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APPLICATION OF AN INTEGRAL OPERATOR FOR P-VALENT FUNCTIONS

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Abstract. By making use of an integral operator defined in an open unit disk, we introduce and study certain new subclasses of p-valent functions. Inclusion relationships are established and integral preserving properties of functions in these subclasses are discussed.

1. Introduction and preliminaries

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k,\ p \in \mathbb{N} = \{1, 2, \cdots\}. \tag{1.1}$$

which are analytic in an open unit disk $\mathbb{U} = \{z : |z| < 1\}$.
Next we define some well known subclasses of p-valent functions as follows:

$$\begin{aligned}
S_p^*(\xi) &= \left\{ f \in A_p : \Re\left(\frac{zf'(z)}{f(z)}\right) > \xi, 0 \le \xi < p, p \in \mathbb{N}, z \in \mathbb{U} \right\};\\
K_p(\rho, \xi) &= \left\{ f \in A_p : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \xi, 0 \le \xi < p, p \in \mathbb{N}, z \in \mathbb{U} \right\};\\
K_p(\rho, \xi) &= \left\{ f \in A_p : \exists g(z) \in S_p^*(\xi) \wedge \Re\left(\frac{zf'(z)}{g(z)}\right) > \rho, 0 \le \rho, \xi < p, p \in \mathbb{N}, z \in \mathbb{U} \right\};\\
K_p^*(\rho, \xi) &= \left\{ f \in A_p : \exists g(z) \in C_p(\xi) \wedge \Re\left(\frac{(zf'(z))'}{g'(z)}\right) > \rho, 0 \le \rho, \xi < p, p \in \mathbb{N}, z \in \mathbb{U} \right\}.
\end{aligned}$$

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The classes $S_p^*(\xi)$, S_p^*, $C_p(\xi)$, C_p, $K_p(\rho,\xi)$, $K_1(\rho,\xi)$, $K_p^*(\rho,\xi)$ and $K_1^*(\rho,\xi)$ were introduced by Patil and Thakare [13], Goodman [14], Owa [15], Aouf [16], Libera [17] and Noor [18, 19] respectively.

Also note that

$f(z) \in C_p(\xi)$ if and only if $\frac{zf'(z)}{p} \in S_p^*(\xi)$, $0 \leq \xi < p, p \in \mathbb{N}, z \in \mathbb{U}\cdot$

Similarly

$f(z) \in K_p(\xi)$ if and only if $\frac{zf'(z)}{p} \in K_p^*(\xi)$, $0 \leq \xi < p, p \in \mathbb{N}, z \in \mathbb{U}\cdot$

For a function $f \in A_p$, we define a differential operator as follow:

$$\begin{aligned}
\Upsilon^0 f(z) &= f(z);\\
\Upsilon^1_\lambda(p,\alpha,\beta,\mu)f(z) &= (\frac{\alpha - p\mu + \beta - p\lambda}{(\alpha+\beta)})f(z) + (\frac{p\mu + p\lambda}{(\alpha+\beta)})\frac{zf'(z)}{p};\\
\Upsilon^1_\lambda(p,\alpha,\beta,\mu)f(z) &= D_{p,\lambda}(\alpha,\beta,\mu)f(z);\\
\Upsilon^2_\lambda(p,\alpha,\beta,\mu)f(z) &= D(\Upsilon^1_\lambda(p,\alpha,\beta,\mu)f(z));\\
&\vdots\\
\Upsilon^n_\lambda(p,\alpha,\beta,\mu)f(z) &= D(\Upsilon^{n-1}_\lambda(p,\alpha,\beta,\mu)f(z))\cdot \qquad (1.2)
\end{aligned}$$

If f is given by (1.1) then from (1.2) we have

$$\Upsilon^n_\lambda(p,\alpha,\beta,\mu)f(z) = z^p + \sum_{k=p+1}^{\infty}\Big(\frac{\alpha + (\mu+\lambda)(k-p) + \beta}{\alpha+\beta}\Big)^n a_k z^k, \qquad (1.3)$$

where $f \in A_p, \alpha, \beta, \mu, \lambda \geq 0, \alpha + \beta \neq 0, n \in \mathbb{N}_o = \{0, 1, \cdots\}$.
This generalizes some well known differential operators available in literature (see for examples [1]-[11]).

Now we define the integral operator for $f(z) \in A_p$ as follows:

$$\mathtt{C}^0_p(\alpha,\beta,\mu,\lambda)f(z) = f(z);$$

$$\mathtt{C}^1_p(\alpha,\beta,\mu,\lambda)f(z) = \frac{\alpha+\beta}{\mu+\lambda} z^{p-(\frac{\alpha+\beta}{\mu+\lambda})}\int_o^z t^{(\frac{\alpha+\beta}{\mu+\lambda})-p-1} f(t)dt;$$

$$\mathtt{C}^2_p(\alpha,\beta,\mu,\lambda)f(z) = \frac{\alpha+\beta}{\mu+\lambda} z^{p-(\frac{\alpha+\beta}{\mu+\lambda})}\int_o^z t^{(\frac{\alpha+\beta}{\mu+\lambda})-p-1} \mathtt{C}^1_p(\alpha,\beta,\mu,\lambda)f(t)dt;$$

$$\vdots$$

$$\mathtt{C}^m_p(\alpha,\beta,\mu,\lambda)f(z) = \frac{\alpha+\beta}{\mu+\lambda} z^{p-(\frac{\alpha+\beta}{\mu+\lambda})}\int_o^z t^{(\frac{\alpha+\beta}{\mu+\lambda})-p-1} \mathtt{C}^{m-1}_p(\alpha,\beta,\mu,\lambda)f(t)dt\cdot$$

This implies

$$\mathtt{C}^m_p(\alpha,\beta,\mu,\lambda)f(z) = z^p + \sum_{k=2}^{\infty}\left(\frac{\alpha+\beta}{\alpha + (\mu+\lambda)(k-p)+\beta}\right)^m a_k z^k, \qquad (1.4)$$

where $\alpha \geq 0, \beta \geq 0, \mu \geq 0, \lambda \geq 0, p \in \mathbb{N}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, f(z) \in A_p, z \in \mathbb{U})$. From (1.4) we have

$$\left(\mu+\lambda\right) z\left(\mathtt{C}_p^{m+1}(\alpha,\beta,\mu,\lambda)f(z)\right)' =$$

$$\left(\alpha+\beta\right)\left(\mathtt{C}_p^{m}(\alpha,\beta,\mu,\lambda)f(z)\right) - \left(\alpha+\beta-p(\mu+\lambda)\right)\left(\mathtt{C}_p^{m+1}(\alpha,\beta,\mu,\lambda)f(z)\right)\cdot \quad (1.5)$$

Using the operator $\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)f(z)$ defined in (1.4), we introduce the following subclasses of p-valent functions:

$$\mathfrak{S}_m^*(p,\xi,\lambda,\alpha,\beta,\mu) = \left\{f \in A_p : \mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)f \in S_p^*(\xi)\right\};$$

$$\mathfrak{C}_m(p,\xi,\lambda,\alpha,\beta,\mu) = \left\{f \in A_p : \mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)f \in C_p(\xi)\right\};$$

$$\mathfrak{K}_m(p,\rho,\xi,\lambda,\alpha,\beta,\mu) = \left\{f \in A_p : \mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)f \in K_p(\rho,\xi)\right\};$$

$$\mathfrak{K}_m^*(p,\rho,\xi,\lambda,\alpha,\beta,\mu) = \left\{f \in A_p : \mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)f \in K_p^*(\rho,\xi)\right\}.$$

2. Inclusion relationships

In this section, we establish various inclusion relationships for the functions belonging to the new subclasses of p-valent functions.

Lemma 2.1.[20, 21] Let $\varphi(\mu,\nu)$ be a complex function, $\phi : D \to \mathbb{C}$, $D \subset \mathbb{C} \times \mathbb{C}$, and let $\mu = \mu_1 + i\mu_1$, $\nu = \nu_1 + i\nu_1$. Suppose that $\varphi(\mu,\nu)$ satisfies the following conditions:

(1) $\varphi(\mu,\nu)$ is continuous in D;
(2) $(1,0) \in D$ and $\Re\{\varphi(1,0)\} > 0$;
(3) $\Re\{\varphi(i\mu_2,\nu_1)\} \leq 0$ for all $(i\mu_2,\nu_1) \in D$ such that $\nu_1 \leq -\frac{1}{2}(1+\mu_2^2)\cdot$

Let $h(z) = 1 + c_1z + c_2z^2 + \cdots$ be analytic in $\mathbb{U}$, such that $(h(z), zh'(z)) \in D$ for all $z \in \mathbb{U}$. If $\Re\{\varphi(h(z), zh'(z))\} > 0 (z \in \mathbb{U})$, then $\Re\{h(z)\} > 0$ for $z \in \mathbb{U}\cdot$

Theorem 2.2. Let $f(z) \in A_p$ and $\alpha,\beta,\mu,\lambda \geq 0, p \in \mathbb{N}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \mathbb{U}\cdot$ Then

$$\mathfrak{S}_m^*(p,\xi,\lambda,\alpha,\beta,\mu) \subseteq \mathfrak{S}_{m+1}^*(p,\xi,\lambda,\alpha,\beta,\mu) \subseteq \mathfrak{S}_{m+2}^*(p,\xi,\lambda,\alpha,\beta,\mu)\cdot$$

Proof. To prove $\mathfrak{S}_m^*(p,\xi,\lambda,\alpha,\beta,\mu) \subseteq \mathfrak{S}_{m+1}^*(p,\xi,\lambda,\alpha,\beta,\mu)$, let $f(z) \in \mathfrak{S}_m^*(p,\xi,\lambda,\alpha,\beta,\mu)$ and assume that

$$\frac{z(\mathtt{C}_p^{m+1}(\alpha,\beta,\mu,\lambda)f(z))'}{\mathtt{C}_p^{m+1}(\alpha,\beta,\mu,\lambda)f(z)} = \xi + (p-\xi)h(z),\ 0 \leq \xi < 1, z \in \mathbb{U}\cdot \quad (2.1)$$

Where $h(z) = 1 + c_1z + c_2z^2 + \cdots$.

Using (1.5) and (2.1), we have

$$\frac{z(\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)f(z))'}{\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)f(z)}-\xi=(p-\xi)h(z)+\frac{(p-\xi)zh'(z)}{(\frac{\alpha+\beta}{\mu+\lambda}-p)+\xi+(p-\xi)h(z)}\cdot \quad (2.2)$$

Taking $h(z)=\mu=\mu_1+i\mu_1$ and $zh'(z)=\nu=\nu_1+i\nu_1$, we define the function $\varphi(\mu,\nu)$ by:

$$\varphi(\mu,\nu)=(p-\xi)\mu+\frac{(p-\xi)\nu}{(\frac{\alpha+\beta}{\mu+\lambda}-p)+\xi+(p-\xi)\mu}\cdot$$

This implies

(i) $\varphi(\mu,\nu)$ is continuous in $D=(\mathbb{C}-\frac{(\frac{\alpha+\beta}{\mu+\lambda}-p)+\xi}{\xi-p})\times\mathbb{C}$,
(ii) $(1,0)\in D$ and $\Re\{\varphi(1,0)\}>1-\xi$,
(iii) For all $(i\mu_2,\nu_1)\in D$ such that $\nu_1\leq-\frac{1}{2}(1+\mu_2^2)$. Therefore

$$\Re\{\varphi(i\mu_2,\nu_1)\}=\Re\{\frac{(p-\xi)\nu}{(\frac{\alpha+\beta}{\mu+\lambda}-p)+\xi+(p-\xi)i\mu_2}\}=\frac{[(\frac{\alpha+\beta}{\mu+\lambda}-p)+\xi](p-\xi)\nu}{((\frac{\alpha+\beta}{\mu+\lambda}-p)+\xi)^2+(p-\xi)^2\mu_2^2},$$

$\Rightarrow$

$$\Re\{\varphi(i\mu_2,\nu_1)\}=\frac{[(\frac{\alpha+\beta}{\mu+\lambda}-p)+\xi](p-\xi)\nu}{((\frac{\alpha+\beta}{\mu+\lambda}-p)+\xi)^2+(p-\xi)^2\mu_2^2}\leq-\frac{[(\frac{\alpha+\beta}{\mu+\lambda}-p)+\xi](p-\xi)(1+\mu_2^2)}{2((\frac{\alpha+\beta}{\mu+\lambda}-p)+\xi)^2+(p-\xi)^2\mu_2^2}<0\cdot$$

Hence, the function $\varphi(\mu,\nu)$ satisfies the conditions of Lemma 2.1. This shows that $\Re\{h(z)\}>0(z\in\mathbb{U})$, that is, $f(z)\in\mathfrak{S}^*_{m+1}(p,\xi,\lambda,\alpha,\beta,\mu)\cdot$

Theorem 2.3. If $f(z)\in A_p$ and $\alpha,\beta,\mu,\lambda\geq 0, p\in\mathbb{N}, m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}, z\in\mathbb{U}\cdot$ Then

$$\mathfrak{C}_m(p,\xi,\lambda,\alpha,\beta,\mu)\subseteq\mathfrak{C}_{m+1}(p,\xi,\lambda,\alpha,\beta,\mu)\subseteq\mathfrak{C}_{m+2}(p,\xi,\lambda,\alpha,\beta,\mu).$$

Proof. Let $f\in\mathfrak{C}_m(p,\xi,\lambda,\alpha,\beta,\mu)\Rightarrow\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)f\in C_p(\xi),\Leftrightarrow\frac{z}{p}(\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)f)'\in S_p^*(\xi)\Leftrightarrow\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)f(\frac{zf'}{p})\in S_p^*(\xi),\Leftrightarrow\frac{zf'}{p}\in\mathfrak{C}_m(p,\xi,\lambda,\alpha,\beta,\mu)\subseteq\mathfrak{C}_{m+1}(p,\xi,\lambda,\alpha,\beta,\mu)$, $\Rightarrow\frac{zf'}{p}\in\mathfrak{C}_{m+1}(p,\xi,\lambda,\alpha,\beta,\mu)\Leftrightarrow\mathtt{C}_p^{m+1}(\alpha,\beta,\mu,\lambda)f(\frac{zf'}{p})\in S_p^*(\xi),\Leftrightarrow\frac{z}{p}(\mathtt{C}_p^{m+1}(\alpha,\beta,\mu,\lambda)f)'\in S_p^*(\xi)\Rightarrow\mathtt{C}_p^{m+1}(\alpha,\beta,\mu,\lambda)f\in C_p(\xi),\Rightarrow f\in\mathfrak{C}_{m+1}(p,\xi,\lambda,\alpha,\beta,\mu)\cdot$

Theorem 2.4. If $f(z)\in A_p$ and $\alpha,\beta,\mu,\lambda\geq 0, p\in\mathbb{N}, m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}, z\in\mathbb{U}\cdot$ Then

$$\mathfrak{K}_m(p,\rho,\xi,\lambda,\alpha,\beta,\mu)\subseteq\mathfrak{K}_{m+1}(p,\rho,\xi,\lambda,\alpha,\beta,\mu)\subseteq\mathfrak{K}_{m+2}(p,\rho,\xi,\lambda,\alpha,\beta,\mu)\cdot$$

Proof. Let $f(z)\in\mathfrak{K}_m(p,\rho,\xi,\lambda,\alpha,\beta,\mu)$ implies

$$\Re\left(\frac{z(\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)f(z))'}{\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)g(z)}\right)>\rho,\ g(z)\in\mathfrak{S}^*_m(p,\xi,\lambda,\alpha,\beta,\mu), 0\leq\rho<1, z\in\mathbb{U}\cdot$$

Since $g(z)\in\mathfrak{S}^*_m(p,\xi,\lambda,\alpha,\beta,\mu)\subseteq\mathfrak{S}^*_{m+1}(p,\xi,\lambda,\alpha,\beta,\mu)$, let

$$\frac{z(\mathtt{C}_p^{m+1}(\alpha,\beta,\mu,\lambda)g(z))'}{\mathtt{C}_p^{m+1}(\alpha,\beta,\mu,\lambda)g(z)}=\xi+(p-\xi)H(z)\cdot \quad (2.3)$$

Suppose that

$$\left(\frac{z(\mathtt{C}_p^{m+1}(\alpha,\beta,\mu,\lambda)f(z))'}{\mathtt{C}_p^{m+1}(\alpha,\beta,\mu,\lambda)g(z)}\right) = \rho + (p-\rho)h(z),\ 0 \le \rho < 1, z \in \mathbb{U}\cdot \tag{2.4}$$

Where $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ Using (1.5), (2.3) and (2.4) we get

$$\frac{z(\mathtt{C}_p^{m}(\alpha,\beta,\mu,\lambda)f(z))'}{\mathtt{C}_p^{m}(\alpha,\beta,\mu,\lambda)g(z)} = \frac{\frac{z(\mathtt{C}_p^{m+1}(\alpha,\beta,\mu,\lambda)zf'(z))'}{\mathtt{C}_p^{m+1}(\alpha,\beta,\mu,\lambda)g(z)} + (\frac{\alpha+\beta}{\mu+\lambda} - p)(\rho + (p-\rho)h(z))}{\xi + (p-\xi)H(z) + (\frac{\alpha+\beta}{\mu+\lambda} - p)}\cdot \tag{2.5}$$

Using (2.3) and (2.4) we have

$$\frac{z((\mathtt{C}_p^{m+1}(\alpha,\beta,\mu,\lambda)zf'(z)))'}{(\mathtt{C}_p^{m+1}(\alpha,\beta,\mu,\lambda)g(z))} = [\rho + (p-\rho)h(z)][\xi + (p-\xi)H(z)] + (p-\rho)zh'(z)\cdot \tag{2.6}$$

Simplifying simultaneously (2.5) and (2.6), we get

$$\frac{z(\mathtt{C}_p^{m}(\alpha,\beta,\mu,\lambda)f(z))'}{\mathtt{C}_p^{m}(\alpha,\beta,\mu,\lambda)g(z)} - \rho = (p-\rho)h(z) + \frac{(p-\rho)zh'(z)}{(\frac{\alpha+\beta}{\mu+\lambda} - p + \xi) + (p-\xi)H(z)}\cdot \tag{2.7}$$

Taking $h(z) = \mu = \mu_1 + i\mu_1$ and $zh'(z) = \nu = \nu_1 + i\nu_1$, we define the function $\varphi(\mu,\nu)$ by

$$\varphi(\mu,\nu) = (p-\rho)\mu + \frac{(p-\rho)\nu}{(\frac{\alpha+\beta}{\mu+\lambda} - p + \xi) + (p-\xi)H(z)}\cdot$$

Clearly conditions (i) and (ii) of Lemma 2.1 in $D = \mathbb{C} \times \mathbb{C}$ are satisfied. For (iii), we proceed as follows.

$$\Re\{\varphi(i\mu_2,\nu_1)\} = \frac{\nu_1(p-\rho)[(\frac{\alpha+\beta}{\mu+\lambda} - p + \xi) + (p-\xi)h_1(x_1,y_1)]}{[(\frac{\alpha+\beta}{\mu+\lambda} - p + \xi) + (p-\xi)h_1(x_1,y_1)]^2 + [(p-\xi)h_2(x_2,y_2)]^2}\cdot$$

Where $H(z) = h_1(x_1,y_1) + ih_2(x_2,y_2)$, $h_1(x_1,y_1)$ and $h_2(x_2,y_2)$ being functions of x and y and $\Re(h_1(x_1,y_1)) > 0$. Since $\nu_1 \le -\frac{1}{2}(1+\mu_2^2)$, implies

$$\Re\{\varphi(i\mu_2,\nu_1)\} = -\frac{1}{2}\frac{(1+\mu_2^2)(p-\rho)[(\frac{\alpha+\beta}{\mu+\lambda} - p + \xi) + (p-\xi)h_1(x_1,y_1)]}{[(\frac{\alpha+\beta}{\mu+\lambda} - p + \xi) + (p-\xi)h_1(x_1,y_1)]^2 + [(p-\xi)h_2(x_2,y_2)]^2} < 0\cdot$$

Applying Lemma 2.1. on $\varphi(\mu,\nu)$, gives $\Re\{h(z)\} > 0 (z \in \mathbb{U})$. This shows that $f(z) \in \mathfrak{K}_{m+1}(p,\rho,\xi,\lambda,\alpha,\beta,\mu)\cdot$

The following theorem can be proved in a similar manner.

Theorem 2.5. If $f(z) \in A_p$ and $\alpha,\beta,\mu,\lambda \ge 0, p \in \mathbb{N}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \mathbb{U}\cdot$ Then

$$\mathfrak{K}^*_m(p,\rho,\xi,\lambda,\alpha,\beta,\mu) \subseteq \mathfrak{K}^*_{m+1}(p,\rho,\xi,\lambda,\alpha,\beta,\mu) \subseteq \mathfrak{K}^*_{m+2}(p,\rho,\xi,\lambda,\alpha,\beta,\mu)\cdot$$

3. Integral operator

For $c > -p$ and $f(z) \in A_p$, the integral operator $L_{c,p} : A_p \to A_p$ is defined by

$$L_{c,p}(f) = \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt\cdot \tag{3.1}$$

The operator $L_{c,p}(f)$ was introduced by Bernardi [12].

Theorem 3.1. Let $c > -p$, $0 \leq \xi < p\cdot$ If $f(z) \in \mathfrak{S}_m^*(p,\xi,\lambda,\alpha,\beta,\mu)$, then $L_c(f) \in \mathfrak{S}_m^*(p,\xi,\lambda,\alpha,\beta,\mu))\cdot$

Proof. Using (3.1) we get

$$z(\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)L_{c,p}f(z))' = (c+p)(\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)f(z)) - c(\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)L_{c,p}f(z))\cdot \quad (3.2)$$

Let

$$\frac{z(\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)L_cf(z))'}{\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)L_cf(z)} = \xi + (p-\xi)h(z)\cdot \quad (3.3)$$

Where $h(z) = 1 + c_1z + c_2z^2 + \cdots\cdot$ By using (3.1) and (3.2) we get

$$\frac{z(\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)f(z))'}{\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)f(z)} - \xi = (p-\xi)h(z) + \frac{(p-\xi)zh'(z)}{\xi+(p-\xi)h(z)+c}\cdot$$

Taking $h(z) = \mu = \mu_1 + i\mu_1$ and $zh'(z) = \nu = \nu_1 + i\nu_1$, we define the function $\varphi(\mu,\nu)$ by

$$\varphi(\mu,\nu) = (p-\xi)\mu + \frac{(p-\xi)\nu}{\xi+c+(p-\xi)\mu}\cdot$$

Clearly conditions (i) and (ii) of Lemma 2.1 in $D = \left(\mathbb{C} - \{\frac{\xi+c}{\xi-p}\}\right) \times \mathbb{C}$ are satisfied. We proceed for (iii) as follows;

$$\Re\{\varphi(i\mu_2,\nu_1)\} = \frac{(\xi+c)(p-\xi)\nu_1}{[\xi+c]^2+[(p-\xi)\mu_2]^2} \leq \frac{-(\xi+c)(p-\xi)(1+\mu_2^2)}{2[\xi+c]^2+2[(p-\xi)\mu_2]^2} < 0\cdot$$

Applying Lemma 2.1. we have $\Re\{h(z)\} > 0 (z \in \mathbb{U})$, that is, $L_c(f) \in \mathfrak{S}_m^*(p,\xi,\lambda,\alpha,\beta,\mu)\cdot$

Theorem 3.2. Let $c > -p$, $0 \leq \xi < p\cdot$ If $f(z) \in \mathfrak{C}_m(p,\xi,\lambda,\alpha,\beta,\mu)$, then $L_c(f) \in \mathfrak{C}_m(p,\xi,\lambda,\alpha,\beta,\mu)\cdot$

Proof. For the proof, use Theorem 3.1 and the fact that $f(z) \in C_p(\xi) \Leftrightarrow \frac{zf'(z)}{p} \in S_p^*(\xi)$.

Theorem 3.3. Let $c > -p$, $0 \leq \xi < p\cdot$ If $f(z) \in \mathfrak{K}_m(p,\rho,\xi,\lambda,\alpha,\beta,\mu)$, then $L_c(f) \in \mathfrak{K}_m(p,\rho,\xi,\lambda,\alpha,\beta,\mu)\cdot$

Proof. As $f(z) \in \mathfrak{K}_m(p,\rho,\xi,\lambda,\alpha,\beta,\mu)$ gives

$$\Re\left(\frac{z(\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)f(z))'}{\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)g(z)}\right) > \rho\cdot$$

Since $g(z) \in \mathfrak{S}_m^*(p,\xi,\lambda,\alpha,\beta,\mu))$ implies $L_c(g(z)) \in \mathfrak{S}_m^*(p,\xi,\lambda,\alpha,\beta,\mu))\cdot$ Let

$$\frac{z(\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)L_c(g(z))'}{\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)L_cg(z)} = \xi + (p-\xi)H(z),\ \Re(H(z)) > 0,\ z \in \mathbb{U}\cdot$$

Also let

$$\left(\frac{z(\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)L_cf(z))'}{\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)L_cg(z)}\right) = \rho + (p-\rho)h(z),\ z \in \mathbb{U}\cdot \quad (3.4)$$

Where $h(z) = 1 + c_1z + c_2z^2 + \cdots$.

After doing calculations, we get

$$\left(\frac{z(\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)f(z))'}{\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)g(z)}\right)=\frac{\frac{z(\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)L_c(zf'(z))'}{\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)L_cg(z)}+c\frac{(\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)L_c(zf'(z))}{\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)L_cg(z)}}{\frac{z(\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)L_c(g(z))'}{\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)L_cg(z)}+c}\cdot$$

Also

$$\frac{z(\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)L_c(zf'(z))'}{\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)L_cg(z)}=[\rho+(p-\rho)h(z)][\xi+(p-\xi)H(z)]+[(p-\rho)zh'(z)]\cdot$$

Hence

$$\left(\frac{z(\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)f(z))'}{\mathtt{C}_p^m(\alpha,\beta,\mu,\lambda)g(z)}\right)-\rho=(p-\rho)h(z)+\frac{(p-\rho)zh'(z)}{\xi+(p-\xi)H(z)+c}. \tag{3.5}$$

Taking $h(z)=\mu=\mu_1+i\mu_1$ and $zh'(z)=\nu=\nu_1+i\nu_1$, we define the function $\varphi(\mu,\nu)$ by

$$\varphi(\mu,\nu)=(p-\rho)\mu+\frac{(p-\rho)\nu}{\xi+(p-\xi)H(z)+c}\cdot \tag{3.6}$$

It is easy to see that the function $\varphi(\mu,\nu)$ satisfies the conditions (i) and (ii) of Lemma 2.1 in $D=\mathbb{C}\times\mathbb{C}$. To verify the condition (iii), we proceed as follows.

$$\Re\{\varphi(i\mu_2,\nu_1)\}=\frac{\nu_1(p-\rho)[(\xi+c)+(p-\xi)h_1(x_1,y_1)]}{[(\xi+c)+(p-\xi)h_1(x_1,y_1)]^2+[(p-\xi)h_2(x_2,y_2)]^2}\cdot$$

Where $H(z)=h_1(x_1,y_1)+ih_2(x_2,y_2)$, $h_1(x_1,y_1)$ and $h_2(x_2,y_2)$ being functions of x and y and $\Re(h_1(x_1,y_1))>0$. By putting $\nu_1\le-\frac{1}{2}(1+\mu_2^2)$, we obtain

$$\Re\{\varphi(i\mu_2,\nu_1)\}=-\frac{1}{2}\frac{(1+\mu_2^2)(p-\rho)[(\xi+c)+(p-\xi)h_1(x_1,y_1)]}{[(\xi+c)+(p-\xi)h_1(x_1,y_1)]^2+[(p-\xi)h_2(x_2,y_2)]^2}<0\cdot$$

By applying Lemma 2.1 we get $\Re\{h(z)\}>0(z\in\mathbb{U})$, that is, $L_c(f)\in\mathfrak{C}_m(p,\xi,\lambda,\alpha,\beta,\mu)\cdot$

References

[1] M. Darus, I. Faisal, *A different approach to normalized analytic aunctions through meromorphic functions defined by extended multiplier transformations operator*, Int. J. Appl. Math. Stat. **23**(11) (2011), 112–121.

[2] M. Darus, I. Faisal, *Characrerization properties for a class of analytic functions defined by generalized Cho and Srivastava operator*, In Proc. 2nd Inter. Conf. Math. Sci., Kuala Lumpur, Malaysia, (2010), pp. 1106-1113.

[3] M.K. Aouf, R.M. El-Ashwah, S.M. El-Deeb, *Some inequalities for certain p–valent functions involving extended multiplier transformations*, Proc. Pakistan Acad. Sci. **46** (2009), 217–221.

[4] F.M. Al-oboudi, *On univalent functions defined by a generalized Salagean operator*, Int. J. Math. Math. Sci. (2004), 1429–1436.

[5] G.S. Salagean, *Subclasses of univalent functions. Lecture Notes in Mathematics 1013*, Springer–Verlag (1983), 362–372.

[6] B.A. Uralegaddi, C. Somanatha, *Certain classes of univalent functions, In: Current Topics in Analytic Function Theory. Eds. H.M. Srivastava and S. Owa.*, World Scientific Publishing Company, Singapore, (1992), 371–374.

[7] N.E. Cho, H.M. Srivastava, *Argument estimates of certain analytic func- tions defined by a class of multiplier transformations*, Math. Comput. Modeling **37** (2003), 39–49.

[8] N.E. Cho, T.H. Kim, *Multiplier transformations and strongly close-to- convex functions*, Bull. Korean Math. Soc. **40** (2003), 399–410.

[9] A. Catas, *On certain classes of p-valent functions defined by multiplier transformations*, in: Proc. Book of the International Symposium on Geometric Function Theory and Applications, Istanbul, Turkey, August(2007), pp. 241–250.

[10] S.S. Kumar, H.C. Taneja, V. Ravichandran, *Classes multivalent functions defined by Dziok–Srivastava linear operator and multiplier transformations*, Kyungpook Math. J. **46** (2006), 97–109.

[11] H.M. Srivastava, K. Suchithra, B. Adolf Stephen, S. Sivasubramanian, *Inclusion and neighborhood properties of certain subclasses of multivalent functions of complex order*, J. Ineq. Pure Appl. Math. **7**(5) (2006), 1–8.

[12] S.D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc., **135** (1969), 429–446.

[13] D.A. Patil, N.K. Thakare, *On convex hulls and extreme points of p-valent starlike and convex classes with applications*, Bull. Math. Soc. Sci. Math. R.S. Roumaine **27**(75), (1983), 145–160.

[14] A.W. Goodman, *On Schwarz?-Chritoffell transformation and p-valent functions*, Trans. Amer. Math. Soc. **68** (1950), 204–223.

[15] S. Owa, *On certain classes of p-valent functions with negative coefficients*, Simon Stevin **59** (1985), 385–402.

[16] M.K. Aouf, *On a class of p-valent close-to-convex functions*, Internat. J. Math. Math. Sci. **11**(2) (1988) 259–266.

[17] R.J. Libera, *Some radius of convexity problems*, Duke Math. J., **31** (1964), 143–158.

[18] K.I. Noor, *On quasi-convex functions and related topics*, Internat. J. Math. Math. Sci., **10** (1987), 241–258.

[19] K.I. Noor, *On some applications of the Ruscheweyh derivatives*, Math. Japon. **36**(5) (1991), 869–874.

[20] S.S. Miller, *Differential inequalities and Carathodory function*, Bull. Amer. Math. Soc. **8** (1975), 79–81.

[21] S.S. Miller, P.T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl., **65** (1978), 289–305.

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