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OPERATORS POSSESSING PROPERTIES (gb) AND (gw)

(COMMUNICATED BY VLADIMER MULLER)

F. LOMBARKIA, H. ZARIOUH

Abstract. This article treatises several problems relevant to property (gb) for bounded linear operators on Banach spaces. Sufficient conditions for an operator T to possess property (gb) and property (gw) are given. We also prove some perturbation results concerning property (b) and property (gw). The theory is exemplified in the case of some special classes of operators.

1. Introduction

Throughout this paper, let $L(X)$ denote the Banach algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space X. For $T \in L(X)$, let T^*, $\ker(T)$, $R(T)$, $\sigma(T)$, $\sigma_a(T)$ and $\sigma_p(T)$ denote the dual, the null space, the range, the spectrum, the approximate point spectrum and the point spectrum of T, respectively. Let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of T defined by $\alpha(T) = \dim\ker(T)$ and $\beta(T) = \operatorname{codim}R(T)$. Recall that an operator $T \in L(X)$ is called an *upper semi-Fredholm* if $\alpha(T) < \infty$ and $R(T)$ is closed, while $T \in L(X)$ is called a *lower semi-Fredholm* if $\beta(T) < \infty$. Let $SF_+(X)$ denote the class of all upper semi-Fredholm operators. If $T \in L(X)$ is an upper or a lower semi-Fredholm operator, then T is called a *semi-Fredholm* operator, and the *index* of T is defined by $\mathrm{ind(T)} = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called a *Fredholm* operator. An operator $T \in L(X)$ is called a *Weyl* operator if it is a Fredholm operator of index zero. Define $SF_+^-(X) = \{T \in SF_+(X) : \mathrm{ind(T)} \leq 0\}$. The classes of operators defined above generate the following spectra: the *Weyl spectrum* $\sigma_W(T)$ of $T \in L(X)$ defined by $\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Weyl opertor}\}$, and the *Weyl essential approximate point spectrum* $\sigma_{SF_+^-}(T)$ of T defined by $\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SF_+^-(X)\}$. For $T \in L(X)$, let $\Delta(T) = \sigma(T) \setminus \sigma_W(T)$ and let $\Delta_a(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. Following Coburn [19], we say that *Weyl's theorem* holds for $T \in L(X)$ if $\Delta(T) = E^0(T)$, where $E^0(T) = \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(T - \lambda I) < \infty\}$. Here and elsewhere in this paper, for $A \subset \mathbb{C}$, $\operatorname{iso}A$ is the set of all isolated points of A.

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According to Rakočević [27], an operator $T \in L(X)$ is said to satisfy *a-Weyl's theorem* if $\Delta_a(T) = E_a^0(T)$, where $E_a^0(T) = \{\lambda \in \text{iso}\sigma_a(T) : 0 < \alpha(T - \lambda I) < \infty\}$. It is well known [27] that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem, but not conversely.

Recall that the *ascent* $a(T)$, of an operator T, is defined by $a(T) = \inf\{n \in \mathbb{N} : \ker(T^n) = \ker(T^{n+1})\}$ and the *descent* $\delta(T)$ of T, is defined by $\delta(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$, with $\inf \emptyset = \infty$. It is well known that if $a(T)$ and $\delta(T)$ are both finite then they are equal, see [25, Corollary 20.5]. An operator $T \in L(X)$ is called *Drazin invertible* if it has a finite ascent and descent. The *Drazin spectrum* $\sigma_D(T)$ of an operator T is defined by $\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\}$. An operator $T \in L(X)$ is called *Browder* if it is Fredholm of finite ascent and descent. The *Browder spectrum* $\sigma_b(T)$ of T is defined by $\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\}$.

Define also the set $LD(X)$ by

$$LD(X) = \{T \in L(X) : a(T) < \infty \text{ and } R(T^{a(T)+1}) \text{ is closed}\}.$$

Following [11], an operator $T \in L(X)$ is said to be *left Drazin invertible* if $T \in LD(X)$. The *left Drazin spectrum* $\sigma_{LD}(T)$ of T is defined by $\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin LD(X)\}$. We say that $\lambda \in \sigma_a(T)$ is a *left pole* of T if $T - \lambda I \in LD(X)$, and that $\lambda \in \sigma_a(T)$ is a left pole of T of finite rank if λ is a left pole of T and $\alpha(T - \lambda I) < \infty$. Let $\Pi_a(T)$ denote the set of all left poles of T and let $\Pi_a^0(T)$ denote the set of all left poles of T of finite rank.

Let $\Pi(T)$ be the set of all poles of the resolvent of T and let $\Pi^0(T)$ be the set of all poles of the resolvent of T of finite rank, that is $\Pi^0(T) = \{\lambda \in \Pi(T) : \alpha(T - \lambda I) < \infty\}$. According to [23], a complex number λ is a *pole* of the resolvent of T if $T - \lambda I$ has ascent and descent finite and greater than 0 (this implies that are equal !).

For $T \in L(X)$ and a nonnegative integer n, define $T_{[n]}$ to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_{[0]} = T$). If for some integer n the range space $R(T^n)$ is closed and $T_{[n]}$ is an upper (resp. a lower) semi -Fredholm operator, then T is called an *upper* (resp. a *lower*) *semi-B-Fredholm* operator. In this case the index of T is defined as the index of the semi-Fredholm operator $T_{[n]}$, see [13]. Moreover, if $T_{[n]}$ is a Fredholm operator, then T is called a *B-Fredholm* operator, see [8]. A *semi-B-Fredholm* operator is an upper or a lower semi-B-Fredholm operator. An operator $T \in L(X)$ is said to be a *B-Weyl* operator if it is a B-Fredholm operator of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ of T is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Weyl operator}\}$.

For $T \in L(X)$, let $\Delta^g(T) = \sigma(T) \setminus \sigma_{BW}(T)$. According to [11], an operator $T \in L(X)$ is said to satisfy *generalized Weyl's theorem* if $\Delta^g(T) = E(T)$, where $E(T) = \{\lambda \in \text{iso}\sigma(T) : 0 < \alpha(T - \lambda I)\}$.

Let $SBF_+(X)$ be the class of all upper semi-B-Fredholm operators, $SBF_+^-(X) = \{T \in SBF_+(X) : \text{ind}(T) \leq 0\}$ and let $SBF_-(X)$ be the class of all lower semi-B-Fredholm operators, $SBF_-^+(X) = \{T \in SBF_-(X) : \text{ind}(T) \geq 0\}$. The *upper B-Weyl spectrum* of T is defined by $\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SBF_+^-(X)\}$, while the *lower B-Weyl spectrum* of T is defined by $\sigma_{SBF_-^+}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SBF_-^+(X)\}$. For $T \in L(X)$, let $\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. According to [11], we say that $T \in L(X)$ obeys *generalized a-Browder's theorem* if $\Delta_a^g(T) = \Pi_a(T)$. According also to [11], an operator $T \in L(X)$ is said to satisfy *generalized a-Weyl's theorem* if $\Delta_a^g(T) = E_a(T)$, where $E_a(T) = \{\lambda \in \text{iso}\sigma_a(T) : \alpha(T - \lambda I) > 0\}$. It is known [11] that an operator obeying generalized a-Weyl's theorem obeys generalized Weyl's theorem, but the converse is not true in general.

Following [7, Definition 2.1], an operator $T \in L(X)$ is said to possess *property* (gw) if $\Delta_a^g(T) = E(T)$ which extends property (w) introduced by Rakočević in [26] to the context of B-Fredholm theory. Recall that an operator $T \in L(X)$ is said to possess *property* (w) if $\Delta_a(T) = E^0(T)$. It is known in [7, Theorem 2.4] that an operator possessing property (gw) satisfies generalized Weyl's theorem, but the converse is not true in general and in [7, Theorem 2.3] that an operator possessing property (gw) possesses property (w), but not conversely. According to [15, Definition 2.1], an operator $T \in L(X)$ is said to possess *property* (b) if $\Delta_a(T) = \Pi^0(T)$ and is said to possess *property* (gb) if $\Delta_a^g(T) = \Pi(T)$ which is an extension to the context of B-Fredholm theory of property (b). It is proved in [15, Theorem 2.3] that an operator possessing property (gb) possesses property (b), but the converse does not hold in general and in [15, Theorem 2.15] that an operator possessing property (gw) possesses property (gb), but the converse does not hold in general. It is also proved in [15, Theorem 2.13] that an operator possessing property (w) possesses property (b), but the converse is not true in general. Weaker variant of property (gb) has been introduced and studied very recently by M. Berkani and H. Zariouh in [16], called property (gab). Recall that an operator $T \in L(X)$ is said to possess *property* (gab) if $\Delta^g(T) = \Pi_a(T)$.

In this paper we give certain sufficient conditions for an operator $T \in L(X)$ to possess property (gw) or property (gb). In the second section, we prove in Theorem 2.1 that if $T \in L(X)$ has no eigenvalues and if f is an analytic function near the spectrum $\sigma(T)$ of T, then $f(T)$ possesses property (gw) or equivalently property (gb). As a consequence, we show in Corollary 2.2 that if there exists a complex number λ_0 satisfies the equality $K(T-\lambda_0 I) = \ker(T-\lambda_0 I) = \{0\}$, then $f(T)$ possesses property (gw) or equivalently property (gb) for all analytic function f near the spectrum $\sigma(T)$. We also show in Theorem 2.6 that if the dual T^* of $T \in L(X)$ has the SVEP on the complement of $\sigma_{SBF_+^-}(T)$ and if $E(T) = \Pi(T)$, then T possesses property (gw) extending [7, Theorem 2.10]. In the third section of this paper, we study the variants of Weyl's theorem for perturbations by algebraic, nilpotent and compact operators. In particular, we show in Theorem 3.1 that if T^* has the SVEP and if $E(T) = \Pi(T)$, then $T + N$ possesses property (gw) for every nilpotent operator N which commutes with T, and in Theorem 3.6, we prove that if $T \in L(X)$ is an operator possessing property (b) and $K \in L(X)$ is a compact operator commuting with T such that $\Pi_a^0(T+K) \subset \sigma_a(T)$, then $T+K$ possesses property (b) extending Theorem 3.3 and Theorem 3.4 of [17]. We finish this section by some remarks concerning some results given by P. Aiena and M. T. Biondi in [3] (see Remark 3.7).

2. Properties (gb) and (gw)

An operator $T \in L(X)$ is said to have the *single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open neighborhood $\mathcal{U}$ of λ_0, the only analytic function $f : \mathcal{U} \longrightarrow X$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\mathcal{U}$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have the SVEP if T has this property at every $\lambda \in \mathbb{C}$ (see [24]). If T^* has the SVEP on the complement of $\sigma_{SBF_+^-}(T)$, then it is known [14, Lemma 2.4] that $\sigma_a(T) = \sigma(T)$ and

$\sigma_{BW}(T) = \sigma_{SBF_+^-}(T)$, this implies that $E(T) = E_a(T)$ and $\Pi(T) = \Pi_a(T)$. A local spectral characterization of operators satisfying generalized a-Browder's theorem is given in [5, Theorem 2.4], more precisely we have $T \in L(X)$ satisfies generalized a-Browder's theorem if and only if T has the SVEP at every $\lambda \notin \sigma_{SBF_+^-}(T)$.

We denote by $\mathcal{H}(\sigma(T))$ the set of all analytic functions defined on an open neighborhood of $\sigma(T)$.

Theorem 2.1. *Let $T \in L(X)$ and let $f \in \mathcal{H}(\sigma(T))$. If $\sigma_p(T) = \emptyset$, then $f(T)$ possesses property (gw), or equivalently property (gb).*

Proof. The hypothesis $\sigma_p(T)$ is empty entails from the proof of [4, Theorem 2.5] that $\sigma_p(f(T))$ is empty. This implies in particular that $f(T)$ has the SVEP on the complement of $\sigma_{SBF_+^-}(f(T))$. Consequently, $f(T)$ satisfies generalized a-Browder's theorem and so $\sigma_{SBF_+^-}(f(T)) = \sigma_{LD}(f(T))$. Since $E_a(f(T)) = E(f(T)) = \emptyset$, then by [11, Theorem 3.1], we conclude that $\sigma_a(f(T)) \subset \sigma_{SBF_+^-}(f(T))$ and since the opposite inclusion holds for every operator, it then follows that $\sigma_a(f(T)) = \sigma_{SBF_+^-}(f(T))$. Hence $f(T)$ possesses property (gw). As we also have $\Pi(f(T)) = \emptyset$, then properties (gw) and (gb) are equivalent for $f(T)$.

Let $T \in L(X)$, the analytic core $K(T)$ of T is defined by $K(T) = \{x \in X :$ there exits $c > 0$ and a sequence $(x_n)_{n \geq 1} \subset X$ such that $Tx_1 = x, Tx_{n+1} = x_n$ for all $n \in \mathbb{N}$ and $\|x_n\| \leq c^n \|x\|$
for all $n \in \mathbb{N}\}$ and the quasinilpotent part $H_0(T)$ of T is defined by

$$H_0(T) = \{x \in X : \lim_{n \to \infty} \|T^n(x)\|^{\frac{1}{n}} = 0\}.$$

(See [1] for information on $K(T)$ and $H_0(T)$).

Corollary 2.2. *Let $T \in L(X)$. If there exists $\lambda_0 \in \mathbb{C}$ such that*

$$K(T - \lambda_0 I) = \{0\} \text{ and } \ker(T - \lambda_0 I) = \{0\},$$

then $f(T)$ possesses property (gw), or equivalently property (gb) for all $f \in \mathcal{H}(\sigma(T))$.

Proof. For all complex number $\lambda \neq \lambda_0$ we have $\mathrm{Ker}(T - \lambda I) \subset K(T - \lambda_0 I)$. Then $\mathrm{Ker}(T - \lambda I) = \{0\}$ for all $\lambda \in \mathbb{C}$, so that $\sigma_p(T) = \emptyset$. From Theorem 2.1, $f(T)$ possesses property (gw), or equivalently property (gb).

Corollary 2.3. *Let $T \in L(X)$. If there exists $\lambda_0 \in \mathbb{C}$ such that $K(T - \lambda_0 I) = \{0\}$ and $\ker(T - \lambda_0 I) = \{0\}$, then*

$$\sigma_a(f(T)) = \sigma_{SF_+^-}(f(T)) = \sigma_{SBF_+^-}(f(T)),$$

and

$$\sigma_{BW}(f(T)) = \sigma_W(f(T)) = \sigma(f(T)),$$

for all $f \in \mathcal{H}(\sigma(T))$.

Proof. We have $\sigma_{SBF_+^-}(f(T)) \subseteq \sigma_{SF_+^-}(f(T)) \subseteq \sigma_a(f(T))$, (this holds for every operator). Since the assumption entails that $\sigma_p(T)$, and hence $\sigma_p(f(T))$, is empty, then the equality of these spectra follows from the proof of Theorem 2.1. Similarly and immediately , we obtain the second assertion.

The conditions of Corollary 2.2 are satisfied by any injective operator for which the hyperrange $T^\infty(X) = \bigcap_{n\geq 0} T^n(X)$ is $\{0\}$. In fact, $K(T) \subseteq T^\infty(X)$ for all $T \in L(X)$, so that $K(T) = \{0\}$. In particular, the conditions of Corollary 2.2 are satisfied by every weighted unilateral right shift operator T on $\ell^p(\mathbb{N})$ $(1 \leq p < \infty)$ defined for each $x = (x_i) \in \ell^p(\mathbb{N})$ by $T(x) = \sum_{i=1}^{\infty} \alpha_i x_i e_{i+1}$, where $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ is a sequence of real numbers such that $0 < \alpha_i \leq 1$ and $(e_i)_{i\in\mathbb{N}}$ is the canonical basis of $\ell^p(\mathbb{N})$, see [24] for details on this class of operators.

Definition 2.4. *An operator $T \in L(X)$ is called a-polaroid (resp. left a-polaroid) if all isolated points of the approximate point spectrum are poles of the resolvent of T, i.e. $iso\sigma_a(T) = \Pi(T)$ (resp. are left poles of T, i.e. $iso\sigma_a(T) = \Pi_a(T)$). Moreover, if $iso\sigma(T) = \Pi(T)$, then we will say that T is polaroid operator.*

From the Definition 2.4, we remark that every a-polaroid operator is polaroid but the converse is not true in general. Indeed, if we consider the operator $T = R \oplus U$ defined on the Banach space $\ell^2(\mathbb{N})\oplus\ell^2(\mathbb{N})$, where R is the unilateral right shift operator on $\ell^2(\mathbb{N})$ and U is defined on $\ell^2(\mathbb{N})$ by $U(x_1, x_2, x_3, ...) = (x_2/2, x_3/3, x_4/4, ...)$, then $\sigma(T) = D(0,1)$ is the closed unit disc in $\mathbb{C}$ and $\sigma_a(T) = C(0,1) \cup \{0\}$, where $C(0,1)$ is the unit circle of $\mathbb{C}$. Consequently, $\text{iso}\sigma(T) = \Pi(T) = \emptyset$ and T is polaroid, but it is not a-polaroid, since $\text{iso}\sigma_a(T) = \{0\}$.

We also observe that every a-polaroid operator is left a-polaroid, but the converse does not hold in general as shown by the following example: let T be defined on the Banach space $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = 0 \oplus R$, where R is the unilateral right shift operator, then $\sigma(T) = D(0,1)$ and $\sigma_a(T) = C(0,1) \cup \{0\}$ and $\Pi_a(T) = \{0\}$. So $\text{iso}\sigma_a(T) = \Pi_a(T)$ and T is left a-polaroid, but T is not a-polaroid, since $\Pi(T) = \emptyset$. However in the following lemma we give sufficient conditions for an operator to be a-polaroid.

Lemma 2.5. *Let $T \in L(X)$ be an operator.*
(i) If $T \in L(X)$ is a left a-polaroid possessing property (gab), then T is an a-polaroid.
(ii) If $T \in L(X)$ is polaroid and its dual has the SVEP on the complement of $\sigma_{SF_+^-}(T)$, then T is an a-polaroid.

Proof. (i) Assume that T is a left a-polaroid possessing property (gab). Then it follows from [16, Corollary 2.7] that $\Pi(T) = \Pi_a(T)$. Since $\Pi_a(T) = \text{iso}\sigma_a(T)$, it follows that is an a-polaroid.
(ii) Assume that T is polaroid and its dual T^* has the SVEP on the complement of $\sigma_{SF_+^-}(T)$. Then it follows from [14, Lemma 2.1] that $\sigma(T) = \sigma_a(T)$. Since $\Pi(T) = \text{iso}\sigma(T)$, it then follows that T is an a-polaroid.

We know from [14, Theorem 2.5] that if T^* has the SVEP on the complement of $\sigma_{SBF_+^-}(T)$, then T possesses property (gb), but this is not true for property (gw). Indeed, let T be defined on the Hilbert space $\ell^2(\mathbb{N})$ by $T(x_1, x_2, x_3, ...) = (x_3/3, x_4/4, x_5/5, ...)$. As T is quasinilpotent, T as well as its adjoint T^*, has the SVEP, but T does not possess property (gw), since $\sigma_a(T) = \sigma_{SBF_+^-}(T) = \{0\}$ and $E(T) = \{0\}$.

The two first parts of the following result have been proved by M. Amouch and M. Berkani in [7, Theorem 2.10] under the hypothesis that T is a polaroid and that T^* (resp. T) has the SVEP. We show here that these results hold also under the weaker hypothesis that $E(T) = \Pi(T)$ (resp. $E(T^*) = \Pi(T)$ and that T^* (resp. T) has the SVEP only on the complement of $\sigma_{SBF_+^-}(T)$ (resp. $\sigma_{SBF_-^+}(T)$). We observe that if T is a polaroid, then $E(T) = \Pi(T)$, but the converse does not hold in general as shown by the following example: let $S : \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N})$ be an injective quasinilpotent operator which is not nilpotent and we define T on the Banach space $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = I \oplus S$, where I is the identity operator on $\ell^2(\mathbb{N})$. Then $\sigma(T) = \sigma_a(T) = \{0, 1\}$ and $\Pi(T) = E(T) = \{1\}$, but T is not polaroid, since $\text{iso}\sigma(T) = \{0, 1\}$.

We also observe that if T is a polaroid, then $E(T^*) = \Pi(T)$, but the converse is not true in general. To see this, consider the operator T defined on the Hilbert space $\ell^2(\mathbb{N})$ by $T(x_1, x_2, x_3, ...) = (x_2/2, x_3/3, x_4/4, ...)$. Then its adjoint T^* is defined by $T^*(x_1, x_2, x_3, ...) = (0, x_1/2, x_2/3, x_3/4, ...)$. So $E(T^*) = \Pi(T) = \emptyset$, but T is not polaroid, since $\text{iso}\sigma(T) = \{0\}$.

Theorem 2.6. *Let $T \in L(X)$ be an operator. Then the following statements hold.*
(i) If $E(T) = \Pi(T)$ and if T^* has SVEP at every $\lambda \notin \sigma_{SBF_+^-}(T)$, then T possesses property (gw).
(ii) If $E(T^*) = \Pi(T)$ and if T has SVEP at every $\lambda \notin \sigma_{SBF_-^+}(T)$, then T^* possesses property (gw).
(iii) If T or T^* has the SVEP and $\text{iso}\sigma_a(T) = \emptyset$, then $T + F$ possesses property (gw) for every finite rank operator $F \in L(X)$ commuting with T.

Proof. (i) Assume that T^* has the SVEP at every $\lambda \notin \sigma_{SBF_+^-}(T)$. Then it follows from [14, Theorem 2.5] that T possesses property (gb), that is $\Delta_a^g(T) = \Pi(T)$. As by hypothesis $E(T) = \Pi(T)$, then $\Delta_a^g(T) = E(T)$ and T possesses property (gw).
(ii) Assume that T has the SVEP at every $\lambda \notin \sigma_{SBF_-^+}(T)$. Then it follows from [14, Theorem 2.5] that T^* possesses property (gb), that is $\Delta_a^g(T^*) = \Pi(T^*)$. Since we know that $\Pi(T^*) = \Pi(T)$ and by hypothesis, $E(T^*) = \Pi(T)$, then $\Delta_a^g(T^*) = E(T^*)$ and T^* possesses property (gw).
(iii) Assume that T has the SVEP, then from [2, Lemma 2.8], $T + F$ has the SVEP. Therefore $T + F$ satisfies generalized a-Browder's theorem, that is $\Delta_a^g(T + F) = \Pi_a(T + F)$. Since $\text{iso}\sigma_a(T) = \emptyset$, from [2, Lemma 2.6], we have $\text{iso}\sigma_a(T + F) = \emptyset$ and consequently, $E(T + F) = \Pi_a(T + F) = \emptyset$. So $\Delta_a^g(T + F) = E(T + F)$ and $T + F$ possesses property (gw). Similarly, if T^* has the SVEP, then $T^* + F^*$ has the SVEP which implies from [14, Corollary 2.7] that $T + F$ satisfies generalized a-Browder's theorem. So $\Delta_a^g(T + F) = \Pi_a(T + F)$, and proceeding as in the first case we see that $T + F$ possesses property (gw).

Remark 2.7. 1)– The following example shows that the condition T^* has the SVEP on the complement of $\sigma_{SBF_+^-}(T)$ assumed in the statement (i) of Theorem 2.6 cannot be replaced by the SVEP for T on the complement of $\sigma_{SBF_+^-}(T)$. Let R be the unilateral right shift operator on $\ell^2(\mathbb{N})$ and let the operator U be defined by $U(x_1, x_2, x_3, ...) = (0, x_2, x_3, ...)$ for all $(x_i) \in \ell^2(\mathbb{N})$. Define T on the Banach space $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = R \oplus U$. Then $\sigma(T) = D(0, 1)$ is the closed unit disc in $\mathbb{C}$ and $\Pi(T) = E(T) = \emptyset$. Moreover, we have $\sigma_a(T) = C(0, 1) \cup \{0\}$, where $C(0, 1)$ is the unit circle of $\mathbb{C}$. This implies that $\sigma_a(T)$ has an empty interior which implies that T has the SVEP. But T does not possess property (gw), since $\sigma_{SBF_+^-}(T) = C(0,1)$.

Now we have $(T^*)^* = T$ has the SVEP on the complement of $\sigma_{SBF_-^+}(T^*)$ and $E((T^*)^*) = \Pi(T^*) = \emptyset$, but $(T^*)^*$ does not possess property (gw). This also shows that in the statement (ii) of Theorem 2.6, we cannot replace the SVEP for T by the SVEP for T^*.

2)– Generally, the converse of the statement (i) of Theorem 2.6 does not hold. Consider the operator $T = R \oplus U$ defined just after Definition 2.4, then $\sigma_a(T) = C(0,1) \cup \{0\}$, $\sigma_{SBF_+^-}(T) = C(0,1) \cup \{0\}$, $\sigma_{BW}(T) = D(0,1)$ and $E(T) = \emptyset$. So T possesses property (gw). But there exists at least one scalar $\lambda_0 \notin \sigma_{SBF_+^-}(T)$ such that T^* does not have the SVEP at λ_0, since the SVEP for T^* on the complement of $\sigma_{SBF_+^-}(T)$ would imply that $\sigma_{BW}(T) = \sigma_{SBF_+^-}(T)$ and this is a contradiction. Now we have $(T^*)^* = T$ possesses property (gw). But T^* does not have the SVEP at $\lambda_0 \notin \sigma_{SBF_-^+}(T^*)$.

The next theorem has been proved in [15, Theorem 2.3], but the authors used in its proof an auxiliary lemma [15, Lemma 2.2]. We give here a direct simple proof without using this lemma.

Theorem 2.8. *Let $T \in L(X)$. If T possesses property (gb), then T possesses property (b).*

Proof. Suppose that T possesses property (gb), then $\Delta_a^g(T) = \Pi(T)$. If $\lambda \in \Delta_a(T)$, then $\lambda \in \Delta_a^g(T)$. As $\Delta_a^g(T) = \Pi(T)$, then λ is a pole of the resolvent of T. Since $T - \lambda I \in SF_+(X)$, it follows that $\alpha(T - \lambda I) < \infty$ and so $\lambda \in \Pi^0(T)$. Hence $\Delta_a(T) \subset \Pi^0(T)$. Since we always have that $\Delta_a(T) \supset \Pi^0(T)$, we have $\Delta_a(T) = \Pi^0(T)$ and T possesses property (b).

In general, the property (gw) is not intermediate between generalized Weyl's theorem and generalized a-Weyl's theorem. Indeed, the operator $T = R \oplus U$ defined as in part 2) of Remark 2.7 possesses property (gw), but it does not satisfy generalized a-Weyl's theorem, since $E_a(T) = \{0\}$ and the operator $T = 0 \oplus R$ defined above satisfies generalized a-Weyl's theorem, since $\sigma_{SBF_+^-}(T) = C(0,1)$, $E_a(T) = \{0\}$, but it does not possess property (gw), since $E(T) = \emptyset$.

We also have that the property (gb) is not intermediate between property (w) and property (b). To see this, the operator $T = 0 \oplus R$ possesses property (w), since $\sigma_{SF_+^-}(T) = C(0,1) \cup \{0\}$, $E^0(T) = \emptyset$, but it does not possesses property (gb), because $\Pi(T) = \emptyset$ and the operator T defined on $\ell^2(\mathbb{N})$ by $T(x_1, x_2, x_3, ...) = (x_2/2, x_3/3, x_4/4, ...)$ possesses property (gb), but it does not possesses property (w), because $\sigma_a(T) = \sigma_{SF_+^-}(T) = \{0\}$ and $E^0(T) = \{0\}$. However, for the class of a-polaroid operators we have the following result.

Proposition 2.9. *Let $T \in L(X)$ be an a-polaroid operator. Then the following statements are equivalent.*
(i) T satisfies generalized a-Browder's theorem,
(ii) T possesses property (gb),
(iii) T possesses property (gw),
(iv) T satisfies generalized a-Weyl's theorem,
(v) T possesses property (w),

(vi) T possesses property (b),
(vii) T satisfies a-Weyl's theorem.

Proof. Since T is an a-polaroid operator, it is easily seen that $\Pi(T) = E(T) = E_a(T) = \Pi_a(T)$. Hence the equivalence between these statements follows directly by [11], [12] and [15].

3. Weyl type theorems for perturbations

The following perturbation result improves Theorem 2.6.

Theorem 3.1. *Let $T \in L(X)$ be an operator and let $N \in L(X)$ be a nilpotent operator commuting with T. Then the following statements hold.*
(i) If $E(T) = \Pi(T)$ and if T^* has SVEP, then $T + N$ possesses property (gw).
(ii) If $E(T^*) = \Pi(T)$ and if T has SVEP, then $T^* + N^*$ possesses property (gw).

Proof. (i) Assume that T^* has the SVEP. As N^* is nilpotent and commutes with T^*, then from [1, Corollary 2.12], $T^* + N^*$ has the SVEP. Therefore $T + N$ possesses property (gb), that is $\sigma_a(T+N) \setminus \sigma_{SBF_+^-}(T+N) = \Pi(T+N)$. On the other hand, we know that $E(T) = E(T+N)$ and $\Pi(T) = \Pi(T+N)$. Since by hypothesis we have $E(T) = \Pi(T)$, it follows that $\sigma_a(T+N) \setminus \sigma_{SBF_+^-}(T+N) = E(T+N)$ and $T + N$ possesses property (gw).
(ii) Assume that T has the SVEP. As N is nilpotent and commutes with T, then from [1, Corollary 2.12], $T+N$ has the SVEP. Therefore T^*+N^* possesses property (gb), that is $\sigma_a(T^* + N^*) \setminus \sigma_{SBF_+^-}(T^* + N^*) = \Pi(T^* + N^*)$. Since by assumption $E(T^*) = \Pi(T)$, we conclude that $E(T^* + N^*) = \Pi(T^* + N^*)$ and so $T^* + N^*$ possesses property (gw).

Remark 3.2. 1)– Generally, Theorem 3.1 does not extend to commuting quasinilpotent perturbations. Indeed, let $X = \ell^2(\mathbb{N})$ and let T and the quasinilpotent operator Q be defined by $T = 0$ and $Q(x_1, x_2, x_3, ...) = (0, x_2/2, x_3/3, x_4/4, ...)$. Then $TQ = QT = 0$, $E(T) = \Pi(T) = \{0\}$ and T^* has the SVEP, but $T + Q = Q$ does not possess property (gw) because $\sigma_a(T + Q) = \{0\}$, $\sigma_{SBF_+^-}(T + Q) = \{0\}$ and $E(T + Q) = \{0\}$.
2)– The hypothesis of commutativity assumed in Theorem 3.1 is crucial. The following example shows that if we do not assume that N commutes with T, the result may fail. Let $X = \ell^2(\mathbb{N})$, and let T and N be defined by

$$T(x_1, x_2, x_3, ...) = (0, x_1/2, x_2/3, ...)\ ,\ N(x_1, x_2, x_3, ...) = (0, -x_1/2, 0, 0, ...).$$

Clearly N is a nilpotent operator which does not commute with T. Moreover, we have $E(T) = \Pi(T) = \emptyset$ and T^* has the SVEP. But $T+N$ does not possess property (gw), since $\sigma_a(T + N) = \{0\}$, $\sigma_{SBF_+^-}(T + N) = \{0\}$ and $E(T + N) = \{0\}$.

Corollary 3.3. *Let $T \in L(X)$ be a polaroid operator and let $N \in L(X)$ be a nilpotent operator commuting with T. If T^* has the SVEP, then $f(T) + N$ and $f(T + N)$ possess property (gw) for every $f \in \mathcal{H}(\sigma(T))$, which is not constant on any component of $\sigma(T)$.*

Proof. Since T^* has the SVEP then $f(T)^* = f(T^*)$ has the SVEP. As T is polaroid, from [12, Theorem 3.3] we have $E(f(T)) = \Pi(f(T))$. It then follows from Theorem 3.1 that $f(T) + N$ possesses property (gw). Now to show that $f(T + N)$ possesses property (gw). As we know that $\sigma(T) = \sigma(T+N)$ and $\Pi(T) = \Pi(T+N)$, then $T+N$ is also polaroid. Therefore $E(f(T+N)) = \Pi(f(T+N))$. Since $f(T+N)^* = f(T^*+N^*)$ has the SVEP, from Theorem 2.6 we have $f(T+N)$ possesses property (gw).

Theorem 3.4. *Let $T \in L(X)$ be an operator and let $K \in L(X)$ be an algebraic operator commuting with T. Then the following assertions hold.*
(i) If $E(T+K) = \Pi(T+K)$ and if T^* has SVEP, then $T+K$ possesses property (gw).
(ii) If $E(T^*+K^*) = \Pi(T+K)$ and if T has SVEP, then T^*+K^* possesses property (gw).

Proof. (i) Suppose that T^* has the SVEP. Since K^* is algebraic and commutes with T^*, it then follows by [6, Theorem 2.14] that T^*+K^* has the SVEP. Hence $T+K$ possesses property (gb), that is $\Delta_a^g(T+K) = \Pi(T+K)$. As by assumption $E(T+K) = \Pi(T+K)$, then $\Delta_a^g(T+K) = E(T+K)$, i.e. $T+K$ possesses property (gw).
(ii) Suppose that T has the SVEP. Since K is algebraic and commutes with T, it then follows by [6, Theorem 2.14] that $T+K$ has the SVEP. Hence T^*+K^* possesses property (gb), that is $\Delta_a^g(T^*+K^*) = \Pi(T^*+K^*)$. As by assumption $E(T^*+K^*) = \Pi(T+K)$, we then have $\Delta_a^g(T^*+K^*) = E(T^*+K^*)$ and T^*+K^* possesses property (gw).

Examples and applications 3.5. 1) Let R be the unilateral right shift operator on $\ell^2(\mathbb{N})$. Then $\sigma_a(R) = C(0,1)$. So $\text{iso}\sigma_a(R) = \emptyset$ and R has the SVEP. Hence if F is a finite rank operator commuting with R, then $R+F$ possesses property (gw).
2) Let T be a not quasinilpotent weighted right shift operator on $\ell^p(\mathbb{N})$, it then follows from [24, Proposition 1.6.15] that $\text{iso}\sigma_a(T) = \emptyset$. Moreover, such operators have the SVEP. Hence if F is a finite rank operator commuting with T, then $T+F$ possesses property (gw).
3) An operator $T \in L(X)$ is said to have the property $H(p)$ if for all $\lambda \in \mathbb{C}$, there exists $p = p(\lambda) \in \mathbb{N}$ such that $H_0(T-\lambda I) = \ker((T-\lambda I)^p)$. It is well known that such operators have the SVEP. Moreover, if T has the property $H(p)$ and K is an algebraic operator commuting with T, then $T+K$ is polaroid (see [6, Theorem 2.14]). So if T has the property $H(p)$ and K is an algebraic operator commuting with T, then T^*+K^* possesses property (gw).
4) Let H be a Hilbert space, an operator T acting on H is called paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for all $x \in H$. Examples of paranormal operators are p-hyponormal operators or log-hyponormal operators (see [22]). It follows from [20, Lemma 2.3] that a paranormal operator is polaroid. Consequently, if F is a finite rank operator commuting with T, then $T+F$ is also polaroid and so $E(T^*+F^*) = \Pi(T+F)$. Moreover, a paranormal operator has the SVEP, see [18, Corollary 2.10]. So if T is a pranormal operator and F is a finite rank operator commuting with T, then T^*+F^* possesses property (gw).
5) Let $\mathcal{A}$ be a semi-simple regular Tauberian commutative Banach algebra and T be a multiplier on $\mathcal{A}$. It is well known that such operators have the SVEP. Also from [1, Theorem 4.36], we know that T is polaroid. If $F \in L(\mathcal{A})$ is a finite rank operator commuting with T, then $T+F$ is also polaroid. Hence T^*+F^* possesses property (gw). On the other hand, since T has the SVEP, T satisfies generalized Browder's theorem that is $\Delta^g(T) = \Pi(T)$. From [10, Theorem 2.8], $T+F$ satisfies generalized Browder's theorem. Now if $\lambda \in \Delta_a^g(T+F)$ be arbitrary, then $T+F-\lambda I$ is an upper

semi-B-Fredholm operator such that $\text{ind}(T+F-\lambda I) \leq 0$. Therefore $T-\lambda I$ is also an upper semi-B-Fredholm operator such that $\text{ind}(T-\lambda I) = \text{ind}(T+F-\lambda I)$. As T is a multiplier, then from [9, Corollary 2.2] we know that $T-\lambda I$ is a B-Weyl operator and so $\text{ind}(T-\lambda I) = \text{ind}(T+F-\lambda I) = 0$. Hence $T+F$ possesses property (gb), see [15, Theorem 2.12]. Since $T+F$ is polaroid, $T+F$ possesses property (gw).

Theorem 3.6. *Let $T \in L(X)$ and let $K \in L(X)$ be a compact operator commuting with T such that $\Pi_a^0(T+K) \subset \sigma_a(T)$. If T possesses property (b), then $T+K$ possesses property (b).*

Proof. From [17, Theorem 3.1], we only have to show that $\Pi_a^0(T+K) = \Pi^0(T+K)$. Let $\lambda \in \Pi_a^0(T+K)$ be arbitrary, then $\lambda \notin \sigma_{SF_+^-}(T+K)$. Since K is a compact operator and commutes with T, we know that $\sigma_{SF_+^-}(T+K) = \sigma_{SF_+^-}(T)$. So $\lambda \notin \sigma_{SF_+^-}(T)$. As by hypothesis $\lambda \in \sigma_a(T)$, then $\lambda \in \Delta_a(T)$ and since T possesses property (b), we have $\lambda \in \Pi^0(T)$ which implies that $\lambda \notin \sigma_b(T)$. As by [1, Corollary 3.49], we have $\sigma_b(T) = \sigma_b(T+K)$, then $\lambda \notin \sigma_b(T+K)$. Since $\lambda \in \sigma(T+K)$, it follows that $\lambda \in \Pi^0(T+K)$. Hence $\Pi_a^0(T+K) \subset \Pi^0(T+K)$. As we always have that $\Pi_a^0(T+K) \supset \Pi^0(T+K)$, then $\Pi_a^0(T+K) = \Pi^0(T+K)$. Hence $T+K$ possesses property (b).

Remark 3.7. 1) It is proved by M. Berkani and H. Zariouh in [17, Theorem 3.3] that if $T \in L(X)$ is an operator possessing property (b) such that $\Pi_a^0(T+F) \subset \sigma_a(T)$, where F is a finite rank operator commuting with T, then $T+F$ possesses property (b). So Theorem 3.6 extends this result to commuting compact perturbations. It extends also [17, Theorem 3.4] which establishes the same result under the stronger hypothesis that $\Delta_a(T+K) \subset \sigma_a(T)$ where K is compact and commutes with T. We observe that if $\Delta_a(T+K) \subset \sigma_a(T)$ then $\Pi_a^0(T+K) \subset \sigma_a(T)$. We also remark that Theorem 3.6 extends a result of P. Aiena and M. T. Biondi [3, Theorem 2.21] which establishes that if T is an operator possessing property (b) and if F is a finite rank operator commuting with T such that $\sigma_a(T) = \sigma_a(T+F)$, then $T+F$ possesses property (b). We observe that if $\sigma_a(T) = \sigma_a(T+F)$ then $\Pi_a^0(T+F) \subset \sigma_a(T)$.
2) It is proved by M. Berkani, M. Sarih and H. Zariouh in [14, Theorem 2.2] that property (b) is satisfied by every operator $T \in L(X)$ whenever its dual T^* has the SVEP on the complement of $\sigma_{SF_+^-}(T)$. This result extends [3, Theorem 2.16] which establishes that if $T \in L(X)$ is an operator satisfying the supplementary condition that $H_0(T-\lambda I) = \ker((T-\lambda I)^p)$ for all $\lambda \in E_a^0(T)$ with $p = p(\lambda) \in \mathbb{N}$ and if T^* has the SVEP, then T possesses property (b).
3) As it had been already mentioned, we have that if T^* has the SVEP and if $F \in L(X)$ is a finite rank operator which commutes with T, then $T^* + F^*$ has the SVEP and so $T+F$ possesses property (b). This result gives a stronger version of [3, Theorem 2.22] where the assumptions that T is a polaroid operator and that T possesses property (b) have been assumed.

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F. Lombarkia
Department of Mathematics, Faculty of Science, University of Batna, B.P 05000, Batna, Algeria
E-mail address: `lombarkiafarida@yahoo.fr`

H. Zariouh
Centre de formation des instituteurs (CFI)-Oujda, B.P 458, Oujda, Morocco
E-mail address: `h.zariouh@yahoo.fr`