

## COMMON FIXED POINTS FOR MAPPINGS SATISFYING $\varphi$ AND $F$ -MAPS IN $G$ -CONE METRIC SPACES

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ABSTRACT. The existence of points of coincidence and common fixed points for three self mappings satisfying generalized contractive conditions related to  $\varphi$  and  $F$ -maps in a  $G$ -cone metric space is proved. Our results extend and generalize several well-known comparable results in the existing literature.

### 1. INTRODUCTION

The study of fixed point theory has been at the centre of vigorous research activity and it has applications in many important areas such as variational and linear inequalities, nonlinear and adaptive control systems, parameterize estimation problems, and fractal image decoding. There has been a number of generalizations of the usual notion of a metric space. One such generalization is a  $G$ -metric space initiated by Mustafa and Sims [14]. They obtained some fundamental results in this structure. Another such generalization proposed by Huang and Zhang [11], replacing the set of real numbers by an ordered Banach space, called cone metric space and established some fixed point theorems for nonlinear mappings in a normal cone metric space. Afterwards, Rezapour and Hambarani [19] studied some interesting fixed point theorems in cone metric spaces without assuming the normality condition. Subsequently, several other authors have generalized the results of Huang and Zhang and have studied fixed point theorems for normal and non-normal cones, coupled fixed point for mappings in cone metric spaces. In [20], Sabetghadam and Masiha introduced the concept of generalized  $\varphi$ -mappings and obtained common fixed points for such mappings. Recently, I. Beg, M. Abbas and T. Nazir [4] introduced  $G$ -cone metric spaces, which is a generalization of  $G$ -metric spaces and cone metric spaces. They later proved some fixed point theorems for mappings satisfying certain contractive conditions. In this paper, we obtain sufficient conditions for existence of points of coincidence and common fixed points for three self mappings satisfying generalized contractive conditions related to  $\varphi$  and  $F$ -maps in  $G$ -cone metric spaces. Finally, some examples are cited in support our

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results.

## 2. DEFINITIONS AND BASIC FACTS

In this section, we recall some basic definitions, standard notations and important results for  $G$ -cone metric spaces that will be needed in the sequel.

Let  $E$  be a real Banach Space and let  $\theta$  denote the zero element in  $E$ . A cone  $P$  is a subset of  $E$  such that

- (i)  $P$  is closed, nonempty and  $P \neq \{\theta\}$ ,
- (ii)  $a, b \in R, a, b \geq 0, x, y \in P$  implies  $ax + by \in P$ ; More generally if  $a, b, c \in R, a, b, c \geq 0, x, y, z \in P \Rightarrow ax + by + cz \in P$ ,
- (iii)  $P \cap (-P) = \{\theta\}$ .

For a given cone  $P \subseteq E$ , we can define a partial ordering  $\leq$  with respect to  $P$  by  $x \leq y$  if and only if  $y - x \in P$ . Let  $A$  be a finite subset of  $E$ . If there exists an element  $x \in A$  such that  $x \leq a$  for all  $a \in A$ , we write  $x = \min A$ . If there is an element  $y \in A$  such that  $a \leq y$  for all  $a \in A$ , we write  $y = \max A$ . It is to be noted that if  $\leq$  is a complete ordering on  $E$  then  $\min A, \max A$  are always exist. The notation  $x < y$  will stand for  $x \leq y$  and  $x \neq y$ , while  $x \ll y$  will stand for  $y - x \in \text{int } P$ , where  $\text{int } P$  denotes the interior of  $P$ . A cone  $P$  is called normal if there is a number  $M > 0$  such that for all  $x, y \in E, 0 \leq x \leq y$  implies  $\|x\| \leq M \|y\|$ . The least positive number satisfying the above inequality is called the normal constant of  $P$ . Razapour and Hambarani [19] proved that there are no normal cones with normal constants  $M < 1$  and for each  $k > 1$  there are cones with normal constants  $M > k$ .

**Definition 2.1.** ([5]) Let  $P$  be a cone. A nondecreasing mapping  $\varphi : P \rightarrow P$  is called a  $\varphi$ -map if

- ( $\varphi_1$ )  $\varphi(\theta) = \theta$  and  $\theta < \varphi(w) < w$  for  $w \in P \setminus \{\theta\}$ ,
- ( $\varphi_2$ )  $w - \varphi(w) \in \text{int } P$  for every  $w \in \text{int } P$ ,
- ( $\varphi_3$ )  $\lim_{n \rightarrow \infty} \varphi^n(w) = \theta$  for every  $w \in P \setminus \{\theta\}$ .

**Definition 2.2.** ([20]) Let  $P$  be a cone and let  $(w_n)$  be a sequence in  $P$ . One says that  $w_n \rightarrow \theta$  if for every  $\epsilon \in P$  with  $\theta \ll \epsilon$  there exists  $n_0 \in N$  such that  $w_n \ll \epsilon$  for all  $n \geq n_0$ .

A nondecreasing mapping  $F : P \rightarrow P$  is called a  $F$ -map if

- ( $F_1$ )  $F(w) = \theta$  if and only if  $w = \theta$ ,
- ( $F_2$ ) for every  $w_n \in P, w_n \rightarrow \theta$  if and only if  $F(w_n) \rightarrow \theta$ ,
- ( $F_3$ ) for every  $w_1, w_2 \in P, F(w_1 + w_2) \leq F(w_1) + F(w_2)$ .

**Definition 2.3.** ([4]) Let  $X$  be a nonempty set. Suppose a mapping  $G : X \times X \times X \rightarrow E$  satisfies:

- ( $G_1$ )  $G(x, y, z) = \theta$  if  $x = y = z$ ,
- ( $G_2$ )  $\theta < G(x, x, y)$ ; whenever  $x \neq y$ , for all  $x, y \in X$ ,
- ( $G_3$ )  $G(x, x, y) \leq G(x, y, z)$ ; whenever  $y \neq z$ ,
- ( $G_4$ )  $G(x, y, z) = G(x, z, y) = G(y, x, z) = \dots$  (Symmetric in all three variables),
- ( $G_5$ )  $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$  for all  $x, y, z, a \in X$ .

Then  $G$  is called a generalized cone metric on  $X$ , and  $X$  is called a generalized cone metric space or more specifically a  $G$ -cone metric space.

The concept of a  $G$ -cone metric space is more general than that of a  $G$ -metric space and a cone metric space.

**Example 2.4.** ([4]) Let  $(X, d)$  be a cone metric space. Define  $G : X \times X \times X \rightarrow E$ , by

$$G(x, y, z) = d(x, y) + d(y, z) + d(z, x).$$

Then  $X$  is a  $G$ -cone metric space.

**Definition 2.5.** ([4]) Let  $X$  be a  $G$ -cone metric space and  $(x_n)$  be a sequence in  $X$ . We say that  $(x_n)$  is:

- (a) Cauchy sequence if for every  $c \in E$  with  $\theta \ll c$ , there is  $n_0$  such that for all  $n, m, l > n_0$ ,  $G(x_n, x_m, x_l) \ll c$ .
- (b) Convergent sequence if for every  $c$  in  $E$  with  $\theta \ll c$ , there is  $n_0$  such that for all  $m, n > n_0$ ,  $G(x_m, x_n, x) \ll c$  for some fixed  $x$  in  $X$ . Here  $x$  is called the limit of a sequence  $(x_n)$  and is denoted by  $\lim_{n \rightarrow \infty} x_n = x$  or  $x_n \rightarrow x$  as  $n \rightarrow \infty$ .

A  $G$ -cone metric space  $X$  is said to be complete if every Cauchy sequence in  $X$  is convergent in  $X$ .

**Proposition 2.6.** ([4]) Let  $X$  be a  $G$ -cone metric space then the following are equivalent.

- (i)  $(x_n)$  converges to  $x$ .
- (ii)  $G(x_n, x_n, x) \rightarrow \theta$ , as  $n \rightarrow \infty$ .
- (iii)  $G(x_n, x, x) \rightarrow \theta$ , as  $n \rightarrow \infty$ .
- (iv)  $G(x_m, x_n, x) \rightarrow \theta$ , as  $m, n \rightarrow \infty$ .

**Lemma 2.7.** ([4]) Let  $X$  be a  $G$ -cone metric space,  $(x_m)$ ,  $(y_n)$ , and  $(z_l)$  be sequences in  $X$  such that  $x_m \rightarrow x$ ,  $y_n \rightarrow y$ , and  $z_l \rightarrow z$ , then  $G(x_m, y_n, z_l) \rightarrow G(x, y, z)$  as  $m, n, l \rightarrow \infty$ .

**Lemma 2.8.** ([4]) Let  $(x_n)$  be a sequence in a  $G$ -cone metric space  $X$  and  $x \in X$ . If  $(x_n)$  converges to  $x$ , and  $(x_n)$  converges to  $y$ , then  $x = y$ .

**Lemma 2.9.** ([4]) Let  $(x_n)$  be a sequence in a  $G$ -cone metric space  $X$  and  $x \in X$ . If  $(x_n)$  converges to  $x$ , then  $(x_n)$  is a Cauchy sequence.

**Lemma 2.10.** ([4]) Let  $(x_n)$  be a sequence in a  $G$ -cone metric space  $X$  and if  $(x_n)$  is a Cauchy sequence in  $X$ , then  $G(x_m, x_n, x_l) \rightarrow \theta$  as  $m, n, l \rightarrow \infty$ .

**Proposition 2.11.** ([12]) If  $E$  is a real Banach space with cone  $P$  and if  $a \leq \lambda a$  where  $a \in P$  and  $0 \leq \lambda < 1$  then  $a = \theta$ .

**Definition 2.12.** ([3]) Let  $T$  and  $S$  be self mappings of a set  $X$ . If  $w = T(x) = S(x)$  for some  $x$  in  $X$ , then  $x$  is called a coincidence point of  $T$  and  $S$  and  $w$  is called a point of coincidence of  $T$  and  $S$ .

**Definition 2.13.** ([13]) The mappings  $T, S : X \rightarrow X$  are weakly compatible, if for every  $x \in X$ , the following holds:

$$T(S(x)) = S(T(x)) \text{ whenever } S(x) = T(x).$$

**Definition 2.14.** A mapping  $T : X \rightarrow X$  in a  $G$ -cone metric space  $X$  is said to be expansive if there is a real constant  $c > 1$  satisfying

$$cG(x, y, z) \leq G(T(x), T(y), T(z))$$

for all  $x, y, z \in X$ .

### 3. MAIN RESULTS

In the following we always suppose that  $E$  is a real Banach space,  $P$  is a non normal cone in  $E$  with  $\text{int } P \neq \emptyset$  and  $\leq$  is a complete ordering on  $E$  with respect to  $P$ . Throughout the paper we denote by  $N$  the set of all positive integers.

We first state a lemma which will play a crucial role in the sequel.

**Lemma 3.1.** ([2]) Let  $X$  be a non empty set and the mappings  $S, T, f : X \rightarrow X$  have a unique point of coincidence  $v$  in  $X$ . If  $(S, f)$  and  $(T, f)$  are weakly compatible, then  $S, T$  and  $f$  have a unique common fixed point.

**Theorem 3.2.** Let  $X$  be a  $G$ -cone metric space and let the mappings  $S, T, f : X \rightarrow X$  satisfy the following condition:

$$\max \left\{ \begin{array}{l} F(G(S(x), T(y), T(y))), \\ F(G(T(x), S(y), S(y))), \\ F(G(T(x), T(y), T(y))), \\ F(G(S(x), S(y), S(y))) \end{array} \right\} \leq \varphi(F(G(f(x), f(y), f(y)))) \quad (3.1)$$

for all  $x, y \in X$ . If  $S(X) \cup T(X) \subseteq f(X)$  and  $f(X)$  is a complete subspace of  $X$ , then  $S, T$  and  $f$  have a unique point of coincidence. Moreover, if  $(S, f)$  and  $(T, f)$  are weakly compatible, then  $S, T$  and  $f$  have a unique common fixed point.

**Proof.** Let  $x_0 \in X$  be arbitrary and choose a point  $x_1 \in X$  such that  $f(x_1) = S(x_0)$  which is possible since  $S(X) \subseteq f(X)$ . Similarly, choose a point  $x_2 \in X$  such that  $f(x_2) = T(x_1)$ . Proceeding in this way, we can define a sequence  $(f(x_n))$  in  $f(X)$  by

$$\begin{aligned} f(x_n) &= S(x_{n-1}), \text{ if } n \text{ is odd} \\ &= T(x_{n-1}), \text{ if } n \text{ is even.} \end{aligned}$$

If  $n \in N$  is odd, then by using (3.1) we obtain

$$\begin{aligned} F(G(f(x_n), f(x_{n+1}), f(x_{n+1}))) &= F(G(S(x_{n-1}), T(x_n), T(x_n))) \\ &\leq \max \left\{ \begin{array}{l} F(G(S(x_{n-1}), T(x_n), T(x_n))), \\ F(G(T(x_{n-1}), S(x_n), S(x_n))), \\ F(G(T(x_{n-1}), T(x_n), T(x_n))), \\ F(G(S(x_{n-1}), S(x_n), S(x_n))) \end{array} \right\} \\ &\leq \varphi(F(G(f(x_{n-1}), f(x_n), f(x_n)))). \end{aligned}$$

If  $n$  is even, then by (3.1), we have

$$\begin{aligned} F(G(f(x_n), f(x_{n+1}), f(x_{n+1}))) &= F(G(T(x_{n-1}), S(x_n), S(x_n))) \\ &\leq \max \left\{ \begin{array}{l} F(G(S(x_{n-1}), T(x_n), T(x_n))), \\ F(G(T(x_{n-1}), S(x_n), S(x_n))), \\ F(G(T(x_{n-1}), T(x_n), T(x_n))), \\ F(G(S(x_{n-1}), S(x_n), S(x_n))) \end{array} \right\} \\ &\leq \varphi(F(G(f(x_{n-1}), f(x_n), f(x_n))))). \end{aligned}$$

Thus for any positive integer  $n$ , it must be the case that,

$$F(G(f(x_n), f(x_{n+1}), f(x_{n+1}))) \leq \varphi(F(G(f(x_{n-1}), f(x_n), f(x_n)))). \quad (3.2)$$

By repeated application of (3.2), we obtain

$$F(G(f(x_n), f(x_{n+1}), f(x_{n+1}))) \leq \varphi^n(F(G(f(x_0), f(x_1), f(x_1)))).$$

Let  $\theta \ll c$  be given, then  $c - \varphi(c) \in \text{int } P$ . By  $(\varphi_3)$ ,  $\lim_{n \rightarrow \infty} \varphi^n(F(G(f(x_0), f(x_1), f(x_1)))) = \theta$ . So, one can find a natural number  $n_0$  such that

$$\varphi^m(F(G(f(x_0), f(x_1), f(x_1)))) \ll c - \varphi(c), \text{ for all } m > n_0.$$

Consequently,  $F(G(f(x_m), f(x_{m+1}), f(x_{m+1}))) \ll c - \varphi(c)$ , for all  $m > n_0$ .

We show that

$$F(G(f(x_m), f(x_{n+1}), f(x_{n+1}))) \ll c \quad (3.3)$$

for a fixed  $m > n_0$  and  $n \geq m$ .

Clearly, (3.3) holds for  $n = m$ . We suppose that (3.3) holds for some  $n \geq m$ . Then by  $(G_5)$  and  $(F_3)$ , we obtain

$$\begin{aligned} F(G(f(x_m), f(x_{n+2}), f(x_{n+2}))) &\leq F(G(f(x_m), f(x_{m+1}), f(x_{m+1}))) \\ &\quad + F(G(f(x_{m+1}), f(x_{n+2}), f(x_{n+2}))) \\ &\leq F(G(f(x_m), f(x_{m+1}), f(x_{m+1}))) \\ &\quad + \max \left\{ \begin{array}{l} F(G(S(x_m), T(x_{n+1}), T(x_{n+1}))), \\ F(G(T(x_m), S(x_{n+1}), S(x_{n+1}))), \\ F(G(T(x_m), T(x_{n+1}), T(x_{n+1}))), \\ F(G(S(x_m), S(x_{n+1}), S(x_{n+1}))) \end{array} \right\} \\ &\leq F(G(f(x_m), f(x_{m+1}), f(x_{m+1}))) \\ &\quad + \varphi(F(G(f(x_m), f(x_{n+1}), f(x_{n+1})))) \\ &\ll c - \varphi(c) + \varphi(c) \\ &= c. \end{aligned}$$

Therefore, by induction (3.3) holds for a fixed  $m > n_0$  and  $n \geq m$ .

Now using  $(F_2)$ , we deduce that  $(f(x_n))$  is a Cauchy sequence in  $f(X)$ . By completeness of  $f(X)$ , there exist  $u, v \in X$  such that  $f(x_n) \rightarrow v = f(u)$ .

Again, by  $(F_2)$ , for a fixed  $\theta \ll c$ , there exists a natural number  $n_0$  such that  $F(G(f(u), f(x_{2n+1}), f(x_{2n+1}))) \ll \frac{c}{2}$  and  $F(G(f(x_{2n}), f(u), f(u))) \ll \frac{c}{2}$  for all  $n > n_0$ .

Then by  $(G_5)$  and  $(F_3)$ , we have

$$\begin{aligned}
F(G(f(u), T(u), T(u))) &\leq F(G(f(u), f(x_{2n+1}), f(x_{2n+1}))) + F(G(f(x_{2n+1}), T(u), T(u))) \\
&= F(G(f(u), f(x_{2n+1}), f(x_{2n+1}))) + F(G(S(x_{2n}), T(u), T(u))) \\
&\leq F(G(f(u), f(x_{2n+1}), f(x_{2n+1}))) \\
&\quad + \max \left\{ \begin{array}{l} F(G(S(x_{2n}), T(u), T(u))), \\ F(G(T(x_{2n}), S(u), S(u))), \\ F(G(T(x_{2n}), T(u), T(u))), \\ F(G(S(x_{2n}), S(u), S(u))) \end{array} \right\} \\
&\leq F(G(f(u), f(x_{2n+1}), f(x_{2n+1}))) + \varphi(F(G(f(x_{2n}), f(u), f(u))) \\
&< F(G(f(u), f(x_{2n+1}), f(x_{2n+1}))) + F(G(f(x_{2n}), f(u), f(u))) \\
&\ll \frac{c}{2} + \frac{c}{2} = c.
\end{aligned}$$

Thus,

$$F(G(f(u), T(u), T(u))) \leq \frac{c}{i} \text{ for all } i \geq 1.$$

So,  $\frac{c}{i} - F(G(f(u), T(u), T(u))) \in P$ , for all  $i \geq 1$ . Since  $\frac{c}{i} \rightarrow \theta$  as  $i \rightarrow \infty$  and  $P$  is closed,  $-F(G(f(u), T(u), T(u))) \in P$ , and hence  $F(G(f(u), T(u), T(u))) = \theta$ . By applying  $(F_1)$ , it follows that  $G(f(u), T(u), T(u)) = \theta$  which implies that  $T(u) = f(u) = v$ .

Similarly, by using

$$F(G(f(u), S(u), S(u))) \leq F(G(f(u), f(x_{2n+2}), f(x_{2n+2}))) + F(G(f(x_{2n+2}), S(u), S(u)))$$

we can show that  $f(u) = S(u) = v$ . Thus,  $f(u) = S(u) = T(u) = v$  and so  $v$  becomes a common point of coincidence of  $S$ ,  $T$  and  $f$ .

For uniqueness, let there exists another point  $w (\neq v) \in X$  such that  $f(x) = S(x) = T(x) = w$  for some  $x \in X$ .

Then,

$$\begin{aligned}
 F(G(v, w, w)) &= F(G(S(u), T(x), T(x))) \\
 &\leq \max \left\{ \begin{array}{l} F(G(S(u), T(x), T(x))), \\ F(G(T(u), S(x), S(x))), \\ F(G(T(u), T(x), T(x))), \\ F(G(S(u), S(x), S(x))) \end{array} \right\} \\
 &\leq \varphi(F(G(f(u), f(x), f(x)))) \\
 &= \varphi(F(G(v, w, w))) \\
 &< F(G(v, w, w))
 \end{aligned}$$

which gives that  $v = w$ .

If  $(S, f)$  and  $(T, f)$  are weakly compatible, then by Lemma 3.1,  $S$ ,  $T$  and  $f$  have a unique common fixed point in  $X$ .

**Corollary 3.3.** *Let  $X$  be a  $G$ -cone metric space and let  $T, f : X \rightarrow X$  satisfy*

$$F(G(T(x), T(y), T(y))) \leq \varphi(F(G(f(x), f(y), f(y))))$$

for all  $x, y \in X$ . If  $T(X) \subseteq f(X)$  and if  $T(X)$  or  $f(X)$  is a complete subspace of  $X$ , then  $T$  and  $f$  have a unique point of coincidence. Moreover, if  $T$  and  $f$  are weakly compatible, then  $T$  and  $f$  have a unique common fixed point.

**Proof.** The proof can be obtained from Theorem 3.2 by taking  $S = T$ .

The following Corollary is an extension of Theorem 2.1 in [1] to  $G$ -cone metric spaces.

**Corollary 3.4.** *Let  $X$  be a complete  $G$ -cone metric space and let  $f : X \rightarrow X$  be an onto expansive mapping i.e.,  $f(X) = X$  and there exists a real constant  $c > 1$  satisfying*

$$cG(x, y, z) \leq G(f(x), f(y), f(z))$$

for all  $x, y, z \in X$ . Then  $f$  has a unique fixed point in  $X$ .

**Proof.** Taking  $T = S = F = I$ , the identity map and  $\varphi : P \rightarrow P$  by  $\varphi(z) = \frac{1}{c}z$  where  $c > 1$ , the conclusion of the Corollary follows from Theorem 3.2.

**Remark 3.5.** *Taking  $S = T$  and  $F = I$  in Theorem 3.2, we obtain the result [18, Theorem 3.1]. Thus, Theorem 3.2 is a generalization of the result [18, Theorem 3.1]. Furthermore, taking  $T = S$ ,  $F = f = I$  and  $\varphi(x) = kx$ ,  $k \in [0, 1)$  in Theorem 3.2, we have Corollary 3.6 which is an extension of Theorem 1 in [11] to  $G$ -cone metric spaces. So, Theorem 3.2 is both an extension and generalization of some results in the existing literature.*

**Corollary 3.6.** *Let  $X$  be a complete  $G$ -cone metric space and let  $T : X \rightarrow X$  satisfies*

$$G(T(x), T(y), T(y)) \leq kG(x, y, y)$$

for all  $x, y \in X$ , where  $0 \leq k < 1$ . Then  $T$  has a unique fixed point in  $X$ .

Now we prove Theorems 3.7 and 3.10 by replacing the condition  $(\varphi_1)$  with the following:

$(\varphi_1')$  there exists  $k \in [0, 1/2)$  such that  $\varphi(w) \leq kw$  for  $w \in P \setminus \{\theta\}$  and  $\varphi(\theta) = \theta$ .

**Theorem 3.7.** Let  $X$  be a  $G$ -cone metric space and let the mappings  $S, T, f : X \rightarrow X$  satisfy one of the following conditions:

$$\begin{aligned} & \max \left\{ \begin{array}{l} F(G(S(x), T(y), T(y))), \\ F(G(T(x), S(y), S(y))) \end{array} \right\} \\ & \leq \varphi \left( \min \left\{ \begin{array}{l} F(G(f(x), S(x), S(x))) + F(G(f(y), T(y), T(y))), \\ F(G(f(x), T(x), T(x))) + F(G(f(y), S(y), S(y))) \end{array} \right\} \right) \end{aligned} \quad (3.4)$$

or

$$\begin{aligned} & \max \left\{ \begin{array}{l} F(G(S(x), T(y), T(y))), \\ F(G(T(x), S(y), S(y))) \end{array} \right\} \\ & \leq \varphi \left( \min \left\{ \begin{array}{l} F(G(f(x), f(x), S(x))) + F(G(f(y), f(y), T(y))), \\ F(G(f(x), f(x), T(x))) + F(G(f(y), f(y), S(y))) \end{array} \right\} \right) \end{aligned} \quad (3.5)$$

for all  $x, y \in X$ . If  $S(X) \cup T(X) \subseteq f(X)$  and  $f(X)$  is a complete subspace of  $X$ , then  $S, T$  and  $f$  have a unique point of coincidence. Moreover, if  $(S, f)$  and  $(T, f)$  are weakly compatible, then  $S, T$  and  $f$  have a unique common fixed point.

**Proof.** Let  $x_0 \in X$  be arbitrary. As in Theorem 3.2, we define a sequence  $(f(x_n))$  in  $f(X)$  by the rule:

$$\begin{aligned} f(x_n) &= S(x_{n-1}), \text{ if } n \text{ is odd} \\ &= T(x_{n-1}), \text{ if } n \text{ is even.} \end{aligned}$$

Suppose the condition (3.4) holds. If  $n \in N$  is odd, then by using (3.4)

$$\begin{aligned} & F(G(f(x_n), f(x_{n+1}), f(x_{n+1}))) = F(G(S(x_{n-1}), T(x_n), T(x_n))) \\ & \leq \max \left\{ \begin{array}{l} F(G(S(x_{n-1}), T(x_n), T(x_n))), \\ F(G(T(x_{n-1}), S(x_n), S(x_n))) \end{array} \right\} \\ & \leq \varphi \left( \min \left\{ \begin{array}{l} F(G(f(x_{n-1}), S(x_{n-1}), S(x_{n-1}))) + F(G(f(x_n), T(x_n), T(x_n))), \\ F(G(f(x_{n-1}), T(x_{n-1}), T(x_{n-1}))) + F(G(f(x_n), S(x_n), S(x_n))) \end{array} \right\} \right) \\ & \leq \varphi(F(G(f(x_{n-1}), S(x_{n-1}), S(x_{n-1}))) + F(G(f(x_n), T(x_n), T(x_n)))) \\ & \leq k F(G(f(x_{n-1}), f(x_n), f(x_n))) + k F(G(f(x_n), f(x_{n+1}), f(x_{n+1}))), \text{ by } (\varphi_1'). \end{aligned}$$

So, it must be the case that

$$F(G(f(x_n), f(x_{n+1}), f(x_{n+1}))) \leq h F(G(f(x_{n-1}), f(x_n), f(x_n)))$$

where  $h = \frac{k}{1-k}$ .



Again, if  $n \in N$  is even, then by using (3.4)

$$\begin{aligned}
& F(G(f(x_n), f(x_{n+1}), f(x_{n+1}))) = F(G(T(x_{n-1}), S(x_n), S(x_n))) \\
& \leq \max \left\{ \begin{array}{l} F(G(S(x_{n-1}), T(x_n), T(x_n))), \\ F(G(T(x_{n-1}), S(x_n), S(x_n))) \end{array} \right\} \\
& \leq \varphi \left( \min \left\{ \begin{array}{l} F(G(f(x_{n-1}), S(x_{n-1}), S(x_{n-1}))) + F(G(f(x_n), T(x_n), T(x_n))), \\ F(G(f(x_{n-1}), T(x_{n-1}), T(x_{n-1}))) + F(G(f(x_n), S(x_n), S(x_n))) \end{array} \right\} \right) \\
& \leq \varphi(F(G(f(x_{n-1}), T(x_{n-1}), T(x_{n-1}))) + F(G(f(x_n), S(x_n), S(x_n)))) \\
& \leq k F(G(f(x_{n-1}), f(x_n), f(x_n))) + k F(G(f(x_n), f(x_{n+1}), f(x_{n+1})))
\end{aligned}$$

which implies that

$$F(G(f(x_n), f(x_{n+1}), f(x_{n+1}))) \leq h F(G(f(x_{n-1}), f(x_n), f(x_n)))$$

where  $h = \frac{k}{1-k}$ .

Thus, for any positive integer  $n$ , we have

$$F(G(f(x_n), f(x_{n+1}), f(x_{n+1}))) \leq h F(G(f(x_{n-1}), f(x_n), f(x_n))) \quad (3.6)$$

where  $0 \leq h < 1$ .

By repeated application of (3.6), we obtain

$$F(G(f(x_n), f(x_{n+1}), f(x_{n+1}))) \leq h^n F(G(f(x_0), f(x_1), f(x_1))). \quad (3.7)$$

Then, for all  $n, m \in N$ ,  $n < m$ , we have by repeated use of  $(G_5)$ ,  $(F_3)$  and (3.7) that

$$\begin{aligned}
F(G(f(x_n), f(x_m), f(x_m))) & \leq F(G(f(x_n), f(x_{n+1}), f(x_{n+1}))) \\
& \quad + F(G(f(x_{n+1}), f(x_{n+2}), f(x_{n+2}))) \\
& \quad + \cdots + F(G(f(x_{m-1}), f(x_m), f(x_m))) \\
& \leq (h^n + h^{n+1} + \cdots + h^{m-1}) F(G(f(x_0), f(x_1), f(x_1))) \\
& \leq \frac{h^n}{1-h} F(G(f(x_0), f(x_1), f(x_1))).
\end{aligned}$$

So, it must be the case that  $F(G(f(x_n), f(x_m), f(x_m))) \rightarrow \theta$  as  $m, n \rightarrow \infty$  and hence by  $(F_2)$ ,  $G(f(x_n), f(x_m), f(x_m)) \rightarrow \theta$  as  $m, n \rightarrow \infty$ . Now for given  $\theta \ll c$ , there exists a natural number  $n_0$  such that

$$G(f(x_n), f(x_m), f(x_m)) \ll \frac{c}{2} \text{ for all } m, n > n_0.$$

For  $n, m, l \in N$ ,  $(G_5)$  implies that

$$\begin{aligned}
G(f(x_n), f(x_m), f(x_l)) & \leq G(f(x_n), f(x_m), f(x_m)) + G(f(x_l), f(x_m), f(x_m)) \\
& \ll \frac{c}{2} + \frac{c}{2} = c
\end{aligned}$$

for all  $n, m, l > n_0$ .

Therefore,  $(f(x_n))$  is a Cauchy sequence in  $f(X)$ . Since  $f(X)$  is complete, there exists  $u, v \in X$  such that  $f(x_n) \rightarrow v = f(u)$ .

Now applying  $(F_2)$ , for a fixed  $\theta \ll c$ , there exists a natural number  $n_1$  such that

$$F(G(f(x_{2n}), f(x_{2n+1}), f(x_{2n+1}))) \ll \frac{c(1-k)}{2k}$$

and

$$F(G(f(u), f(x_{2n+1}), f(x_{2n+1}))) \ll \frac{c(1-k)}{2}$$

for all  $n > n_1$ .

Hence for  $n > n_1$ , we have

$$\begin{aligned} F(G(f(u), T(u), T(u))) &\leq F(G(f(u), f(x_{2n+1}), f(x_{2n+1}))) + F(G(f(x_{2n+1}), T(u), T(u))) \\ &= F(G(f(u), f(x_{2n+1}), f(x_{2n+1}))) + F(G(S(x_{2n}), T(u), T(u))) \\ &\leq F(G(f(u), f(x_{2n+1}), f(x_{2n+1}))) \\ &\quad + \max \left\{ \begin{array}{l} F(G(S(x_{2n}), T(u), T(u))), \\ F(G(T(x_{2n}), S(u), S(u))) \end{array} \right\} \\ &\leq F(G(f(u), f(x_{2n+1}), f(x_{2n+1}))) \\ &\quad + \varphi \left( \min \left\{ \begin{array}{l} F(G(f(x_{2n}), S(x_{2n}), S(x_{2n}))) + F(G(f(u), T(u), T(u))), \\ F(G(f(x_{2n}), T(x_{2n}), T(x_{2n}))) + F(G(f(u), S(u), S(u))) \end{array} \right\} \right) \\ &\leq F(G(f(u), f(x_{2n+1}), f(x_{2n+1}))) \\ &\quad + \varphi(F(G(f(x_{2n}), S(x_{2n}), S(x_{2n}))) + F(G(f(u), T(u), T(u)))) \\ &\leq F(G(f(u), f(x_{2n+1}), f(x_{2n+1}))) \\ &\quad + k F(G(f(x_{2n}), f(x_{2n+1}), f(x_{2n+1}))) + k F(G(f(u), T(u), T(u))) \end{aligned}$$

which gives that

$$\begin{aligned} F(G(f(u), T(u), T(u))) &\leq \frac{1}{1-k} F(G(f(u), f(x_{2n+1}), f(x_{2n+1}))) \\ &\quad + \frac{k}{1-k} F(G(f(x_{2n}), f(x_{2n+1}), f(x_{2n+1}))) \\ &\ll \frac{c}{2} + \frac{c}{2} = c. \end{aligned}$$

Thus,

$$F(G(f(u), T(u), T(u))) \leq \frac{c}{i}, \text{ for all } i \geq 1$$

which implies that  $F(G(f(u), T(u), T(u))) = \theta$  and therefore,  $T(u) = f(u) = v$ . Similarly, we can prove that  $f(u) = S(u) = v$ . Thus,  $f(u) = S(u) = T(u) = v$  and so  $v$  becomes a common point of coincidence of  $S$ ,  $T$  and  $f$ .

Now we show that  $S$ ,  $T$  and  $f$  have unique point of coincidence. For this we suppose that there exists another point  $w \in X$  such that  $f(x) = S(x) = T(x) = w$

for some  $x \in X$ . Then,

$$\begin{aligned} F(G(v, w, w)) &= F(G(S(u), T(x), T(x))) \\ &\leq \max \left\{ \begin{array}{l} F(G(S(u), T(x), T(x))), \\ F(G(T(u), S(x), S(x))) \end{array} \right\} \\ &\leq \varphi \left( \min \left\{ \begin{array}{l} F(G(f(u), S(u), S(u))) + F(G(f(x), T(x), T(x))), \\ F(G(f(u), T(u), T(u))) + F(G(f(x), S(x), S(x))) \end{array} \right\} \right) \\ &= \theta \end{aligned}$$

which gives that  $F(G(v, w, w)) = \theta$  and hence  $v = w$ . If  $(S, f)$  and  $(T, f)$  are weakly compatible, then by Lemma 3.1,  $S, T$  and  $f$  have a unique common fixed point in  $X$ .

If  $S, T$  and  $f$  satisfy condition (3.5), then by the same technique as given above we can obtain the desired conclusion.

**Corollary 3.8.** *Let  $X$  be a  $G$ -cone metric space and let  $T, f : X \rightarrow X$  satisfy*

$$F(G(T(x), T(y), T(y))) \leq \varphi(F(G(f(x), T(x), T(x))) + F(G(f(y), T(y), T(y))))$$

or

$$F(G(T(x), T(y), T(y))) \leq \varphi(F(G(f(x), f(x), T(x))) + F(G(f(y), f(y), T(y))))$$

for all  $x, y \in X$ . If  $T(X) \subseteq f(X)$  and if  $T(X)$  or  $f(X)$  is a complete subspace of  $X$ , then  $T$  and  $f$  have a unique point of coincidence. Moreover, if  $T$  and  $f$  are weakly compatible, then  $T$  and  $f$  have a unique common fixed point.

**Proof.** The proof can be obtained from Theorem 3.7 by taking  $S = T$ .

The following Corollary is an extension of the result [11, Theorem 3] to  $G$ -cone metric spaces.

**Corollary 3.9.** *Let  $X$  be a  $G$ -cone metric space and let  $T, f : X \rightarrow X$  satisfy*

$$G(T(x), T(y), T(y)) \leq k(G(f(x), T(x), T(x)) + G(f(y), T(y), T(y)))$$

or

$$G(T(x), T(y), T(y)) \leq k(G(f(x), f(x), T(x)) + G(f(y), f(y), T(y)))$$

for all  $x, y \in X$ , where  $0 \leq k < \frac{1}{2}$ . If  $T(X) \subseteq f(X)$  and if  $T(X)$  or  $f(X)$  is a complete subspace of  $X$ , then  $T$  and  $f$  have a unique point of coincidence. Moreover, if  $T$  and  $f$  are weakly compatible, then  $T$  and  $f$  have a unique common fixed point.

**Proof.** Putting  $S = T, F = I, \varphi(w) = kw, 0 \leq k < \frac{1}{2}$  in Theorem 3.7, one can obtain the desired result.

Now, using the same methods as in proof of Theorem 3.7 one can prove the next result.

**Theorem 3.10.** *Let  $X$  be a  $G$ -cone metric space and let the mappings  $S, T, f : X \rightarrow X$  satisfy one of the following conditions:*

$$\begin{aligned} & \max \left\{ \begin{array}{l} F(G(S(x), T(y), T(y))), \\ F(G(T(x), S(y), S(y))) \end{array} \right\} \\ & \leq \varphi \left( \min \left\{ \begin{array}{l} F(G(f(x), T(y), T(y))) + F(G(f(y), S(x), S(x))), \\ F(G(f(x), S(y), S(y))) + F(G(f(y), T(x), T(x))) \end{array} \right\} \right) \\ & \qquad \qquad \qquad \text{or} \\ & \max \left\{ \begin{array}{l} F(G(S(x), T(y), T(y))), \\ F(G(T(x), S(y), S(y))) \end{array} \right\} \\ & \leq \varphi \left( \min \left\{ \begin{array}{l} F(G(f(x), f(x), T(y))) + F(G(f(y), f(y), S(x))), \\ F(G(f(x), f(x), S(y))) + F(G(f(y), f(y), T(x))) \end{array} \right\} \right) \end{aligned}$$

for all  $x, y \in X$ . If  $S(X) \cup T(X) \subseteq f(X)$  and  $f(X)$  is a complete subspace of  $X$ , then  $S, T$  and  $f$  have a unique point of coincidence. Moreover, if  $(S, f)$  and  $(T, f)$  are weakly compatible, then  $S, T$  and  $f$  have a unique common fixed point.

**Corollary 3.11.** *Let  $X$  be a  $G$ -cone metric space and let  $T, f : X \rightarrow X$  satisfy*

$$F(G(T(x), T(y), T(y))) \leq \varphi(F(G(f(x), T(y), T(y))) + F(G(f(y), T(x), T(x))))$$

or

$$F(G(T(x), T(y), T(y))) \leq \varphi(F(G(f(x), f(x), T(y))) + F(G(f(y), f(y), T(x))))$$

for all  $x, y \in X$ . If  $T(X) \subseteq f(X)$  and if  $T(X)$  or  $f(X)$  is a complete subspace of  $X$ , then  $T$  and  $f$  have a unique point of coincidence. Moreover, if  $T$  and  $f$  are weakly compatible, then  $T$  and  $f$  have a unique common fixed point.

**Proof.** Taking  $S = T$  in Theorem 3.10 one can obtain the desired result.

The following Corollary is the result [4, Theorem 3.4]. Also, it is an extension of the result [11, Theorem 3] to  $G$ -cone metric spaces.

**Corollary 3.12.** *Let  $X$  be a complete  $G$ -cone metric space and let  $T : X \rightarrow X$  be a mapping satisfying one of the conditions:*

$$G(T(x), T(y), T(y)) \leq k(G(x, T(y), T(y)) + G(y, T(x), T(x)))$$

or

$$G(T(x), T(y), T(y)) \leq k(G(x, x, T(y)) + G(y, y, T(x)))$$

for all  $x, y \in X$ ,  $0 \leq k < \frac{1}{2}$ . Then  $T$  has a unique fixed point.

**Proof.** The proof follows from Theorem 3.10 by taking  $S = T$ ,  $F = f = I$ ,  $\varphi(w) = kw$ , where  $k \in [0, 1/2)$  is a constant.

**Note:** It is worth mentioning that for the cases  $S = T$  it is sufficient to assume that  $\leq$  is a partial ordering on  $E$  with respect to  $P$  instead of a complete ordering.

We give some examples in support our results.

**Example 3.13.** Let  $E = R$  and  $P = \{x \in R : x \geq 0\}$  be a cone in  $E$ . Let  $X = [1, \infty)$  and define  $G : X \times X \times X \rightarrow E$  by

$$G(x, y, z) = |x - y| + |y - z| + |z - x|, \text{ for all } x, y, z \in X.$$

Then  $X$  is a complete  $G$ -cone metric space. Define  $T, S, f : X \rightarrow X$  by  $T(x) = S(x) = 3x - 2$ ,  $f(x) = 4x - 3$ , for all  $x \in X$ . Also, define  $\varphi, F : P \rightarrow P$  by  $\varphi(w) = \frac{3}{4}w$  and  $F(w) = \frac{1}{2}w$ , for all  $w \in P$ .

Now,

$$\begin{aligned} \max \left\{ \begin{array}{l} F(G(S(x), T(y), T(y))), \\ F(G(T(x), S(y), S(y))), \\ F(G(T(x), T(y), T(y))), \\ F(G(S(x), S(y), S(y))) \end{array} \right\} &= F(G(T(x), T(y), T(y))) \\ &= F(2 |T(x) - T(y)|) \\ &= |T(x) - T(y)| \\ &= 3 |x - y| \\ &= \frac{3}{4} |4x - 4y| \\ &= \frac{3}{4} |f(x) - f(y)| \\ &= \frac{3}{8} G(f(x), f(y), f(y)) \\ &= \frac{3}{4} F(G(f(x), f(y), f(y))) \\ &= \varphi(F(G(f(x), f(y), f(y))), \text{ for all } x, y \in X. \end{aligned}$$

Thus the condition (3.1) of Theorem 3.2 is satisfied. Furthermore, we have

(i)  $S(X) \cup T(X) \subseteq f(X)$  and  $f(X)$  is complete, since  $f(X) = X$ ,

(ii)  $(S, f)$  and  $(T, f)$  are weakly compatible.

Hence we have all the conditions of Theorem 3.2 and we see that 1 is the unique common fixed point for  $S, T$  and  $f$  in  $X$ .

**Example 3.14.** Let  $E, P, X, G$  are same as in Example 3.13. Define  $T, S, f : X \rightarrow X$  by  $T(x) = S(x) = \frac{x+1}{2}$ ,  $f(x) = 2x - 1$ , for all  $x \in X$ . Also, define  $\varphi, F : P \rightarrow P$  by  $\varphi(w) = \frac{1}{3}w$  and  $F(w) = \frac{1}{4}w$ , for all  $w \in P$ .

$$\begin{aligned}
& \text{Then,} \\
& \varphi \left( \min \left\{ \begin{array}{l} F(G(f(x), S(x), S(x))) + F(G(f(y), T(y), T(y))), \\ F(G(f(x), T(x), T(x))) + F(G(f(y), S(y), S(y))) \end{array} \right\} \right) \\
& \quad = \varphi (F(G(f(x), T(x), T(x))) + F(G(f(y), T(y), T(y)))) \\
& \quad = \varphi (F(2 | T(x) - f(x) |) + F(2 | T(y) - f(y) |)) \\
& \quad = \varphi (F(3 | x - 1 |) + F(3 | y - 1 |)) \\
& \quad = \varphi \left( \frac{3}{4} | x - 1 | + \frac{3}{4} | y - 1 | \right) \\
& \quad = \frac{1}{4} (| x - 1 | + | y - 1 |).
\end{aligned}$$

Again,

$$\begin{aligned}
& \max \left\{ \begin{array}{l} F(G(S(x), T(y), T(y))), \\ F(G(T(x), S(y), S(y))) \end{array} \right\} = F(G(T(x), T(y), T(y))) \\
& \quad = F(2 | T(x) - T(y) |) \\
& \quad = F(| x - y |) \\
& \quad = \frac{1}{4} | x - y | \\
& \quad \leq \frac{1}{4} (| x - 1 | + | y - 1 |) \\
& = \varphi \left( \min \left\{ \begin{array}{l} F(G(f(x), S(x), S(x))) + F(G(f(y), T(y), T(y))), \\ F(G(f(x), T(x), T(x))) + F(G(f(y), S(y), S(y))) \end{array} \right\} \right).
\end{aligned}$$

Thus we see that the condition (3.4) of Theorem 3.7 is satisfied. Also, we have

(i)  $S(X) \cup T(X) \subseteq f(X)$  and  $f(X)$  is complete, since  $f(X) = X$ ,

(ii)  $(S, f)$  and  $(T, f)$  are weakly compatible.

Hence all the conditions of Theorem 3.7 are satisfied and 1 is the unique common fixed point for  $S, T$  and  $f$  in  $X$ .

**Example 3.15.** Let  $E, P, X, G$  are same as in Example 3.13. Define  $T, S, f : X \rightarrow X$  by  $T(x) = S(x) = \frac{x}{2} + 1$ ,  $f(x) = x$ , for all  $x \in X$ . Also, define  $\varphi, F : P \rightarrow P$  by  $\varphi(w) = F(w) = \frac{1}{3}w$ , for all  $w \in P$ . Then we have all the conditions of Theorem 3.10 and we see that 2 is the unique common fixed point for  $S, T$  and  $f$  in  $X$ .

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