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HARMONIC MAPS ON GENERALIZED WARPED PRODUCT MANIFOLDS

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Abstract. In this paper, we present some new properties for harmonic maps between generalized warped product manifolds .

1. Introduction

Consider a smooth map $\phi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds. The energy functional of ϕ is defined by

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 dv_g . E(\phi) = \int_M e(\phi) dv_g. \tag{1.1}$$

(or over any compact subset $K \subset M$), where $e(\phi) = \frac{1}{2}|d\phi|^2$, denote the energy density of ϕ.

A map is called harmonic if it is a critical point of the energy functional $E(\phi)$ (or $E(K)$ for all compact subsets $K \subset M$), the Euler-Lagrange equation associated to (1.1) is

$$\tau(\phi) = Tr_g \nabla d\phi, \tag{1.2}$$

$\tau(\phi)$ is called the tension field of ϕ. For any smooth variation $\{\phi_t\}_{t\in I}$ of ϕ with $\phi_0 = \phi$ and $V = \left.\frac{d\phi_t}{dt}\right|_{t=0}$, we have

$$\left.\frac{d}{dt} E(\phi_t)\right|_{t=0} = -\int_M h(\tau(\phi), V)\, dv_g. \tag{1.3}$$

Then, we have

Theorem 1.1. *A smooth map $\phi : (M^m, g) \to (N^n, h)$ is harmonic if and only if*

$$\tau(\phi) = Tr_g \nabla d\phi = 0. \tag{1.4}$$

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If $(x^i)_{1\leq i\leq m}$ and $(y^\alpha)_{1\leq\alpha\leq n}$ denote local coordinates on M and N respectively, then equation (1.4) takes the form

$$\tau(\phi)^\alpha=\left(\Delta\phi^\alpha+g^{ij}\,{}^N\Gamma^\alpha_{\beta\gamma}\frac{\partial\phi^\beta}{\partial x^i}\frac{\partial\phi^\gamma}{\partial x^j}+g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial\phi^\alpha}{\partial x^j}\right)=0,\quad 1\leq\alpha\leq n, \tag{1.5}$$

where $\Delta\phi^\alpha=\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ij}\frac{\partial\phi^\alpha}{\partial x^j}\right)$ is the Laplace operator on (M^m,g) and ${}^N\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols on N. One can refer to [1], [3], [4], [5] and [8] for background on harmonic maps.

2. Some results on generalized warped product manifolds

In this section, we give the definition and some geometric properties of generalized warped product manifolds.

Definition 2.1. *Let (M^m,g) and (N^n,h) be two riemannian manifolds, and $f:M\times N\to\mathbb{R}$ be a smooth positive function. The generalized warped metric on $M\times_f N$ is defined by*

$$G_f=\pi^*g+(f)^2\eta^*h \tag{2.1}$$

where $\pi:(x,y)\in M\times N\to x\in M$ and $\eta:(x,y)\in M\times N\to y\in N$ are the canonical projections. For all $X,Y\in T(M\times N)$, we have

$$G_f(X,Y)=g(d\pi(X),d\pi(Y)+(f)^2h(d\eta(X),d\eta(Y))$$

and we denote by $X\wedge_{G_{f^2}}Y$, the linear map :

$$Z\in\mathcal{H}(M)\times\mathcal{H}(N)\to\Big(X\wedge_{G_{f^2}}Y\Big)Z=G_{f^2}\Big(Z,Y\Big)X-G_{f^2}\Big(Z,X\Big)Y \tag{2.2}$$

Proposition 2.2. *Let (M^m,g) and (N^n,h) be two Riemannian manifolds. If $\overline{\nabla}$ denote the Levi-Civita connection on $(M\times_f N,G_f)$, then for all $X_1,Y_1\in\mathcal{H}(M)$ and $X_2,Y_2\in\mathcal{H}(N)$ whe have :*

$$\begin{aligned}\overline{\nabla}_XY &= \nabla_XY+X(\ln f)(0,Y_2)+Y(\ln f)(0,X_2)\\ &\quad -\frac{1}{2}h(X_2,Y_2)(grad_Mf^2,\frac{1}{f^2}grad_Nf^2)\end{aligned} \tag{2.3}$$

where $X=(X_1,X_2)$, $Y=(Y_1,Y_2)$ and $\nabla_XY=(\nabla^M_{X_1}Y^1,\nabla^N_{X_2}Y^2)$

In the general case, the geometry of product manifolds is considered in [6].

Proof follows from the Kozul formula and the following formulas:

$$\begin{aligned}X(f^2).h(Y_2,Z_2) &= 2X(\ln f)G_f((0,Y_2),Z)\\ Y(f^2).h(X_2,Z_2) &= 2Y(\ln f)G_f((0,X_2),Z)\\ Z(f^2).h(X_2,Y_2) &= h(X_2,Y_2)G_f((grad_Mf^2,\frac{1}{f^2}grad_Nf^2),Z)\\ G_f(\nabla_XY,Z) &= g(\nabla^M_{X_1}Y_1,Z_1)\circ\pi+f^2.h(\nabla^N_{X_2}Y_2,Z_2)\circ\eta\end{aligned}$$

where $Z=(Z_1,Z_2)\in\mathcal{H}(M)\times\mathcal{H}(N)$

Remarks. *:*

(1) *If* $f:(x,y)\in M\times N\mapsto f(x,y)=f(x)$ *, then*

$$\begin{aligned}\overline{\nabla}_X Y &= \nabla_X Y + X_1(\ln f)\Big(0,Y_2\Big)+Y_1(\ln f)\Big(0,X_2\Big)\\ &\quad -\frac{1}{2}h(X_2,Y_2)\Big(grad_M(f^2),0\Big)\end{aligned}$$

is the Levi-Civita connection of warped product manifolds.

(2) *If* $f:(x,y)\in M\times N\mapsto f(x,y)=f(y)$ *, then*

$$\begin{aligned}\overline{\nabla}_X Y &= \nabla_X Y + X_2(\ln f)\Big(0,Y_2\Big)+Y_2(\ln f)\Big(0,X_2\Big)\\ &\quad -\frac{1}{2}h(X_2,Y_2)\Big(0,\frac{1}{f^2}grad_N(f^2)\Big)\\ &= \Big(\nabla^M_{X_1}Y_1,\widehat{\nabla}_{X_2}Y_2\Big)\end{aligned}$$

is the Levi-Civita connection of product Riemannian manifolds (M,g) *and* (N,f^2h)*, where*

$$\widehat{\nabla}_{X_2}Y_2=\nabla^N_{X_2}Y_2+X_2(\ln f)Y_2+Y_2(\ln f)X_2-h(X_2,Y_2)grad_N\ln f$$

From Proposition 2.2, we obtain

Corollary 2.3. *For all* $X_1,Y_1\in\mathcal{H}(M)$ *and* $X_2,Y_2\in\mathcal{H}(N)$*, we have:*

$$\begin{aligned}\overline{\nabla}_{(X_1,0)}(Y_1,0) &= (\nabla^M_{X_1}Y_1,0).\\ \overline{\nabla}_{(X_1,0)}(0,Y_2) &= X_1(\ln f)(0,Y_2).\\ \overline{\nabla}_{(0,X_2)}(Y_1,0) &= Y_1(\ln f)(0,X_2).\\ \overline{\nabla}_{(0,X_2)}(0,Y_2) &= (0,\nabla^N_{X_2}Y_2)+X_2(\ln f)(0,Y_2)+Y_2(\ln f)(0,X_2)\\ &\quad -\frac{1}{2}h(X_2,Y_2)(grad_M f^2,\frac{1}{f^2}grad_N f^2).\end{aligned}$$

From Proposition 2.2, Corollary 2.3 and formula of curvature tensor, we obtain

Proposition 2.4. *Let* (M^m,g) *and* (N^n,h) *be two Riemannian manifolds and* $f:M\times N\to\mathbb{R}$ *be smooth positive function. If* R *and* $\overline{R}$ *denote the curvatures tensors of product manifold* $(M\times N,G)$ *and generalized warped product manifold* $(M\times_f N,G_f)$ *respectively, then*

$$\begin{aligned}\overline{R}(X,Y)Z - R(X,Y)Z &= (\nabla^M_{Y_1} grad_M \ln f + Y_1(\ln f) grad_M \ln f, 0) \wedge_{G_f} (0, X_2)\\ &- (\nabla^M_{X_1} grad_M \ln f + X_1(\ln f) grad_M \ln f, 0) \wedge_{G_f} (0, Y_2)\\ &+ \frac{1}{f^2}\Big[(0, \nabla^N_{Y_2} grad_N \ln f - Y_2(\ln f) grad_N \ln f) \wedge_{G_f} (0, X_2)\\ &- (0, \nabla^N_{X_2} grad_N \ln f - X_2(\ln f) grad_N \ln f) \wedge_{G_f} (0, Y_2)\\ &- (f^2 \mid grad_M \ln f \mid^2 + \mid grad_N \ln f \mid^2)(0, X_2) \wedge_{G_f} (0, Y_2)\Big]\\ &+ \Big[X_1(Z_2(\ln f)) + X_2(Z_1(\ln f))\Big](0, Y_2)\\ &- \Big[Y_1(Z_2(\ln f)) + Y_2(Z_1(\ln f))\Big](0, X_2) \qquad (2.4)\end{aligned}$$

for all $X, Y, Z \in \mathcal{H}(M) \times \mathcal{H}(N)$, where $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$ and $Z = (Z_1, Z_2)$.

Proposition 2.5. *Let (M^m, g) and (N^n, h) be two Riemannian manifolds and $f : M \times N \to \mathbb{R}$ be smooth positive function. The Ricci curvature from generalized warped product manifolds $(M \times_f N, G_f)$ is given by the following formulas:*

$$\begin{aligned}Ric((X_1,0),(Y_1,0)) &= Ric^M(X_1,Y_1) - ng(\nabla^M_{X_1} grad_M \ln f + X_1(\ln f) grad_M \ln f, Y_1)\\ Ric((X_1,0),(0,Y_2)) &= -nX_1(Y_2(\ln f))\\ Ric((0,X_2),(Y_1,0)) &= h(X_2, grad_N(Y_1(\ln f))) - nX_2(Y_1(\ln f))\\ Ric((0,X_2),(0,Y_2)) &= Ric^N(X_2,Y_2) + (2-n)h(\nabla^N_{X_2} grad_N \ln f, Y_2)\\ &+ (2-n)\big[h(X_2,Y_2) \mid grad_N \ln f \mid^2 - X_2(\ln f) h(grad_N \ln f, Y_2)\big]\\ &+ h(X_2,Y_2)\big[nf^2 \mid grad_M \ln f \mid^2 - \Delta_N(\ln f) - f^2 \Delta_M(\ln f)\big]\end{aligned}$$

for all $X_1, Y_1 \in \mathcal{H}(M)$ and $X_2, Y_2 \in \mathcal{H}(N)$.

The proof follows from Proposition 2.4 and the Ricci formulas.

3. Harmonic maps on generalized warped product manifolds

Let $(M^m, g), (N^n, h)$ and (P^p, ℓ) be Riemannian manifolds of dimensions m, n and p respectively, $f : M \times N \to \mathbb{R}$ be smooth positive function, and $(M \times_f N, G_f)$ be the generalized warped product manifold.

3.1. Harmonicity conditions of $\phi : (P, \ell) \longrightarrow (M \times_f N, G_f)$. .

Proposition 3.1. *If $\varphi : P \to M$ and $\psi : P \to N$ are regular maps, then the tension field of*

$$\begin{aligned}\phi : (P^p, \ell) &\longrightarrow (M \times_f N, G_f)\\ x &\longmapsto (\varphi(x), \psi(x))\end{aligned}$$

is given by the following relation:

$$\begin{aligned} \tau(\phi) &= \Big(\tau(\varphi),\tau(\psi)\Big) + 2\Big(0, d\psi(grad_P(\ln f\circ\phi))\Big) \\ &\quad -e(\psi)\Big(grad_M f^2, \frac{1}{f^2} grad_N f^2\Big) \end{aligned} \tag{3.1}$$

Proof. Choose a local orthonormal frame $(e_i)_i$ with respect to ℓ on M. Then by definition of tension field, we have

$$\begin{aligned} \tau(\phi) &= tr_\ell \nabla d\phi \\ &= \nabla_{e_i} d\phi(e_i) - d\phi(\nabla^P_{e_i} e_i) \\ &= \overline{\nabla}_{(d\varphi(e_i), d\psi(e_i))}(d\varphi(e_i), d\psi(e_i)) - \Big(d\varphi(\nabla^P_{e_i} e_i), d\psi(\nabla^P_{e_i} e_i)\Big) \\ &= \nabla_{(d\varphi(e_i), d\psi(e_i))}(d\varphi(e_i), d\psi(e_i)) + 2(d\varphi(e_i), d\psi(e_i))(\ln f)(0, d\psi(e_i)) \\ &\quad -e(\psi)\Big(grad_M f^2\circ\varphi, \frac{1}{f^2} grad_N f^2\circ\psi\Big) - \Big(d\varphi(\nabla^P_{e_i} e_i), d\psi(\nabla^P_{e_i} e_i)\Big) \\ &= \Big(\tau(\varphi),\tau(\psi)\Big) + 2\Big(0, d\psi(grad_P(\ln f\circ\phi))\Big) \\ &\quad -e(\psi)\Big((grad_M f^2)\circ\phi, (\frac{1}{f^2} grad_N f^2)\circ\phi\Big) \end{aligned}$$

□

From Proposition 3.1, we have

Remarks. *:*

- *If f is a constant function, then the tension field of ϕ is given by*

$$\tau(\phi) = \Big(\tau(\varphi),\tau(\psi)\Big)$$

and ϕ is harmonic map if and only if φ et ψ are harmonic maps.

- *If $P = M$ and $\psi = y_0$ is constant, then the tension field of $\phi : x \in M \longmapsto (\varphi(x), y_0) \in M\times N$ is given by*

$$\tau(\phi) = (\tau(\varphi), 0)$$

- *If $P = N$ and $\varphi = x_0$ is constant then the tension field of $\phi : y \in N \longmapsto (x_0, \psi(y)) \in M\times N$ is given by*

$$\tau(\phi) = (0,\tau(\psi)) + 2\Big(0, d\psi(grad_M(\ln f\circ\phi))\Big) - e(\psi)\Big(grad_M f^2, \frac{1}{f^2} grad_N f^2\Big)$$

- *If $P = N$ and $\psi = Id_N$, then $e(\psi) = \frac{n}{2}$ and then the tension field of $\phi : y \in N \longmapsto (\varphi(y), y) \in M\times N$ is given by*

$$\tau(\phi) = \Big(\tau(\varphi) - \frac{n}{2} grad_M f^2, (2-n) grad_N f^2\Big)$$

From definition of conformal map and Proposion 3.1, we deduce

Proposition 3.2. *Let $\varphi : M \to M$ be conformal map with dilatation λ, then the tension field of*

$$\begin{array}{rcl} \phi : (M,g) & \longrightarrow & (M \times_f M, G_f) \\ x & \longmapsto & (\varphi(x), \varphi(x)) \end{array}$$

is given by

$$\begin{aligned} \tau(\phi) &= (2-m)\Big(d\varphi(grad \ln \lambda), d\varphi(grad \ln \lambda)\Big) + 2\Big(0, d\varphi(grad(\ln f \circ \varphi))\Big) \\ &\quad -\frac{m}{2}\lambda^2\Big(gradf^2, \frac{1}{f^2}gradf^2\Big) \circ \varphi \end{aligned}$$

For more details on conformal maps, we can refer to [2], [7]

3.2. **Harmonicity conditions of $\phi : (M \times_f N, G_f) \longrightarrow (P, \ell)$.** .

Let $\phi : (x,y) \in (M \times_f N, G_f) \longrightarrow \phi(x,y) \in (P,k)$ be smooth map. If we denote by

$$\begin{array}{rcl} \phi_N = \phi_N^x : (N,h) & \longrightarrow & (P,k) \\ y & \longmapsto & \phi_N^x(y) = \phi(x,y) \end{array}$$

and

$$\begin{array}{rcl} \phi_M = \phi_M^y : (M,g) & \longrightarrow & (P,k) \\ x & \longmapsto & \phi_M^y(x) = \phi(x,y) \end{array}$$

then for all $X \in \mathcal{H}(M)$, $Y \in \mathcal{H}(N)$ and $(x,y) \in M \times N$, we have:

$$\begin{cases} d_{(x,y)}\phi(X,0) = d_x\phi_M^y(X) = d_x\phi_M(X) \\ d_{(x,y)}\phi(0,Y) = d_y\phi_N^x(Y) = d_y\phi_N(Y) \end{cases}$$

Proposition 3.3. *The tension field of $\phi : (M \times_f N, G_f) \longrightarrow (P,k)$ is given by:*

$$\begin{aligned} \tau(\phi) &= \tau(\phi_M) + nd\phi_M(grad_M \ln f) \\ &\quad + \frac{1}{f^2}\Big\{\tau(\phi_N) + (n-2)d\phi_N(grad_N \ln f)\Big\}. \end{aligned} \tag{3.2}$$

Proof. Any local orthonormal frame $\{E_i, i = \overline{1,m}\}$ and $\{F_j, j = \overline{1,n}\}$ on (M^m, g) and (N^n, h) respectively, induces a local orthonormal frame on $(M \times_f N, G_f)$ by:

$$\{U_i = (E_i, 0), \quad U_{m+j} = (0, \frac{1}{f}F_j) : \quad i = \overline{1,m}, \quad j = \overline{1,n}\}$$

Using formula of tension field, we have :

$$\begin{aligned}
\tau(\phi) &= tr_{G_f}\nabla^P d\phi \\
&= \sum_{k=1}^{m+n}\nabla^P d\phi(U_k,U_k) \\
&= \sum_{i=1}^{m}\Big\{\nabla^P_{d\phi(E_i,0)}d\phi(E_i,0)-d\phi(\overline{\nabla}_{(E_i,0)}(E_i,0))\Big\} \\
&\quad +\sum_{j=1}^{n}\Big\{\frac{1}{f}\nabla^P_{d\phi(0,F_j)}\frac{1}{f}d\phi(0,F_j)-d\phi(\frac{1}{f}\overline{\nabla}_{(0,F_j)}\frac{1}{f}(0,F_j))\Big\} \\
&= \sum_{i=1}^{m}\Big\{\nabla^P_{d\phi_M(E_i)}d\phi_M(E_i)-d\phi_M(\nabla^M_{E_i}E_i)\Big\} \\
&\quad +\sum_{j=1}^{n}\frac{1}{f}\Big[-\frac{1}{f^2}F_j(f)d\phi_N(F_j)+\frac{1}{f}\nabla_{d\phi_N(F_j)}d\phi_N(F_j)\Big] \\
&\quad -d\phi\Big[-\frac{1}{f^2}(0,grad_N\ln f)+\sum_{j=1}^{n}\frac{1}{f^2}(0,\nabla^N_{F_i}F_i)\Big] \\
&\quad -d\phi\Big[\frac{2}{f^2}(0,grad_N\ln f)-\frac{n}{2f^2}(grad_M f^2,\frac{1}{f^2}grad_N f^2)\Big] \\
&= \tau(\phi_M)-\frac{1}{f^2}d\phi_N(grad_N\ln f)+\sum_{j=1}^{n}\frac{1}{f^2}\nabla_{d\phi_N(F_j)}d\phi_N(F_j) \\
&\quad -\sum_{j=1}^{n}\frac{1}{f^2}d\phi_N(\nabla^N_{F_i}F_i)+nd\phi_M(grad_M\ln f)-\frac{1-n}{f^2}d\phi_N(grad_N\ln f)
\end{aligned}$$

hence

$$\tau(\phi)=\tau(\phi_M)+nd\phi_M(grad_M\ln f)+\frac{1}{f^2}\Big\{\tau(\phi_N)+(n-2)d\phi_N(grad_N\ln f)\Big\}.$$

□

Particular cases :

- If $f\in C^\infty(N)$ (i.e: $f(x,y)=f(y)$), then
$$\tau(\phi)=\tau(\phi_M)+nd\phi_M(grad_M\ln f)+\frac{1}{f^2}\tau(\phi_N).$$
- If $f\in C^\infty(M)$ (i.e: $f(x,y)=f(x)$), then
$$\tau(\phi)=\tau(\phi_M)+\frac{1}{f^2}\Big(\tau(\phi_N)+(n-2)d\phi_N(grad_N\ln f)\Big).$$
- Let $\phi=\pi:(x,y)\in M\times_f N\to x\in M$, then $\tau(\pi)=n.grad_M\ln f$ and π is harmonic map if and only if f is constant on M,(i.e: $f(x,y)=f(y)$) .
- Let $\phi=\eta:(x,y)\in M\times_f N\to y\in N$, then $\tau(\eta)=\frac{n-2}{f^2}grad_N\ln f$ and η is harmonic map if and only if f is constant on N, (i.e: $f(x,y)=f(x)$), or $dimN=2$.

- Let $\varphi : (M, g) \longrightarrow (P, k)$ be a smooth map and $\phi(x, y) = \varphi(x)$, then

$$\tau(\phi) = \tau(\varphi) + n.d\varphi(grad_M \ln f)$$

therefore, if φ is a conformal map with dilatation λ, then

$$\tau(\phi) = (2 - m)d\varphi(grad_M \ln \lambda) + n.d\varphi(grad_M \ln f)$$

and ϕ is a harmonic map if and only if $f = C(y).\lambda^{\frac{m-2}{n}}$.

- Let $\psi : (N, h) \longrightarrow (P, k)$ be a smooth map and $\phi(x, y) = \psi(y)$, then

$$\tau(\phi) = \frac{1}{f^2}\Big(\tau(\psi) + (n-2).d\psi(grad_N \ln f)\Big)$$

therefore, if ψ is a conformal map with dilatation λ, then

$$\tau(\phi) = \frac{(n-2)}{f^2}\Big(d\psi(grad_N \ln f) - d\psi(grad_N \ln \lambda)\Big)$$

and ϕ is a harmonic map if and only if $f = C(x).\lambda$ or $dimN = 2$.

- Let $\varphi : (M, g) \longrightarrow \mathbb{R}$ and $\psi : (N, h) \longrightarrow \mathbb{R}$ are a smooth functions, if $\phi(x, y) = \varphi(x)\psi(y)$, then

$$\tau(\phi) = \psi\Big\{\tau(\varphi) + nd\varphi(grad_M \ln f)\Big\} + \frac{\varphi}{f^2}\Big\{\tau(\psi) + (n-2)d\psi(grad_N \ln f)\Big\}$$

3.3. **Harmonicity conditions of $\phi : (M \times_f N, G_f) \longrightarrow (P \times_\alpha B, G_\alpha)$.** .

Let (M^m, g), (N^n, h), (P^p, ℓ) and (Q^q, ρ) be Riemannian manifolds of dimensions m, n, p and q respectively, $f : M \times N \to \mathbb{R}$ (resp $\alpha : M \times N \to \mathbb{R}$) be smooth positive functions, and $(M \times_f N, G_f)$ (resp $(P \times_\alpha Q, G_\alpha)$) be the generalized warped product manifolds of (M^m, g) and (N^n, h) (respectively (P^p, ℓ) and (Q^q, ρ)).

Proposition 3.4. *Let $\varphi : (M, g) \longrightarrow (P, \ell)$ and $\psi : (N, h) \longrightarrow (Q, \rho)$ be a smooth maps. The tension field of*

$$\begin{array}{rcl} \phi : (M \times_f N, G_f) & \longrightarrow & (P \times_\alpha Q, G_\alpha) \\ (x, y) & \longmapsto & (\varphi(x), \psi(y)) \end{array}$$

is given by:

$$\begin{array}{rcl} \tau(\phi) & = & (\tau(\varphi), 0) + n(d\varphi(grad_M \ln f), 0) \\ & & +\dfrac{1}{f^2}\Big[(0, \tau(\psi)) + (n-2)(0, d\psi(grad_N \ln f)) \\ & & +2(0, d\psi(grad_N(\ln \alpha \circ \psi))) - e(\psi)(grad_P \alpha^2, \dfrac{1}{\alpha^2} grad_Q \alpha^2)\Big] \end{array}$$

Proof. Let $(E_i)_i$ (resp$(F_j)_j$) be an orthonormal frame on (M, g) (resp(N, h)). If $\overline{\nabla}$(resp$\widetilde{\nabla}$) denote the Levi-Civita connection on the generalized warped product manifolds $(M\times_f N, G_f)$ and $(P\times_\alpha Q, G_\alpha)$ respectively, then we have

$$\begin{aligned}
\tau(\phi) &= tr_{G_f}\widetilde{\nabla}d\phi\\
&= \big(\nabla^P_{d\varphi(E_i)}d\varphi(E_i),0\big)-\big(d\varphi(\nabla^M_{E_i}E_i),0\big)\\
&\quad+\frac{1}{f}\widetilde{\nabla}_{(0,d\psi(F_i)}\frac{1}{f}(0,d\psi(F_i)-d\phi(\frac{1}{f}\overline{\nabla}_{(0,F_i)}\frac{1}{f}(0,F_i))\\
&= (\tau(\varphi),0)+\frac{1}{f^2}(0,\tau(\psi))+\frac{(n-2)}{f^2}(0,d\psi(grad_N\ln f))\\
&\quad-\frac{e(\psi)}{f^2}(grad_P\alpha^2,\frac{1}{\alpha^2}grad_B\alpha^2)+\frac{2}{f^2}(0,d\psi(grad_N(\ln\alpha\circ\psi)))\big]\\
&\quad+n(d\varphi(grad_M\ln f),0)
\end{aligned}$$

□

From Proposition 3.4, follows:

Proposition 3.5. *The tension field of*

$$\begin{aligned}
\phi = Id_{M\times_f N} : (M\times_f N, G_f) &\longrightarrow (M\times_\alpha N, G_\alpha)\\
(x,y) &\longmapsto (x,y)
\end{aligned}$$

is given by

$$\tau(\phi)=n\Big(grad_M\ln f-\frac{1}{2f^2}grag_M\alpha^2,0\Big)+\frac{n-2}{f^2}\Big(0,grad_N\ln(\frac{f}{\alpha})\Big).$$

So, $Id_{M\times_f N}$ *is harmonic if and only if* $\begin{cases}\big(l_1^2(x)-1\big)\alpha^2=l_2(y)\\ \wedge\\ f=l_1(x).\alpha\end{cases}$

where $l_1\in C^\infty(M)$ *and* $l_2\in C^\infty(N)$.

Proposition 3.6. *If* $\varphi: M\to M$ *and* $\psi: N\to N$ *are harmonic maps, then the tension fields of*

$$\begin{aligned}
\phi_1 : (M\times_f N, G_f) &\longrightarrow (M\times N, G)\\
(x,y) &\longmapsto (\varphi(x),\psi(y))
\end{aligned}$$

and

$$\begin{aligned}
\phi_2 : (M\times N, G) &\longrightarrow (M\times_\alpha N, G_\alpha)\\
(x,y) &\longmapsto (\varphi(x),\psi(y))
\end{aligned}$$

are given by the following formulas:

$$\tau(\phi_1)=n(d\varphi(grad_M\ln f),0)+\frac{(n-2)}{f^2}(0,d\psi(grad_N\ln f))$$

$$\tau(\phi_2)=2(0,d\psi(grad_N(\ln\alpha\circ\psi)))-e(\psi)(grad_M\alpha^2,\frac{1}{\alpha^2}grad_N\alpha^2)$$

Proposition 3.7. *Let* $\varphi: M\to P$ *and* $\psi: N\to Q$ *are conformal maps with dilation* λ *and* μ *respectively. The des tension field of*

$$\begin{aligned}
\phi : (M\times_f N, G_f) &\longrightarrow P\times Q\\
(x,y) &\longmapsto (\varphi(x),\psi(y))
\end{aligned}$$

is given by

$$\begin{aligned}\tau(\phi) &= ((2-m)d\varphi(grad_M \ln\lambda) + nd(\varphi(grad_M \ln f)), 0) \\ &+ \frac{2-n}{f^2}\Big[(0, d\psi(grad_N ln\frac{\mu}{f}))\Big]\end{aligned}$$

and ϕ is harmonic map if and only $\begin{cases} f^n\lambda^{2-m}(x) = \alpha(y) \\ \wedge \\ \mu(y) = f\beta(x) \end{cases}$

therefore, ϕ is harmonic map if and only if the function f takes the form

$$f : (x, y) \in M \times N \to f(x, y) = l_1(x)l_2(y),$$

where $l_1 \in C^\infty(M)$ and $l_2 \in C^\infty(N)$.

Example

$$\begin{aligned}\phi : (\mathbb{R}^n - \{0\} \times_f \mathbb{R}^n - \{0\}, G_f) &\longrightarrow (\mathbb{R}^n - \{0\} \times \mathbb{R}^n - \{0\}, G) \\ (x, y) &\longmapsto (\frac{x}{|x|^2}, \frac{y}{|y|^2})\end{aligned}$$

then ϕ is harmonic if and only if

$$f(x, y) = |x|^{\frac{2(2-m)}{n}}|y|^{-2}$$

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