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LEIBNIZ RULES AND INTEGRAL ANALOGUES FOR FRACTIONAL DERIVATIVES VIA A NEW TRANSFORMATION FORMULA

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Abstract. Recently, Tremblay et al. invented a very interesting transformation formula for fractional derivatives of arbitrary order. Also, the authors have obtained a new generalized Leibniz rule and a corresponding integral analogue for the fractional derivatives of the product of two functions. In this paper, we apply the new transformation formula on the classical generalized Leibniz rule and the corresponding integral analogue due to Osler and on those established by the authors. Some special cases are given.

1. Introduction

The fractional derivative of arbitrary order α (integral, rational, irrational or complex) is an extension of the familiar nth derivative $D^n_{g(z)}F(z) = d^nF(z)/(dg(z))^n$ of the function $F(z)$ with respect to $g(z)$ to non-integral values of n and denoted by $D^\alpha_{g(z)}F(z)$. The concept has been introduced in many ways to generalize classical results of the nth order derivative to fractional order. For a general survey of the different approach used to define fractional derivatives the reader should read [29]. Many examples of the use of fractional derivatives appear in the literature : ordinary [12] and partial differential equations [6, 8, 27], integral equations [7, 8, 11], differential equations of non-integer order. Many others applications have been investigated through various field of science and engineering[1, 9, 17, 20, 28, 29]. Particularly, the Leibniz rule has been effective in the summation of infinite series just as his integral analogue in the evaluation of definite integrals [19, 21, 24, 25].

Studies of a Leibniz rule for derivatives of arbitrary order date back to 1832 when Liouville [16, p.117] gave the case

$$D^\alpha_z u(z)v(z) = \sum_{n=0}^{\infty} \left(\begin{array}{c} \alpha \\ n \end{array} \right) D^{\alpha-n}_z u(z) D^n_z v(z). \tag{1.1}$$

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Liouville used a fractional derivative based on the fact that $D^n e^{az} = a^n e^{az}$, $n = 0, 1, 2, ...$, could be generalized for arbitrary α by $D^\alpha e^{az} = a^\alpha e^{az}$. In 1867 and 1868 A.K. Grunwald [10, pp.406-468] and A.V. Letnikov [15] found (1.1) by starting with the well-known Riemann-Liouville integral representation for fractional derivative

$$D_z^\alpha f(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z \frac{f(\zeta)\mathrm{d}\zeta}{(z-\zeta)^{\alpha+1}} \tag{1.2}$$

which is valid for $Re(\alpha) < 0$.

Considering a derivative of arbitrary order $\alpha \in \mathbb{C}$ related with the Cauchy integral formula [2, 3, 4, 18, 21, 22]

$$D_z^\alpha z^p f(z) = \frac{\Gamma(1+\alpha)}{2\pi i} \int_0^{(z+)} f(\xi)\xi^p(\xi - z)^{-\alpha-1}\mathrm{d}\xi \tag{1.3}$$

where the contour is a single loop beginning at $\xi = 0$ encloses the point $\xi = z$ once in the positive direction and returns to $\xi = 0$ without cutting the branch line for $\xi^p(\xi - z)^{-\alpha-1}$, Osler [21] obtained a more general form of (1.1)

$$D_z^\alpha z^{p+q} u(z)v(z) = \sum_{n=-\infty}^{\infty} \begin{pmatrix} \alpha \\ \gamma + n \end{pmatrix} D_z^{\alpha-\gamma-n} z^p u(z) D_z^{\gamma+n} z^q v(z) \tag{1.4}$$

which yields for α not a negative integer, γ an arbitrary complex number, $Re(p) > -1$, $Re(q) > -1$ and $Re(p+q) > -1$.

Hereafter in 1972, Osler [24] presents a further extension of (1.2) based on the generalization of the Taylor series for fractional derivatives [23] and the concept of fractional derivatives with respect to a function $g(z)$. Thereby, he found:

Theorem A. *(i) Let $u(z)$ and $v(z)$ be analytic in the simply connected $\mathcal{R}$. (ii) Let $g(z)$ be regular and univalent function for $g^{-1}(\mathcal{R})$ such that $g^{-1}(0)$ is an interior or a boundary point of $\mathcal{R}$. Then, for $0 < a \leq 1$, $\alpha \in \mathbb{C}$, $\alpha \neq$ negative integer, $\gamma \in \mathbb{C}$, $Re(p) > -1$, $Re(q) > -1$ and $Re(p+q) > -1$, the following Leibniz rule holds true*

$$D_{g(z)}^\alpha g(z)^{p+q} u(z)v(z) = a \sum_{n=-\infty}^{\infty} \begin{pmatrix} \alpha \\ \gamma + an \end{pmatrix} D_{g(z)}^{\alpha-\gamma-an} g(z)^p u(z) D_{g(z)}^{\gamma+an} g(z)^q v(z). \tag{1.5}$$

Letting $a \to 0^+$ in (1.5), Osler [25] obtained the integral analogue of the Leibniz rule, namely:

Theorem B. *Assume the hypothesis of Theorem A, then the following integral analogue holds true*

$$D_{g(z)}^\alpha g(z)^{p+q} u(z)v(z) = \int_{-\infty}^{\infty} \begin{pmatrix} \alpha \\ \gamma + \omega \end{pmatrix} D_{g(z)}^{\alpha-\gamma-\omega} g(z)^p u(z) D_{g(z)}^{\gamma+\omega} g(z)^q v(z)\, d\omega. \tag{1.6}$$

At this point, we need the following definition of fractional derivative in the complex plane using a Pochhammer's contour of integration introduced in [14] (see also [13, 30]).

Definition 1.1. *Let $f(z)$ be analytic in a simply connected region $\mathcal{R}$. Let $g(z)$ be regular and univalent on $\mathcal{R}$ and let $g^{-1}(0)$ be an interior point of $\mathcal{R}$ then if α is* *not a negative integer, p is not an integer, and z is in* $\mathcal{R} - \{g^{-1}(0)\}$, *we define the fractional derivative of order* α *of* $g(z)^p f(z)$ *with respect to* $g(z)$ *by*

$$D^{\alpha}_{g(z)} g(z)^p f(z)$$
$$= \frac{e^{-i\pi p}\Gamma(1+\alpha)}{4\pi \sin(\pi p)} \int_{C(z+,g^{-1}(0)+,z-,g^{-1}(0)-;F(a),F(a))} \frac{f(\xi)g(\xi)^p g'(\xi)}{(g(\xi)-g(z))^{\alpha+1}} d\xi \qquad (1.7)$$

For non-integer α *and p, the functions* $g(\xi)^p$ *and* $(g(\xi)-g(z))^{-\alpha-1}$ *in the integrand have two branch lines which begin respectively at* $\xi = z$ *and* $\xi = g^{-1}(0)$, *and both pass through the point* $\xi = a$ *without crossing the Pochhammer contour* $P(a) = \{C_1 \cup C_2 \cup C_3 \cup C_4\}$ *at any other point as shown in Figure 1.* $F(a)$ *denotes the principal value of the integrand in (1.7) at the beginning and ending point of the Pochhammer contour* $P(a)$ *which is closed on Riemann surface of the multiple-valued function* $F(\xi)$.

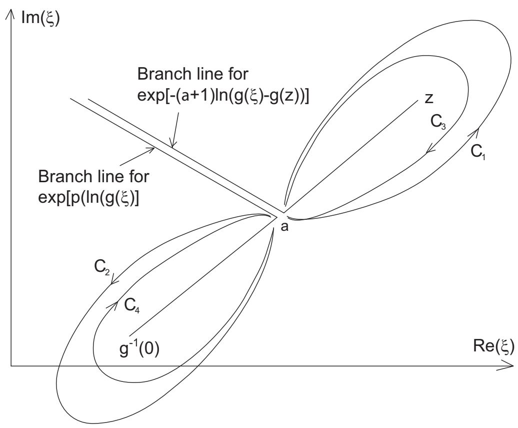

FIGURE 1. Pochhammer's contour

Making use of this less restrictive definition for fractional derivatives Lavoie et al. in [14] have shown that $Re(p) > -1$ and $Re(q) > -1$ are unnecessary conditions in (1.6). Moreover, in the definition used by Osler for fractional derivatives with respect to an arbitrary function $g(z)$, the function $f(z)$ must be analytic at $\xi = g^{-1}(0)$. One of the most important advantage of using the Pochhammer's contour representation for fractional derivatives is the fact that we can allow $f(z)$ to have an essential singularity at $\xi = g^{-1}(0)$. For a complete study on the properties of fractional derivative defined on Pochhammer's contour the reader should read [13, 14, 30].

Recently, the authors [31] obtained two new results involving the fractional derivatives of arbitrary order. Explicitly, they established a new generalized Leibniz type rule for fractional derivatives as well as the corresponding integral analogue. These two results are stated respectively as Theorem C and Theorem D below.

Theorem C. *(i) Let* $\mathcal{R}$ *be a simply connected region containing the origin. (ii) Let* $u(z)$ *and* $v(z)$ *satisfy the conditions of Definition 1.1 for the existence of the fractional derivative. (iii) Let* $\mathcal{U} \subset \mathcal{R}$ *being the region of analyticity of the function*

$u(z)$ and $\mathcal{V} \subset \mathcal{R}$ being the one for the function $v(z)$. (iv) Let $g(z)$ be a regular and univalent function for $z \in g^{-1}(\mathcal{R})$ then for $z \neq g^{-1}(0)$, $z \in \mathcal{U} \cap \mathcal{V}$, $Re(1-\beta) > 0$ and for $0 < a \leq 1$, the following product rule holds

$$D^{\alpha}_{g(z)} g(z)^{\alpha+\beta-1} u(z) v(z) = \frac{g(z)\sin(\beta\pi)\Gamma(1+\alpha)}{\sin((\alpha+\beta)\pi)\sin((\beta-\mu-\nu)\pi)\sin((\mu+\nu)\pi)} \cdot \sum_{n=-\infty}^{\infty} a \frac{\sin((\mu+an)\pi)\sin((\alpha+\beta-\mu-an)\pi)}{\Gamma(2+\alpha+\nu-an)\Gamma(-\nu+an)} \cdot D^{\alpha+\nu+1-an}_{g(z)} g(z)^{\alpha+\beta-\mu-1-an} u(z) D^{-\nu-1+an}_{g(z)} g(z)^{\mu-1+an} v(z). \tag{1.8}$$

Theorem D. *Assuming the hypotheses of Theorem C, the following integral analogue of (1.8) holds*

$$D^{\alpha}_{g(z)} g(z)^{\alpha+\beta-1} u(z) v(z) = \frac{g(z)\sin(\beta\pi)\Gamma(1+\alpha)}{\sin((\alpha+\beta)\pi)\sin((\beta-\mu-\nu)\pi)\sin((\mu+\nu)\pi)} \cdot \int_{-\infty}^{\infty} \frac{\sin((\mu+\omega)\pi)\sin((\alpha+\beta-\mu-\omega)\pi)}{\Gamma(2+\alpha+\nu-\omega)\Gamma(-\nu+\omega)} \cdot D^{\alpha+\nu+1-\omega}_{g(z)} g(z)^{\alpha+\beta-\mu-1-\omega} u(z) D^{-\nu-1+\omega}_{g(z)} g(z)^{\mu-1+\omega} v(z)\, d\omega. \tag{1.9}$$

Moreover, Tremblay et al. [32] found a really interesting transformation formula for fractional derivatives. Namely, they obtained the following result:

Theorem E. *Let $f(z)$ be a function that satisfies the conditions, listed in the definition (1.1), for the existence of the fractional derivative $D^{\alpha}_{z-b}(z-b)^p f(z)$ with $g(z) = z-b$ and using a Pochhammer contour $P(a) = C_1 \cup C_2 \cup -C_1 \cup -C_2$ laid out around the points $g^{-1}(0) = b$ and z (see Figure 1). If $f(b) \neq 0$ and $p \neq -1, -2, ...$ then we have*

$$D^{\alpha}_{z-b}(z-b)^p f(z) = \frac{\Gamma(1+p)}{\Gamma(-\alpha)} D^{-p-1}_{z-b}(z-b)^{-\alpha-1} f(w+b-z)\bigg|_{w=z} \tag{1.10}$$

for $z \in \mathcal{R} - \{b\}$. Note that we must have $w \to z$ in the right side of (1.10) after the evaluation of the fractional derivative, the point w must be near the point z inside of the loop C_1.

Or, in a more general form, they found

Theorem F. *Let $f(z)$ be a function that satisfies the conditions, listed in the definition (1.1) for the existence of the fractional derivative $D^{\alpha}_{g(z)}(g(z))^p f(z)$ and using a Pochhammer contour $P(a) = C_1 \cup C_2 \cup -C_1 \cup -C_2$ laid out around the points $g^{-1}(0)$ and z (see Figure 1). If $f(g^{-1}(0)) \neq 0$ and $p \neq -1, -2, ...$ then we have*

$$D^{\alpha}_{g(z)}(g(z))^p f(z) = \frac{\Gamma(1+p)}{\Gamma(-\alpha)} D^{-p-1}_{g(z)}(g(z))^{-\alpha-1} f\left(g^{-1}(g(w)-g(z))\right)\bigg|_{w=z} \tag{1.11}$$

for $z \in \mathcal{R} - \{g^{-1}(0)\}$. Note that we must have $w \to z$ in the right side of (1.11) after the evaluation of the fractional derivative, the point w must be near the point z inside of the loop C_1.

In this paper, we apply the new transformation formula (Theorem F) for fractional derivatives to both Leibniz rules for the fractional derivatives of the product of two functions (Theorem A and Theorem C) as well as their integral analogues. In section 3, we give some special cases involving special functions of mathematical physics for each the new formulas obtained.

2. New forms of Leibniz rules and of integral analogues

In this section, we use the general transformation formula (Theorem F) for fractional derivatives in order to obtain new expressions for the Leibniz rules (Theorems A and C) as well as their integral analogues (Theorems B and D). These new expressions are stated as Theorem 2.1 to Theorem 2.4 below.

Theorem 2.1. *Let $u(z)$ and $v(z)$ be functions that satisfy the conditions, listed in the definition (1.1) for the existence of the fractional derivative $D^{\alpha}_{g(z)}(g(z))^p u(z)v(z)$, $D^{\alpha}_{g(z)}(g(z))^p u(z)$ and $D^{\alpha}_{g(z)}(g(z))^p v(z)$ and using a Pochhammer contour $P(a) = C_1 \cup C_2 \cup -C_1 \cup -C_2$ laid out around the points $g^{-1}(0)$ and z (see Figure 1) Then, for $0 < a \leq 1$, $\alpha \in \mathbb{C}$, $\alpha \neq$ negative integer, $\gamma \in \mathbb{C}$, $Re(p+q) > -1$, $p \neq -1, -2, ...$, $q \neq -1, -2, ...$ and the following Leibniz rule holds true*

$$D^{\alpha}_{g(z)} g(z)^{p+q} u(z)v(z) = a \sum_{n=-\infty}^{\infty} \binom{\alpha}{\gamma + an} \frac{\Gamma(1+p)\Gamma(1+q)}{\Gamma(-\alpha+\gamma+an)\Gamma(-\gamma-an)} \tag{2.1}$$
$$\cdot D^{-p-1}_{g(z)} g(z)^{-\alpha+\gamma+an-1} u\left(g^{-1}(g(w)-g(z))\right)\Big|_{w=z}$$
$$\cdot D^{-q-1}_{g(z)} g(z)^{-\gamma-an-1} v\left(g^{-1}(g(w)-g(z))\right)\Big|_{w=z}.$$

Note that we must have $w \to z$ after the evaluation of the fractional derivatives, the point w must be near the point z inside of the loop C_1.

Proof. Applying the transformation formula for fractional derivatives (1.11) on each operator of fractional derivatives involved in the R.H.S. of the Leibniz rule (1.5), we obtain (2.1). □

Theorem 2.2. *Assume the hypotheses of Theorem 2.1 then the following integral analogue of the Leibniz rule holds true*

$$D^{\alpha}_{g(z)} g(z)^{p+q} u(z)v(z) = \int_{-\infty}^{\infty} \binom{\alpha}{\gamma + \omega} \frac{\Gamma(1+p)\Gamma(1+q)}{\Gamma(-\alpha+\gamma+\omega)\Gamma(-\gamma-\omega)} \tag{2.2}$$
$$\cdot D^{-p-1}_{g(z)} g(z)^{-\alpha+\gamma+\omega-1} u\left(g^{-1}(g(w)-g(z))\right)\Big|_{w=z}$$
$$\cdot D^{-q-1}_{g(z)} g(z)^{-\gamma-\omega-1} v\left(g^{-1}(g(w)-g(z))\right)\Big|_{w=z} d\omega.$$

Note that we must have $w \to z$ after the evaluation of the fractional derivatives, the point w must be near the point z inside of the loop C_1.

Proof. Applying the transformation formula for fractional derivatives (1.11) on each operator of fractional derivatives involved in the in the R.H.S. of the integral analogue of the Leibniz rule (1.6), we obtain (2.2). □

Theorem 2.3. *(i) Let $\mathcal{R}$ be a simply connected region containing the origin. (ii) Let $u(z)$ and $v(z)$ satisfy the conditions of Definition 1.1 for the existence of the fractional derivative. (iii) Let $\mathcal{U} \subset \mathcal{R}$ being the region of analyticity of the function $u(z)$ and $\mathcal{V} \subset \mathcal{R}$ being the one for the function $v(z)$. (iv) Let $g(z)$ be a regular and univalent function for $z \in g^{-1}(\mathcal{R})$ then for $z \neq g^{-1}(0)$, $z \in \mathcal{U} \cap \mathcal{V}$, $Re(1-\beta) > 0$, $\alpha+\beta-\mu-an \neq -1,-2,...$, $\mu+an \neq -1,-2,...$ and for $0 < a \leq 1$, the following product rule holds*

$$D^{\alpha}_{g(z)} g(z)^{\alpha+\beta-1} u(z)v(z) = \frac{g(z)\sin(\beta\pi)\Gamma(1+\alpha)}{\sin((\alpha+\beta)\pi)\sin((\beta-\mu-\nu)\pi)\sin((\mu+\nu)\pi)}$$
$$\cdot \sum_{n=-\infty}^{\infty} a \frac{\sin((\mu+an)\pi)\sin((\alpha+\beta-\mu-an)\pi)}{\Gamma(2+\alpha+\nu-an)\Gamma(-\nu+an)} \frac{\Gamma(\alpha+\beta-\mu-an)\Gamma(\mu+an)}{\Gamma(-\alpha-\nu-1+an)\Gamma(\nu+1-an)} \tag{2.3}$$
$$\cdot D^{-\alpha-\beta+\mu+an}_{g(z)} g(z)^{-\alpha-\nu-2+an} u\left(g^{-1}(g(w)-g(z))\right)\Big|_{w=z}$$
$$\cdot D^{-\mu-an}_{g(z)} g(z)^{\nu-an} v\left(g^{-1}(g(w)-g(z))\right)\Big|_{w=z}.$$

Note that we must have $w \to z$ after the evaluation of the fractional derivatives, the point w must be near the point z inside of the loop C_1.

Proof. Applying the transformation formula for fractional derivatives (1.11) on each operator of fractional derivatives involved in the R.H.S. of the Leibniz rule (1.8), we obtain (2.3). □

Theorem 2.4. *Assume the hypotheses of Theorem 2.3 then the following integral analogue of the new Leibniz rule (1.8) holds true*

$$D^{\alpha}_{g(z)} g(z)^{\alpha+\beta-1} u(z)v(z) = \frac{g(z)\sin(\beta\pi)\Gamma(1+\alpha)}{\sin((\alpha+\beta)\pi)\sin((\beta-\mu-\nu)\pi)\sin((\mu+\nu)\pi)}$$
$$\cdot \int_{-\infty}^{\infty} \frac{\sin((\mu+\omega)\pi)\sin((\alpha+\beta-\mu-\omega)\pi)}{\Gamma(2+\alpha+\nu-\omega)\Gamma(-\nu+\omega)} \frac{\Gamma(\alpha+\beta-\mu-\omega)\Gamma(\mu+\omega)}{\Gamma(-\alpha-\nu-1+\omega)\Gamma(\nu+1-\omega)} \tag{2.4}$$
$$\cdot D^{-\alpha-\beta+\mu+\omega}_{g(z)} g(z)^{-\alpha-\nu-2+\omega} u\left(g^{-1}(g(w)-g(z))\right)\Big|_{w=z}$$
$$\cdot D^{-\mu-\omega}_{g(z)} g(z)^{\nu-\omega} v\left(g^{-1}(g(w)-g(z))\right)\Big|_{w=z} d\omega.$$

Note that we must have $w \to z$ after the evaluation of the fractional derivatives, the point w must be near the point z inside of the loop C_1.

Proof. Applying the transformation formula for fractional derivatives (1.11) on each operator of fractional derivatives involved in the R.H.S. of the integral analogue of the new Leibniz rule (1.9), we obtain (2.4). □

Remark 2.5. *Theorem 2.1 to Theorem 2.4 have been obtained by applying the transformation formula (1.11) on each of the fractional derivative operators appearing in the R.H.S. of Theorem A to Theorem D. We could also obtain similar expressions by simply using the transformation formula on just one fractional derivative operator involved in the Theorem A to Theorem D.*

3. Some special cases

In this section, we examine some interesting special cases which can be obtained from the main formulas (2.1), (2.2), (2.3) and (2.4) by choosing specific functions $u(z)$, $v(z)$, $g(z)$ and parameters. These different forms of formulas will imply special functions of the mathematical physics such those appearing in table 1.

Remark 3.1. *In the following examples, the fractional derivatives $D^{\alpha}_{g(z)}g(z)^p f(g(z))$ encountered can be computed by using the fundamental formula*

$$D^{\alpha}_{g(z)}(g(z))^p = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)}g(z)^{p-\alpha}$$

and by differentiating the power series $\sum_n f_n(g(z))^n$ term by term. We get

$$D^{\alpha}_{g(z)}(g(z))^p f(g(z)) = \sum_n \frac{\Gamma(1+p+n)}{\Gamma(1+p-\alpha+n)} f_n(g(z))^{p-\alpha+n}.$$

Example 1. *Setting $u(z)=1$, $v(z)=(1+z)^{\lambda}$ and $g(z)=1-z$ in Theorem 2.1 and using the fractional derivatives representation for the Jacobi function (see Table 1), the following hypergeometric representation of Jacobi function (*[26, p. 254, eq. (2)]*)*

$$P^{(\alpha,\beta)}_{\mu}(z) = \frac{\Gamma(1+\alpha+\mu)}{\Gamma(1+\mu)\Gamma(1+\alpha)}\left(\frac{z+1}{2}\right)^{\mu} {}_2F_1\left[\begin{array}{ccc} -\mu, & -\beta-\mu; & \\ & & \frac{z-1}{z+1} \\ & 1-\alpha; & \end{array}\right] \tag{3.1}$$

and the fact that

$$D^{-q-1}_{1-z}(1-z)^{-\gamma-an-1}(2+w-z)^{\lambda}\Big|_{w=z} =$$
$$(1+z)^{\lambda}(1-z)^{q-\gamma-an}\frac{\Gamma(-\gamma-an)}{\Gamma(1+q-\gamma-an)}\ {}_2F_1\left[\begin{array}{ccc} -\lambda, & -\gamma-an; & \\ & & \frac{z-1}{z+1} \\ & 1+q-\alpha-an; & \end{array}\right], \tag{3.2}$$

we obtain for $0<a\le 1$

$$P^{(p+q-\alpha,\alpha-\lambda)}_{\lambda}(z) = \frac{a\Gamma(1+p)\Gamma(1+q)\Gamma(1+p+q-\alpha+\lambda)}{\Gamma(1+p+q)}$$
$$\cdot \sum_{n=-\infty}^{\infty}\left(\begin{array}{c}\alpha\\ \gamma+an\end{array}\right)\frac{\Gamma(1-q+\gamma+an)P^{(-q+\gamma+an,-\lambda+\gamma+an)}_{\lambda}(z)}{\Gamma(p-\alpha+\gamma+1+an)\Gamma(q-\gamma+1-an)\Gamma(1+\lambda-q+\gamma+an)}. \tag{3.3}$$

Example 2. *Putting $u(z)=\sin z$, $v(z)=1$ and $g(z)=z$ in Theorem 2.2 and making use of the elementary trigonometric identity*

$$\sin(w-z) = \sin w\cos z - \sin z\cos w, \tag{3.4}$$

we observe that

$$\begin{aligned} &D^{-p-1}_{z} z^{-\alpha+\gamma-1+\omega}\sin(w-z)\Big|_{w=z} = \\ &\sin z\cdot D^{-p-1}_{z} z^{-\alpha+\gamma-1+\omega}\cos z - \cos z\cdot D^{-p-1}_{z} z^{-\alpha+\gamma-1+\omega}\sin z \end{aligned} \tag{3.5}$$

Table 1
Some special functions expressed in terms of fractional derivatives

Name	Fractional derivative representation
Gauss hypergeometric function	${}_2F_1(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)}{\Gamma(\beta)}z^{1-\gamma}D_z^{\beta-\gamma}z^{\beta-1}(1-z)^{-\alpha}$
Degenerate hypergeometric function	${}_1F_1(\alpha;\beta;z)=\frac{\Gamma(\beta)}{\Gamma(\alpha)}z^{1-\beta}D_z^{\alpha-\beta}z^{\alpha-1}\mathrm{e}^z$
Generalized hypergeometric function	${}_{p+1}F_{q+1}(\alpha,a_1,...,a_p;\gamma,b_1,...,b_q;z)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)}z^{1-\gamma}D_z^{\alpha-\gamma}z^{\alpha-1}\ {}_pF_q(a_1,...,a_p;b_1,...,b_q;z)$
Bessel function	$J_\mu(z)=\frac{z^{-\mu}}{2^{\mu-1}\sqrt{\pi}}D_{z^2}^{-\mu+1/2}\sin z=\frac{z^{-\mu}}{2^\mu\sqrt{\pi}}D_{z^2}^{-\mu-1/2}\frac{\cos z}{z}=\frac{z^{-\mu}}{2^{\mu-\nu}}D_{z^2}^{-\mu+\nu}z^\nu J_\nu(z)$
Modified Bessel function	$I_\mu(z)=\frac{z^{-\mu}}{2^\mu\sqrt{\pi}}D_{z^2}^{-\mu-1/2}\frac{\cosh z}{z}$
Struve function	$H_\mu(z)=\frac{z^{-\mu}}{2^\mu\sqrt{\pi}}D_{z^2}^{-\mu-1/2}\frac{\sin z}{z}$
Modified Struve function	$L_\mu(z)=\frac{z^{-\mu}}{2^\mu\sqrt{\pi}}D_{z^2}^{-\mu-1/2}\frac{\sinh z}{z}$
Legendre function of the 1st kind	$P_\mu(z)=\frac{1}{\Gamma(1+\mu)}D_{1-z}^{\mu}(1-z^2)^\mu$
Associated Legendre function (1st kind)	$P_\mu^\nu(z)=\frac{(1-z^2)^{\nu/2}}{\Gamma(1+\mu)2^\mu}D_{1-z}^{\mu+\nu}(1-z^2)^\mu$
Associated Legendre function (2nd kind)	$Q_\mu^\nu(z)=\frac{\mathrm{e}^{i\pi\mu}\sqrt{\pi}\Gamma(1+\mu+\nu)\Gamma(-\mu-\nu)}{\Gamma(-1-2\nu)\Gamma(3/2+\nu)2^{\nu+1}}(1-z^2)^{\mu/2}D_{1-z}^{-1+\mu-\nu}(1-z^2)^{-1-\nu}$
Jacobi function	$P_\mu^{(\alpha,\beta)}(z)=\frac{\Gamma(1+\alpha+\mu)}{2^\mu\Gamma(1+\mu)\Gamma(1+\alpha+\beta+\mu)}(1-z)^{-\alpha}D_{1-z}^{\beta+\mu}(1-z)^{\alpha+\beta+\mu}(1+z)^\mu$
	$P_\mu^{(\alpha,\beta)}(z)=\frac{\Gamma(1+\beta+\mu)\mathrm{e}^{i\pi\mu}}{2^\mu\Gamma(1+\mu)\Gamma(1+\alpha+\beta+\mu)}(1+z)^{-\beta}D_{1+z}^{\alpha+\mu}(1+z)^{\alpha+\beta+\mu}(1-z)^\mu$
	$P_\mu^{(\alpha,\beta)}(z)=\frac{(z-1)^{-\alpha}(z+1)^{-\beta}}{2^\mu\Gamma(1+\mu)}D_{1+z}^{\mu}(z-1)^{\alpha+\mu}(z+1)^{\beta+\mu}$
Laguerre function	$L_\mu^{(\alpha)}(z)=\frac{z^{-\alpha}}{\Gamma(1+\mu)}\mathrm{e}^zD_z^\mu z^{\alpha+\mu}\mathrm{e}^{-z}=\frac{\Gamma(1+\mu+\alpha)}{\Gamma(1+\mu)\Gamma(-\mu)}z^{-\alpha}D_z^{-\alpha-\mu-1}z^{-\mu-1}\mathrm{e}^z$
Incomplete gammma function	$\gamma(\alpha,z)=\Gamma(\alpha)\mathrm{e}^{-z}D_z^{-\alpha}\mathrm{e}^z$
Psi function	$\Psi(\xi)=-\gamma+\ln(z)-\Gamma(\xi)z^{1-\xi}D_z^{1-\xi}\ln(z)$
Whittaker function	$M_{\mu;\nu}(z)=\frac{\Gamma(1+2\nu)}{\Gamma(1/2+\nu-\mu)}\mathrm{e}^{-z/2}z^{-3/2-\nu}D_z^{-1/2-\nu-\mu}z^{\nu-\mu-1/2}\mathrm{e}^z$

and thus, we have

$$ {}_2F_3\left[\begin{array}{ccc} \frac{2+p+q}{2}, & \frac{3+p+q}{2}; & \\ & & -z^2/4 \\ \frac{2+p+q-\alpha}{2}, & \frac{3+p+q-\alpha}{2}, & 3/2; \end{array}\right] = \frac{\Gamma(1+p)\Gamma(1+q)\Gamma(2+p+q-\alpha)}{\Gamma(2+p+q)} $$

$$ \left[\int_{-\infty}^{\infty} \frac{\binom{\alpha}{\gamma+\omega}\frac{\Gamma(-\alpha+\gamma+\omega)\sin z}{z\Gamma(1+p-\alpha+\gamma+\omega)}\, {}_2F_3\left[\begin{array}{ccc} \frac{-\alpha+\gamma+\omega}{2}, & \frac{1-\alpha+\gamma+\omega}{2}; & \\ & & -z^2/4 \\ \frac{1+p-\alpha+\gamma+\omega}{2}, & \frac{2+p-\alpha+\gamma+\omega}{2}, & 1/2; \end{array}\right]}{\Gamma(-\alpha+\gamma+\omega)\Gamma(1+q-\gamma-\omega)}\, d\omega \right. $$

$$ \left. -\int_{-\infty}^{\infty} \frac{\binom{\alpha}{\gamma+\omega}\frac{\Gamma(1-\alpha+\gamma+\omega)\cos z}{\Gamma(2+p-\alpha+\gamma+\omega)}\, {}_2F_3\left[\begin{array}{ccc} \frac{1-\alpha+\gamma+\omega}{2}, & \frac{2-\alpha+\gamma+\omega}{2}; & \\ & & -z^2/4 \\ \frac{2+p-\alpha+\gamma+\omega}{2}, & \frac{3+p-\alpha+\gamma+\omega}{2}, & 3/2; \end{array}\right]}{\Gamma(-\alpha+\gamma+\omega)\Gamma(1+q-\gamma-\omega)}\, d\omega \right]. \tag{3.6} $$

Example 3. *If $u(z) = 1$, $v(z) = L_k^{(a+b+1)}(z)$ and $g(z) = z$ in Theorem 2.3 where $L_k^{a+b+1}(z)$ are the generalized Laguerre polynomials of degree k* [26, p.200, eq.(1)] *defined by*

$$ L_k^{(\alpha)}(z) = \frac{(1+\alpha)_k}{k!}\left[\begin{array}{cc} -k; & \\ & z \\ 1+\alpha; & \end{array}\right] \tag{3.7} $$

and employing the following well known addition property for the generalized Laguerre polynomials [26, p.209, eq. (3)]

$$ L_k^{(a+b+1)}(x+y) = \sum_{i=0}^{k} L_i^{(a)}(x)L_{k-i}^{(b)}(y)\ , \tag{3.8} $$

we have

$$ D_z^{-\mu-an} z^{\nu-an} L_k^{(a+b+1)}(w-z)\Big|_{w=z} = \sum_{i=0}^{k} L_i^{(a)}(z) D_z^{-\mu-an} z^{\nu-an} L_{k-i}^{(b)}(-z) \tag{3.9} $$

and then we get, for $0 < a \leq 1$,

$$ {}_2F_2\left[\begin{array}{ccc} -k, & \alpha+\beta; & \\ & & z \\ 2+a+b, & \beta; & \end{array}\right] = \frac{a\ k!\Gamma(1-\alpha-\beta)\Gamma(1+\alpha)\Gamma(2-\beta+\mu+\nu)\Gamma(-\mu-\nu)}{(2+a+b)_k\Gamma(1-\beta)} $$

$$ \cdot\sum_{n=-\infty}^{\infty} \frac{\sum_{i=0}^{k} L_i^{(a)}(z)\frac{(1+\beta)_{k-i}}{(k-i)!}\, {}_2F_2\left[\begin{array}{ccc} -k+i, & 1+\nu-an; & \\ & & z \\ 1+b, & 1+\mu+\nu; & \end{array}\right]}{\Gamma(2+\alpha+\nu-an)\Gamma(-\nu+an)\Gamma(1-\mu-an)\Gamma(1-\alpha-\beta+\mu+an)}. \tag{3.10} $$

Example 4. *Now, if we set $u(z) = B_k(z)$, $v(z) = 1$ and $g(z) = z$ in Theorem 2.4. where $B_k(z)$ are the Bell polynomials of degree k* [5] *defined as follow:*

$$ B_k(z) = \sum_{i=0}^{k} S(k,i) z^i \tag{3.11} $$

and $S(k,i)$ are the Stirling number of the second kind [5] *defined by*

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\begin{pmatrix} k \\ i \end{pmatrix} i^n. \tag{3.12}$$

Considering the addition property for the Bell polynomials

$$B_k(x+y) = \sum_{i=0}^{k}\begin{pmatrix} k \\ i \end{pmatrix} B_i(x)B_{k-i}(y)\ , \tag{3.13}$$

we observe that

$$D_z^{-\alpha-\beta+\mu+an} z^{-\alpha-\nu-2+an} B_k(w-z)\Bigg|_{w=z}$$
$$= \sum_{i=0}^{k}\begin{pmatrix} k \\ i \end{pmatrix} B_i(z) D_z^{-\alpha-\beta+\mu+an} z^{-\alpha-\nu-2+an} B_{k-i}(-z). \tag{3.14}$$

We thus obtain

$$\sum_{i=0}^{k}\frac{S(k,i)\ (\alpha+\beta)_i\ z^i}{(\beta)_i} = \frac{\Gamma(1-\alpha-\beta)\Gamma(1+\alpha)\Gamma(2-\beta+\mu+\nu)\Gamma(-\mu-\nu)}{\Gamma(1-\beta)}$$
$$\cdot\int_{-\infty}^{\infty}\frac{\displaystyle\sum_{i=0}^{k}\begin{pmatrix} k \\ i \end{pmatrix} B_i(z)\sum_{j=0}^{k-i}\frac{S(k-i,j)(-\alpha-\nu-1+\omega)_j}{(\beta-\mu-\nu-1)_j}(-z)^j}{\Gamma(2+\alpha+\nu-\omega)\Gamma(-\nu+\omega)\Gamma(1-\mu-\omega)\Gamma(1-\alpha-\beta+\mu+\omega)}d\omega. \tag{3.15}$$

Remark 3.2. *It is important to note that several restrictions are to be imposed on the parameters involved in each of the preceding examples. So, we have to be careful when manipulating these last expressions. These are mentioned in the statements of Theorem 2.1 to Theorem 2.4.*

4. Conclusion

The usefulness of generalized Leibniz rule and the corresponding integral analogue to obtain new series expansion or definite integrals is a well known fact. We, thus, have presented here 4 new expressions. We found these new relationships by applying a new transformation formula for fractional derivatives given recently by Tremblay et al. [32]. Many examples have also been given in section 3.

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