property of its generating function f, the theorem follows from [7, chapter II, formula (7.1)] and lemma 2.2.

Corollary 2.4. *Suppose that $T = (t_{nv})$ is a positive normal matrix. Suppose that $X_n = (n\,|t_{nn}|)^{-1}$ and satisfied (14). If a sequence (λ_n) holds for $k \geq 1$ and the conditions (9), (15) are also satisfied, then the summability $|R, p_n|_k$ of the series $\sum_{n=1}^{\infty} \lambda_n X_n A_n(t)$ at any point is a local property of f.*

Proof. The proof follows from theorem 2.1 by putting

$$t_{nv} = \frac{p_v}{P_n}, \quad \hat{t}_{nv} = \frac{p_n P_{v-1}}{P_n P_{n-1}}, \quad \Delta_v \hat{t}_{nv} = \frac{-p_n p_v}{P_n P_{n-1}}.$$

It is not difficult to check that the conditions (16)-(19) are all satisfied. □

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