

## MODIFIED MULTI-STEP ITERATIVE METHODS FOR A FAMILY OF $p$ -STRONGLY PSEUDOCONTRACTIVE MAPS

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ABSTRACT. In this paper, we prove the convergence of modified multi-step iteration to the common fixed points of a family of  $p$ -strongly pseudo-contractive maps. Consequently, we generalize the recent results of Xue and Fan [Zhiqun Xue and Ruiqin Fan, Some comments on Noor's iterations in Banach spaces, Appl.Math.Comput. 206(2008), 12-15] which in turn are corrections of the results of Rafiq [Arif Rafiq, Modified Noor iterations for non-linear equations in Banach spaces, Appl.Math.Comput.182(2006), 589-595]. Consequently, we extend and refine previously known results in this area.

### 1. INTRODUCTION

Throughout this paper,  $X$  denotes a real Banach space and  $X^*$ , the dual of  $X$ ; and  $I$  denotes the identity operator on  $X$ .

The map  $J : X \rightarrow 2^{X^*}$  defined by

$$Jx = \{f \in X^* : \langle x, f \rangle = \|x\|^2, \|f\|^2 = \|x\|\}, \forall x \in X,$$

is called the normalized duality mapping. Let  $y \in Y$  and  $j(y) \in J(y)$  be the single valued normalized duality mapping.

Let  $K$  be a nonempty subset of  $X$ . A map  $T : K \rightarrow K$  is strongly pseudocontractive if there exists  $k \in (0, 1)$  and  $j(x - y) \in J(x - y)$  such that

$$\langle Tx - Ty, j(x - y) \rangle \leq k\|x - y\|^2, \forall x, y \in K$$

A map  $S : K \rightarrow K$  is called strongly accretive if there exists  $k \in (0, 1)$  and  $j(x - y) \in J(x - y)$  such that

$$\langle Sx - Sy, j(x - y) \rangle \geq k\|x - y\|^2, \forall x, y \in K.$$

The importance of accretive operators stems from its applications in solving evolution equations like heat, wave or schrodinger equations. It is still of research interest in constructing approximation of solutions to the equations  $Sx = 0$ , (see [11]). Clearly, the pseudocontractive maps are characterized by the fact that the

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mapping  $T$  is pseudo-contractive if and only if  $(I - T)$  is strongly accretive. Let  $K$  be a nonempty closed convex subset of  $X$  and  $T : K \rightarrow K$  be a mapping. For an arbitrary  $x_0 \in K$ . The sequence  $\{x_n\}_{n=0}^\infty \subset K$  defined by

$$\begin{aligned} x_{n+1} &= (1 - a_n)x_n + a_n T y_n^1 \\ y_n^1 &= (1 - b_n)x_n + b_n T y_n^2 \\ y_n^2 &= (1 - c_n)x_n + c_n T x_n, n \geq 0. \end{aligned} \quad (1.1)$$

where  $\{a_n\}_{n=0}^\infty, \{b_n\}_{n=0}^\infty$  and  $\{c_n\}_{n=0}^\infty$  are three real sequences satisfying  $a_n, b_n, c_n \in [0, 1]$  is called three-step or Noor iteration iterations [8]. If  $c_n \equiv 0$ , (1) reduces to Ishikawa iterative sequence, i.e.  $\{x_n\}_{n=0}^\infty \subset K$  (see[6]) defined by

$$\begin{aligned} x_{n+1} &= (1 - a_n)x_n + a_n T y_n^1 \\ y_n^1 &= (1 - b_n)x_n + b_n T x_n, n \geq 0. \end{aligned} \quad (1.2)$$

If  $b_n = c_n = 0$ , then (1) becomes Mann iteration (see[9]). It is the sequence  $\{x_n\}_{n=0}^\infty \subset K$  defined by

$$x_{n+1} = (1 - a_n)x_n + a_n T x_n, n \geq 0. \quad (1.3)$$

In recent years, several authors have used the Mann, Ishikawa and the three-step(Noor iteration) iterations to establish the existence and approximation of the fixed points of strongly pseudocontractive mappings (e.g see[1,2,3,4,7,9-13]).

In 2006, Rafiq [14] introduced the following new type of iterative scheme which he called the modified three-step iterative process :

Let  $T_1, T_2, T_3 : K \rightarrow K$  be three mappings. For any given  $x_0 \in K$ , the modified three-step iteration  $\{x_n\}_{n=0}^\infty \subset K$  is defined by

$$\begin{aligned} x_{n+1} &= (1 - a_n)x_n + a_n T_1 y_n^1 \\ y_n^1 &= (1 - b_n)x_n + b_n T_2 y_n^2 \\ y_n^2 &= (1 - c_n)x_n + c_n T_3 x_n, \end{aligned} \quad (1.4)$$

where  $\{a_n\}_{n=0}^\infty, \{b_n\}_{n=0}^\infty$  and  $\{c_n\}_{n=0}^\infty$  are three real sequence in  $[0, 1]$  satisfying certain conditions.

Observe that if  $T_1 = T_2 = T_3 = T$  in (4), we have (1).

Rafiq [14] proved the following :

**Theorem 1.1** Let  $X$  be a real Banach space and  $K$  be a nonempty closed convex subset of  $X$ . Let  $T_1, T_2, T_3$  be strongly pseudocontractive self maps of  $K$  with  $T_1(K)$  bounded and  $T_1, T_3$  be uniformly continuous. Let  $\{x_n\}_{n=0}^\infty$  be the sequence defined by (4), where  $\{a_n\}_{n=0}^\infty, \{b_n\}_{n=0}^\infty$  and  $\{c_n\}_{n=0}^\infty$  are the three real sequences in  $[0,1]$  satisfying the conditions,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} c_n = 0, \sum_{n=0}^{\infty} a_n = \infty.$$

If  $F(T_1) \cap F(T_2) \cap F(T_3) \neq \phi$ , then the sequence  $\{x_n\}_{n=0}^\infty$  converges strongly to the common fixed point of  $T_1, T_2, T_3$ .

Recently, Xue and Fan [17] showed that the above Theorem of Rafiq and its proof are incorrect. The authors stated and proved a corrected version which is stated below.

**Theorem 1.2** Let  $X$  be a real Banach Space and  $K$  be a nonempty closed convex subset of  $X$ . Let  $T_1, T_2$  and  $T_3$  be strongly pseudocontractive self maps of  $K$  with

$T_1(K)$  bounded and  $T_1, T_2$  and  $T_3$  uniformly continuous. Let  $\{x_n\}_{n=0}^\infty$  be defined by (4), where  $\{a_n\}_{n=0}^\infty, \{b_n\}_{n=0}^\infty$  and  $\{c_n\}_{n=0}^\infty$  are three real sequences in  $[0,1]$  such that: (i)  $a_n, b_n \rightarrow 0$  as  $n \rightarrow \infty$  and (ii)  $\sum_{n=0}^\infty a_n = \infty$ . If  $F(T_1) \cap F(T_2) \cap F(T_3) \neq \phi$ , then the sequence  $\{x_n\}_{n=0}^\infty$  converges strongly to the common fixed point of  $T_1, T_2$  and  $T_3$ .

In year 2004, Rhoades and Soltuz[15] introduced the multi-step procedure. We generalize this to the modified multi-step iterative scheme define as follows:

$$\begin{aligned} x_{n+1} &= (1 - a_n - a'_n)x_n + a_n T_1 y_n^1 + a'_n u_n \\ y_n^l &= (1 - b_n^l)x_n + b_n^l T_{l+1} y_n^{l+1} \\ y_n^{p-1} &= (1 - b_n^{p-1})x_n + b_n^{p-1} T_p x_n, \end{aligned} \tag{1.5}$$

where  $\{a_n\}_{n=0}^\infty, \{b_n^i\}_{n=0}^\infty$  and  $\{b_n^{i+1}\}_{n=0}^\infty$  are three sequences in  $[0, 1], l = 1, 2, 3 \dots, p - 2; p \geq 2$  and  $\{u_n\}$  is a bounded sequence in  $X$ .

It may be noted that the iteration schemes (1)-(4) may be viewed as special cases of (5). For example, if  $p = 3$  and  $a'_n = 0$  in (5) we obtain iteration (4). Taking  $p = 2$  we obtain (3). It is worth mentioning that other important iteration schemes introduced recently by Das and Debata [5] and Kim et al. [8] are all special cases of our iteration scheme. We would like to emphasize that the multi-step iteration can be viewed as the predictor-corrector methods for solving the nonlinear equations in Banach spaces. For the convergence analysis of the predictor-corrector and multi-step iterative methods for solving the variational inequalities and optimization problems,(see Noor [10]) and the references therein.

In this paper, we give convergence results for iteration (5) for a family of  $p$ - strongly pseudocontractive maps in real Banach spaces.

We will use the following results.

**Lemma 1 [6]** Let  $X$  be a real Banach space, then for all  $x, y \in X$ , there exists  $j(x + y) \in J(x + y)$  so that

$$\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, j(x + y) \rangle .$$

**Lemma 2 [16]** Let  $\{\alpha_n\}$  be a non- negative sequence which satisfies the following inequality

$$\alpha_{n+1} \leq (1 - \lambda_n)\alpha_n + \delta_n,$$

where  $\lambda_n \in (0, 1), \forall n \in N, \sum_{n=1}^\infty \lambda_n = \infty$  and  $\delta_n = o(\lambda_n)$ . Then  $\lim_{n \rightarrow \infty} \alpha_n = 0$ .

## 2. RESULTS

**Theorem 2.1.** Let  $K$  be a nonempty closed convex subset of  $X$  and  $T_l, l = 1, 2, \dots, p; p \geq 2$  be uniformly continuous self maps of  $K$  with  $T_1(K)$  bounded such that  $\cap_{l=1}^{p \geq 2} F(T_l) \neq \phi$ . Suppose either  $T_1$  or  $T_2$  or  $\dots$  or  $T_p$  be  $p$ - strongly pseudo-contractive mapping and  $\{x_n\}$  be a sequence defined by (5) where  $\{a_n\}, \{a'_n\}, \{b_n^i\}, \{b_n^{p-1}\}$  are real sequences in  $[0,1], l = 1, 2, \dots, p; p \geq 2$  and  $\{u_n\}$  is a bounded sequence in  $X$  satisfying:

- (i)  $0 \leq a_n + a'_n < 1$
- (ii)  $a_n, a'_n, b_n^1 \rightarrow 0$  as  $n \rightarrow \infty$

- (iii)  $\sum_{n=1}^{\infty} a_n = \infty$
- (iv)  $\lim_{n \rightarrow \infty} \frac{a'_n}{a_n} = 0,$

then, the sequence  $\{x_n\}_{n=0}^{\infty}$  converges strongly to the unique common fixed point of  $T_l$ ,  $l = 1, 2, \dots, p$ ;  $p \geq 2$ .

**Proof.** Since either of  $T_l$  ( $l = 1, 2, \dots, p$ ) is strongly pseudocontractive, then there exist a common constant  $k = \max\{k_1, k_2, \dots, k_p\}$  so that

$$\langle T_l x - T_l y, j(x - y) \rangle \leq k \|x - y\|^2.$$

In view of the definition of strongly pseudo-contractivity of  $T_l$  any common fixed point of  $T_1, T_2, \dots, T_p$ ;  $p \geq 2$  in particular, is a fixed point of  $T_1$ . Let  $\rho$  be such common fixed point. However, since  $T_1$  has bounded range, we denote

$D_1 = \|x_0 - \rho\| + \sup_{n \geq 0} \|T_1 y_n^1 - \rho\| + \|u_n - \rho\|$ . We prove by induction that  $\|x_n - \rho\| \leq D_1$  for all  $n$ . It is clear that,  $\|x_0 - \rho\| \leq D_1$ . Assume that  $\|x_n - \rho\| \leq D_1$  holds. We will prove that  $\|x_{n+1} - \rho\| \leq D_1$ . Indeed, from (5), we obtain

$$\begin{aligned} \|x_{n+1} - \rho\| &\leq \|(1 - a_n - a'_n)(x_n - \rho) + a_n(T_1 y_n^1 - \rho) + a'_n(u_n - \rho)\| \\ &\leq (1 - a_n - a'_n)\|x_n - \rho\| + a_n\|T_1 y_n^1 - \rho\| + a'_n\|u_n - \rho\| \\ &\leq (1 - a_n - a'_n)D_1 + a_n D_1 + a'_n D_1 = D_1, \end{aligned}$$

Hence the sequence  $\{x_n\}$  is bounded.

Using the uniform continuity of  $T_p$ , we have  $\{T_p x_n\}$  is bounded. Denote  $D_2 = \max\{D_1, \sup\{\|T_p x_n - \rho\|\}\}$ , then

$$\begin{aligned} \|y_n^{p-1} - \rho\| &= \|(1 - b_n^{p-1})(x_n - \rho) + b_n^{p-1}(T_p x_n - \rho)\| \\ &= \|(1 - b_n^{p-1})(x_n - \rho) + b_n^{p-1}(T_p x_n - \rho)\| \\ &\leq (1 - b_n^{p-1})\|x_n - \rho\| + b_n^{p-1}\|T_p x_n - \rho\| \\ &\leq (1 - b_n^{p-1})D_1 + b_n^{p-1}D_2 \\ &\leq (1 - b_n^{p-1})D_2 + b_n^{p-1}D_2 = D_2. \end{aligned}$$

By the virtue of the uniform continuity of  $T_{p-1}$ , we get that  $\{T_{p-1} y_n^{p-1}\}$  is bounded. Set  $D_3 = \sup_{n \geq 0} \{\|T_{p-1} y_n^{p-1} - \rho\|\} + D_2$ , then

$$\begin{aligned} \|y_n^{p-2} - \rho\| &= \|(1 - b_n^{p-2})(x_n - \rho) + b_n^{p-2}(T_{p-1} y_n^{p-1} - \rho)\| \\ &= \|(1 - b_n^{p-2})(x_n - \rho) + b_n^{p-2}(T_{p-1} y_n^{p-1} - \rho)\| \\ &\leq (1 - b_n^{p-2})\|x_n - \rho\| + b_n^{p-2}\|T_{p-1} y_n^{p-1} - \rho\| \\ &\leq (1 - b_n^{p-2})D_1 + b_n^{p-2}D_3 \\ &\leq (1 - b_n^{p-2})D_3 + b_n^{p-2}D_3 = D_3. \end{aligned}$$

By the virtue of the uniform continuity of  $T_{p-2}$ , we get that  $\{T_{p-2} y_n^{p-2}\}$  is bounded. Recursively, we have in that  $\{y_{l+1}\}$  is bounded and by uniform continuity of  $T_{l+1}$ ,  $\{T_{l+1} y_n^{l+1}\}$  is bounded. Thus,  $\{y_i\}$  is bounded for  $l = 1, 2, \dots, p-1$ .

Now, set

$$D = \max\{D_1, D_2, \dots, D_{p-1}\}.$$

Applying Lemma 1 and (5), we have

$$\begin{aligned}
 \|x_{n+1} - \rho\|^2 &= \|(1 - a_n - a'_n)(x_n - \rho) + a_n(T_1 y_n^1 - \rho) + c_n(u_n - \rho)\|^2 \\
 &\leq (1 - a_n)^2 \|x_n - \rho\|^2 \\
 &\quad + 2\langle a_n(T_1 y_n^1 - \rho) + a'_n(u_n - \rho), j(x_{n+1} - \rho) \rangle \\
 &= (1 - a_n)^2 \|x_n - \rho\|^2 + 2a_n \langle T_1 y_n^1 - \rho, j(x_{n+1} - \rho) \rangle \\
 &\quad + 2a'_n \langle u_n - \rho, j(x_{n+1} - \rho) \rangle \\
 &\leq (1 - a_n)^2 \|x_n - \rho\|^2 + 2a_n \langle T_1 y_n^1 - T_1 x_{n+1}, j(x_{n+1} - \rho) \rangle \quad (6) \\
 &\quad + 2a_n \langle T_1 x_{n+1} - \rho, j(x_{n+1} - \rho) \rangle + 2D^2 a'_n \\
 &\leq (1 - a_n)^2 \|x_n - \rho\|^2 + 2a_n k \|x_{n+1} - \rho\|^2 \\
 &\quad + 2a_n \|T_1 y_n^1 - T_1 x_{n+1}\| \|x_{n+1} - \rho\| + 2D^2 a'_n \\
 &\leq (1 - a_n)^2 \|x_n - \rho\|^2 + 2a_n k \|x_{n+1} - \rho\|^2 \\
 &\quad + 2a_n \sigma_n D + 2D^2 a'_n,
 \end{aligned}$$

where  $\sigma_n = \|T_1 y_n^1 - T_1 x_{n+1}\|$ . Indeed, since

$$\begin{aligned}
 \|y_n^1 - x_{n+1}\| &= \|y_n^1 - x_n + x_n - x_{n+1}\| \\
 &\leq \|y_n^1 - x_n\| + \|x_n - x_{n+1}\| \\
 &= b_n^1 \|x_n - T_2 y_n^2\| + a_n \|x_n - T_1 y_n^1\| + a' \|x_n - u_n\| \\
 &\leq 2D(a_n + 2a'_n).
 \end{aligned}$$

This implies that  $\lim_{n \rightarrow \infty} \|x_{n+1} - y_n^1\| = 0$ , since  $\lim_{n \rightarrow \infty} a_n = 0$ ,  $\lim_{n \rightarrow \infty} a'_n = 0$ ,  $\lim_{n \rightarrow \infty} b_n^1 = 0$ . Since  $T_1$  is uniformly continuous, we have

$$\sigma_n = \|T_1 x_{n+1} - T_1 y_n^1\| \rightarrow 0, \text{ (as } n \rightarrow \infty \text{)}.$$

And, since  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ , then there exist a positive integer  $N$  such that  $a_n \leq \min\{\frac{1}{2k}, \frac{1-k}{(1-k)^2+k^2}\}$  for all  $n \geq N$ . It follows from (6) that

$$\begin{aligned}
 \|x_{n+1} - \rho\|^2 &\leq \frac{(1-a_n)^2}{1-2a_n k} \|x_n - \rho\|^2 + \frac{2a_n D \sigma_n}{1-2a_n k} + \frac{2D^2 a'_n}{1-2a_n k} \\
 &\leq \frac{(1-a_n)^2}{1-2a_n k} \|x_n - \rho\|^2 + \frac{2a_n D^2 \sigma_n}{1-2a_n k} + \frac{2D^2 a'_n}{1-2a_n k} \quad (7) \\
 &\leq (1 - \frac{2-2k-a_n}{1-2a_n k}) \|x_n - \rho\|^2 + \frac{2a_n D^2}{1-2a_n k} (\sigma_n + \frac{a'_n}{a_n}) \\
 &\leq (1 - (1-k)a_n) \|x_n - \rho\|^2 + \frac{2a_n D^2}{1-2a_n k} (\sigma_n + \frac{a'_n}{a_n}).
 \end{aligned}$$

Set  $\alpha_n = \|x_n - q\|$ ,  $\lambda_n = (1-k)a_n$  and  $\delta_n = \frac{2a_n D^2}{1-2a_n k} (\sigma_n + \frac{a'_n}{a_n})$ . Applying Lemma 2, we obtain  $\|x_n - q\| \rightarrow 0$  as  $n \rightarrow \infty$ . This complete the proof.

**Remark 2.2** Theorem 2.1 extends Theorem 2.1 of Xue and Fan[17] in the sense that, we replace modified three-step iterative scheme by a more general modified multi-step iterative scheme.

**Theorem 2.3** Let  $X, K, T_1, T_2, \dots, T_p, \{x_n\}, \{a_n\}, \{a'_n\}$  and  $\sigma_n$  be as in Theorem 2.1. Suppose there exists a sequence  $\{t_n\}$  with  $\lim_{n \rightarrow \infty} t_n = 0$  and  $a'_n = t_n a_n$  for any  $n \geq 0$ . Then the sequence  $\{x_n\}$  defined in (5) converges strongly to the unique common fixed point of  $T_1, T_2, \dots, T_p$  which is the unique fixed point of  $T_1$ .

**Proof:** Just as in the proof of Theorem 2.1, we have from (7)

$$\begin{aligned} \|x_{n+1} - \rho\|^2 &\leq (1 - (1 - k)a_n)\|x_n - \rho\|^2 + \frac{2a_n D^2}{1 - 2a_n k}(\sigma_n + \frac{a'_n}{a_n}) \\ &= (1 - (1 - k)a_n)\|x_n - \rho\|^2 + \frac{2a_n D^2}{1 - 2a_n k}(\sigma_n + \frac{t_n a_n}{a_n}) \\ &= (1 - (1 - k)a_n)\|x_n - \rho\|^2 + \frac{2a_n D^2}{1 - 2a_n k}(\sigma_n + t_n) \end{aligned}$$

Put  $\alpha_n = \|x_n - \rho\|$ ,  $\lambda_n = (1 - k)a_n$  and  $\delta_n = \frac{2b_n D^2}{1 - 2b_n k}(\sigma_n + t_n)$ . Then, Lemma 2 ensures that  $\|x_n - \rho\| \rightarrow 0$  as  $n \rightarrow \infty$ . This complete the proof.

For  $p = 3$  and  $a'_n = 0$ , we can recover the result of Xue and Fan[17] from the following result.

**Theorem 2.4** *Let  $K$  be a nonempty closed convex subset of  $X$  and  $T_l$ ,  $l = 1, 2, 3$  be uniformly continuous self maps of  $K$  with  $T_1(K)$  bounded such that  $\cap_{l=1}^3 F(T_l) \neq \phi$ . Suppose either  $T_1$  or  $T_2$  or  $T_3$  be strongly pseudo-contractive mapping and  $\{x_n\}$  be a sequence defined by (5) where  $\{a_n\}, \{b_n\}, \{c_n\}$  are real sequences in  $[0, 1]$ , satisfying:*

- (i)  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ ,
- (ii)  $\sum_{n=1}^{\infty} a_n = \infty$ ,

*then, the sequence  $\{x_n\}_{n=0}^{\infty}$  converges strongly to the unique common fixed point of  $T_l$ ,  $l = 1, 2, 3$ .*

For  $p = 2$ ,  $a'_n = 0$ ,  $T_1 = T_2 = T$  we can recover the result of Ciric and Ume [3] from the following result.

**Theorem 2.5** *Let  $K$  be a nonempty closed convex subset of  $X$  and  $T_l$ ,  $l = 1, 2$  be uniformly continuous self maps of  $K$  with  $T_1(K)$  bounded such that  $\cap_{l=1}^2 F(T_l) \neq \phi$ . Suppose either  $T_1$  or  $T_2$  be strongly pseudo-contractive mapping and  $\{x_n\}$  be a sequence defined by (5) where  $\{a_n\}, \{b_n\}, \{c_n\}$  are real sequences in  $[0, 1]$ , satisfying:*

- (i)  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ ,
- (ii)  $\sum_{n=1}^{\infty} a_n = \infty$ ,

*then, the sequence  $\{x_n\}_{n=0}^{\infty}$  converges strongly to the unique common fixed point of  $T_l$ ,  $l = 1, 2$ .*

**Theorem 2.6** *Let  $X$  be a real Banach space. Let  $S_1, S_2, \dots, S_p$ ;  $p \geq 2$  be uniformly continuous and  $p$ -strongly accretive operators. For a fixed  $f \in X$ , define  $T_l x = x - S_l x + f$  for all  $l = 1, 2, \dots, p$ ;  $p \geq 2$ . For arbitrary  $x_0 \in X$ , let modified multi-step iteration sequence  $\{x_n\}$  be defined by*

$$\begin{aligned} x_{n+1} &= (1 - a_n - a'_n)x_n + a_n(f + (I - S_1)y_n^1) + a'_n u_n \\ y_n^l &= (1 - b_n^l)x_n + b_n^l(f + (I - S_{l+1})y_n^{l+1}) \\ y_n^{p-1} &= (1 - b_n^{p-1})x_n + b_n^{p-1}(f + (I - S_p)x_n), \end{aligned} \tag{8}$$

satisfying the following conditions:

- (i)  $0 \leq a_n + a'_n < 1$
- (ii)  $a_n, a'_n, b_n^1 \rightarrow 0$  as  $n \rightarrow \infty$
- (iii)  $\sum_{n=1}^{\infty} a_n = \infty$

$$(iv) \lim_{n \rightarrow \infty} \frac{a'_n}{a_n} = 0,$$

then, the sequence  $\{x_n\}_{n=0}^{\infty}$  converges strongly to the solution of  $S_l x = f$ ,  $l = 1, 2, \dots, p; p \geq 2$ .

**Proof:** Obviously, if  $x^* \in X$  is a solution of the equation  $S_i x = f$  ( $i = 1, 2, \dots, p; p \geq 2$ ), then  $x^*$  is the common fixed point of  $T_i$  ( $i = 1, 2, \dots, p; p \geq 2$ ). It is easy to prove that  $T_i$  ( $i = 1, 2, \dots, p; p \geq 2$ ) is uniformly continuous and strongly pseudo-contractive. Thus, Theorem 2.6 follows from Theorem 2.1.

**Remark 2.7**

- (i) We replaced modified Noor iteration scheme by more general modified multi-step iteration scheme.
- (ii) Our method of proof is easy and purely analytical.

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