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ON BASES IN THE SPACE OF VECTOR VALUED ENTIRE DIRICHLET SERIES OF TWO COMPLEX VARIABLES

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Abstract. The space of all entire functions represented by vector valued Dirichlet series of two complex variables is considered in this paper. It is equipped with two equivalent topologies. The main result of this paper is concerned with finding the conditions for a base in $X$to become a proper base and certain continuous linear operators which are used to determine the proper bases in$X$.

1. **Introduction**

Consider

$$f(s_1, s_2) = \sum_{m,n=0}^{\infty} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2),\ (s_j = \sigma_j + it_j\ ,\ j = 1, 2) \tag{1}$$

where $a'_{m,n}s$ belong to a commutative Banach algebra $(E, ||.||)$;$0 = \lambda_0 < \lambda_1 < ... < \lambda_m \to \infty$ as $m \to \infty$,$0 = \mu_0 < \mu_1 < ... < \mu_n \to \infty$ as $n \to \infty$ and

$$\lim_{m+n\to\infty} \sup \frac{\log(m+n)}{\lambda_m + \mu_n} = D < +\infty\,, \tag{2}$$

$$\lim_{m+n\to\infty} \sup \frac{\log(||a_{m,n}||)}{\lambda_m + \mu_n} = -\infty\ . \tag{3}$$

Then the Dirichlet series given in (1) represents a vector valued entire function

$f(s_1, s_2)$ in two complex variables (see[3]). Let Xdenote the space of all entire functions $f : C^2 \to E$defined as above by the vector valued Dirichlet series (1). In [1] and [2], S.Daoud obtained the properties of space of entire functions defined by Dirichlet
series of two complex variables. In this paper we study the properties of the space X of all entire functions defined by vector valued Dirichlet series.

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2. **Different Topologies and Bases in** X

Let us assume that $\{\sigma_1^{(k)}\}$and $\{\sigma_2^{(k)}\}$ are two non-decreasing sequences of positive numbers, $\sigma_1^{(k)} \to \infty$and $\sigma_2^{(k)} \to \infty$ with $k \to \infty$.For each $f \in X$given by (1), we define

$$||f;\sigma_1^{(k)},\sigma_2^{(k)}|| \; = \sum_{m,n=0}^{\infty} ||a_{m,n}|| \exp(\lambda_m \sigma_1^{(k)} + \mu_n \sigma_2^{(k)}). \tag{4}$$

Now from (3) we obtain

$$\lim_{m+n\to\infty} \inf \;\; \log ||a_{m,n}||^{-\,1/\,(\lambda_m+\mu_n)} = +\infty \; .$$

Then for an arbitrarily large number K , we have for $m+n \; > \; N'$,

$$||a_{m,n}|| \; < \; \exp[-K(\lambda_m + \mu_n)]\,.$$

Hence $||f;\sigma_1^{(k)},\sigma_2^{(k)}|| \; = \sum_{m,n=0}^{\infty} ||a_{m,n}|| \exp(\lambda_m \sigma_1^{(k)} + \mu_n \sigma_2^{(k)})$

$$= \sum_{m+n=0}^{N'} ||a_{m,n}|| \exp(\lambda_m \sigma_1^{(k)} + \mu_n \sigma_2^{(k)}) \; + \sum_{m+n\,\geq N'}^{\infty} ||a_{m,n}|| \exp(\lambda_m \sigma_1^{(k)} + \mu_n \sigma_2^{(k)})$$

$$< O(1) + \textstyle\sum_{m+n\,\geq N'}^{\infty} \exp[\lambda_m(\sigma_1^{(k)} - K) + \mu_n(\sigma_2^{(k)} - K)]$$

$< \; \infty$, since $K >> \; \sigma_1^{(k)}, \sigma_2^{(k)}$ can be chosen.

Then the series on right hand side of (4) converges and $||f;\sigma_1^{(k)},\sigma_2^{(k)}||$ defines a norm on X for each $k \, = \, 1, 2.....$Further, from (4), it follows that

$$||f;\sigma_1^{(k)},\sigma_2^{(k)}|| \; \leq \; ||f;\sigma_1^{(k+1)},\sigma_2^{(k+1)}||, \;\; \text{for all} \;\; k \geq 1.$$

With these countable number of norms $||f;\sigma_1^{(k)},\sigma_2^{(k)}||$, $(k \geq 1)$,
we define a metric topology on $X$with metric ρ defined as

$$\rho(f,g) = \sum_{k\geq 1} \frac{1}{2^k} . \frac{||f-g;\sigma_1^{(k)},\sigma_2^{(k)}||}{1+||f-g;\sigma_1^{(k)},\sigma_2^{(k)}||} \;,\;\; f,g \in \; X.$$

For each $f \in \; X$ and$0 < \sigma_1,\, \sigma_2 < \infty$, put

$$M(f;\sigma_1,\sigma_2) = \sup_{-\infty\,<t_1,t_2\,<\,\infty} ||f(\sigma_1 + it_1, \sigma_2 + it_2)|| \tag{5}$$

Then $M(f,\sigma_1,\sigma_2\,)$ defines a family of norms on X. Using this we define a metric topology generated by norms (5) as

$$\Im(f,g) = \sum_{j=1}^{\infty} \frac{1}{2^j} \, \frac{M(f-g,\sigma_1^{(j)},\sigma_2^{(j)})}{1+M(f-g,\sigma_1^{(j)},\sigma_2^{(j)})} \;, \qquad\qquad f,g \in \; X$$

where $\{\sigma_1^{(j)}\}$ and $\{\sigma_2^{(j)}\}$ are two non decreasing sequences of positive numbers, $\sigma_1^{(j)} \to \infty$ and $\sigma_2^{(j)} \to \infty$ as $j \to \infty$. It was proved earlier in [1] that for $\alpha > 0$

$$M(f;\sigma_1,\sigma_2) \leq \ ||f;\sigma_1,\sigma_2|| \ \leq K(\alpha)\ M(f;\sigma_1+\alpha,\sigma_2+\alpha)$$

where $K = \sum_{m,n=0}^{\infty} \exp[-\alpha(\lambda_m+\mu_n)]$ .Hence the two topologies defined on X by ρ and $\Im$ are equivalent.

Now we give the characterizations of certain types of bases in X.

A sequence$\{f_{m,n} : m,n \geq 0\} \subset X$ is said to be a base for X if for each $f \in X$, there exists a unique sequence $\{C_{m,n} : m,n \geq 0\} \subset E$,such that $f = \sum_{m,n=0}^{\infty} C_{m,n}\, f_{m,n}$ where the convergence of the infinite series being with respect to the topology on X.

Here $\{C_{m,n}\}$ are called the base functions. If $e_{m,n} \in X,\ e_{m,n}(s_1,s_2) = \exp(s_1\lambda_m + \mu_n s_2)\ ,\ m,n \geq 1$then each $f \in X$can be expressed as in (1) with coefficients $\{a_{m,n}\}$satisfying

$$\lim_{m+n\to\infty} \sup \frac{\log(||a_{m,n}||)}{\lambda_m+\mu_n} = -\infty\ .$$

Therefore $\{e_{m,n}\}$is a base in$X$.

A base $\{f_{m,n}\}$will be called a genuine base for X if the corresponding base functions satisfy (3).A sequence $\{f_{m,n}\}$will be called an absolute base for X if it is a base in $X$and the infinite series corresponding to each $f \in X$is absolutely convergent with respect to the topology on X.A sequence $\{f_{m,n}\}$will be called a proper base for X if it is a genuine and an absolute base for X.

3. **Main Results**

We now give the characterization of proper bases. We prove

Theorem 3.1. *Let $\{C_{m,n}\}$ be an arbitrary sequence contained in E satisfying*

$$\lim_{m+n\to\infty} \sup \frac{\log(||C_{m,n}||)}{\lambda_m+\mu_n} = -\infty\,. \tag{6}$$

and $\{\alpha_{m,n} : m,n \geq 0\} \subset X$, then the series $\Sigma M(C_{m,n}\,\alpha_{m,n}\,;\sigma_1,\sigma_2)$ converges if and only if

$$\lim_{m+n\to\infty} \frac{\log M(\alpha_{m,n};\sigma_1,\sigma_2)}{\lambda_m+\mu_n} < \infty\,. \tag{7}$$

for each σ_1,σ_2.

Proof. Let us assume that equation (7) is satisfied. Then for $\sigma_1,\sigma_2 > 0$ there exists a constant $\varepsilon_1 = \varepsilon_1(\sigma_1,\sigma_2) > 0$ such that

$$\log M(\alpha_{m,n};\sigma_1,\sigma_2) < \varepsilon_1(\lambda_m+\mu_n) \tag{8}$$

for $m+n \geq N(\varepsilon_1)$.

Let $\varepsilon_2 > \varepsilon_1$, then using (6) we find that exists $N_1 = N_1(\varepsilon_2)$, such that

$$||C_{m,n}|| \ \leq \ \exp\{-\varepsilon_2(\lambda_m+\mu_n)\}\ \ ;\ m+n \geq N_1. \tag{9}$$

From (8) and (9),we get

$$M(C_{m,n}\,\alpha_{m,n}\,;\sigma_1,\sigma_2) = ||C_{m,n}||\, M(\alpha_{m,n};\sigma_1,\sigma_2) \leq \exp(\varepsilon_1-\varepsilon_2)(\lambda_m+\mu_n)$$

$\forall m+n \geq N_2 = \max(N, N_1)$. In an earlier paper, we proved the following result in [?] :

$$\mu(f,\sigma_1,\sigma_2) \leq M(f,\sigma_1,\sigma_2) \leq K\mu(f,\sigma_1+\alpha,\sigma_2+\alpha) \text{ where } K = K(\alpha)$$

and $\mu(f,\sigma_1,\sigma_2) = \max\limits_{m,n\geq 0}\{||a_{m,n}|| \exp(\lambda_m\sigma_1+\mu_n\sigma_2)\}$ denotes the maximum term of the entire function $f(s_1,s_2)$. Using the above result, we find that $\sum_{m+n=0}^{\infty} M(C_{m,n}\,\alpha_{m,n}\,;\sigma_1,\sigma_2)$ converges for every σ_1,σ_2.

Conversely, suppose that (7) is false. Hence for some $\sigma_1,\sigma_2 > 0$, there exist sequences $\{m_p\}$ and $\{n_q\}$ such that

$$\log M(\alpha_{m_p,n_q};\sigma_1,\sigma_2) > (p+q)(\lambda_{m_p}+\mu_{n_q})\,, \quad p,q \geq 0. \tag{10}$$

Now we define the sequence $\{C_{m,n}\} \subset E$ such that

$$\log ||C_{m,n}|| = \begin{cases} \lambda_m+\mu_n - \log M(\alpha_{m,n};\sigma_1,\sigma_2) & for\, m=m_p; n=n_q \\ -\infty & for\, m\neq m_p; n\neq n_q \end{cases}$$

Then from (10), we have $\lim\limits_{m+n\to\infty} \sup \frac{\log(||C_{m,n}||)}{\lambda_m+\mu_n} = -\infty$ .Hence for the sequence defined above, equation (6) holds but

$$M(C_{m,n}\alpha_{m,n};\sigma_1,\sigma_2) = \begin{cases} C_{m_p,n_q}M(\alpha_{m_p,n_q};\sigma_1,\sigma_2) = \exp(\lambda_{m_p}+\mu_{n_q}), m=m_p, n=n_q, \\ 0 \ \text{ for } m\neq m_p\ ,\ n\neq n_q \end{cases}$$

and so $\Sigma M(C_{m,n}\,\alpha_{m,n}\,;\sigma_1,\sigma_2)$ does not converge. Hence equation (7) is true. This completes the proof of Theorem 1. □

Remark 1. *Theorem 3.1 above generalizes Theorem 3.2 of* [1].
Next we prove

Theorem 3.2. *Let $\{C_{m,n}\}$ be an arbitrary sequence contained in E and $\{\alpha_{m,n} : m,n \geq 0\} \subset X$ such that the series $\Sigma M(C_{m,n}\,\alpha_{m,n}\,;\sigma_1,\sigma_2)$ converges. Then*

$$\lim_{m+n\to\infty} \sup \frac{\log(||C_{m,n}||)}{\lambda_m+\mu_n} = -\infty\,. \tag{11}$$

if and only if

$$\lim_{\sigma_1,\sigma_2\to\infty} \left\{ \lim_{m+n\to\infty} \inf \frac{\log M(\alpha_{m,n};\sigma_1,\sigma_2)}{\lambda_m+\mu_n} \right\} = +\infty. \tag{12}$$

Proof. Let us assume that the equation (12) holds and (11) does not hold. Therefore for each $\sigma_1,\sigma_2 > 0$, series $\Sigma M(C_{m,n}\,\alpha_{m,n}\,;\sigma_1,\sigma_2)$ converges but

$$\lim_{m+n\to\infty} \sup \frac{\log(||C_{m,n}||)}{\lambda_m+\mu_n} \neq -\infty\,.$$

Now there exist sequences $\{m_p\},\{n_q\}$ of positive integers such that

$$\log(||C_{m,n}||) > \alpha(\lambda_{m_p}+\mu_{n_q}); \alpha > -\infty$$

By (12) we can find σ_1,σ_2, such that

$$\lim_{m+n\to\infty} \inf \frac{\log M(\alpha_{m,n};\sigma_1,\sigma_2)}{\lambda_m+\mu_n} > 1-\alpha$$

and $\frac{\log M(C_{m_p,n_q}\alpha_{m_p,n_q};\sigma_1,\sigma_2)}{\lambda_{m_p}+\mu_{n_q}} > 1\,, \quad p,q \geq 0\,.$
From this we find that $\Sigma M(C_{m,n}\,\alpha_{m,n}\,;\sigma_1,\sigma_2)$ does not converge .This proves the first part of the result .Conversely, let us take (11) to be true but (12) be not true. Then for some $\beta > 0$ and for each $\sigma_1, \sigma_2 > 0$

$$\lim_{\sigma_1,\sigma_2\to\infty}\left\{\lim_{m+n\to\infty}\inf\frac{\log M(\alpha_{m,n};\sigma_1,\sigma_2)}{\lambda_m+\mu_n}\right\} < \beta < +\infty.$$

Since $M(\alpha_{m,n};\sigma_1,\sigma_2)$ is monotonically increasing in $\sigma_1, \sigma_2 > 0$ for each fixed pair of integers(m, n) then there exist sequences $\{m_p\}, \{n_q\}$ such that

$$\log M(\alpha_{m_p,n_q};\sigma_1,\sigma_2) < \beta(\lambda_{m_p}+\mu_{n_q}).$$

Now we define $\{C_{m,n}\} \subset E$ as follows

$$\log||C_{m,n}|| = \begin{cases} -\beta(\lambda_m+\mu_n) & \text{for } m = m_p; n = n_q \\ -\infty & \text{for } m \neq m_p; n \neq n_q \end{cases}$$

Then for given σ_1, σ_2

$$\sum_{m,n=0}^{\infty}||C_{m,n}||\,M(\alpha_{m,n};\sigma_1,\sigma_2) \leq \sum_{m,n=0}^{\infty}\exp[-\beta(\lambda_m+\mu_n)] < +\infty$$

and therefore $\sum_{m,n=0}^{\infty} M(C_{m,n}\,\alpha_{m,n}\,;\sigma_1,\sigma_2)$converges for positive real numbersσ_1, σ_2. Now from the above, we get,

$$\lim_{m+n\to\infty}\sup\frac{\log(||C_{m,n}||)}{\lambda_m+\mu_n} = -\beta$$

that is, (11) does not hold. Hence (11) implies (12) and this proves the theorem.

Remark 2. *Theorem 3.2 above generalizes Lemma 3.3 of* [1]. *Our next result characterizes proper bases.*

Theorem 3.3. *Let*$\{\alpha_{m,n} : m, n \geq 0\}$*be an absolute base in*$X$*, then* $\{\alpha_{m,n}\}$ *is a proper base if and only if* (7) *and* (8) *hold good.*

Proof. Earlier we have shown that if $\{\alpha_{m,n} : m, n \geq 0\}$ is a proper base then

$$\lim_{m+n\to\infty}\sup\frac{\log M(\alpha_{m,n};\sigma_1,\sigma_2)}{\lambda_m+\mu_n} < \infty\,.$$

for each σ_1, σ_2.The result now follows on using the definitions of genuine and proper bases and the results in Theorems 3.1and 3.2.
Now we obtain the characterizations of linear continuous operators .We prove

Theorem 3.4. *Let* $\{\alpha_{m,n} : m, n \geq 0\} \subset X$*. Suppose* T *is a linear operator from* X*into* X*, such that* $T(\delta_{m,n}) = \alpha_{m,n} \quad m, n \geq 0$ *where* $\delta_{m,n}(s_1, s_2) = \exp(\lambda_m s_1 + \mu_n s_2)$*. Then* T *is a continuous operator on* X*if for each* $\sigma_1, \sigma_2 \geq 0$

$$\lim_{m+n\to\infty}\sup\frac{\log M(\alpha_{m,n};\sigma_1,\sigma_2)}{\lambda_m+\mu_n} < \infty\,. \tag{13}$$

Conversely, if equation (13) *holds, then there exists a continuous linear operator* $T : X \to X$ *such that*

$$T(\delta_{m,n}) = \alpha_{m,n} \quad m, n \geq 0.$$

Proof. Suppose that T is a continuous linear operator from X into X with $T(\delta_{m,n}) = \alpha_{m,n}$.Then for given $\sigma_1, \sigma_2 > 0$, there exists a positive constant K and numbers$\sigma_1^{'} > 0, \sigma_2^{'} > 0$ such that

$$M(T\delta_{m,n};\sigma_1,\sigma_2) = M(\alpha_{m,n};\sigma_1,\sigma_2) \leq KM(\delta_{m,n};\sigma_1^{'},\sigma_2^{'})$$

$$\leq K\exp\{(\lambda_m+\mu_n)\sigma\} \quad , \quad \sigma = \max(\sigma_1^{'},\sigma_2^{'}).$$

Hence

$$\lim_{m+n\to\infty}\sup\frac{\log M(\alpha_{m,n};\sigma_1,\sigma_2)}{\lambda_m+\mu_n} < \infty\,.$$

Thus equation (13) follows.
Conversely, assume that equation (13) is true. Let $\alpha \in X$, then α is represented by

$$\alpha = \sum_{m,n=0}^{\infty} a_{m,n}\delta_{m,n}$$

where the coefficients $a_{m,n}$'s satisfy equation (3).Since equation (13) holds, therefore there exists an $A_0 = A_0(\sigma_1,\sigma_2)$, depending on σ_1,σ_2 such that
$\lim_{m+n\to\infty} \frac{\log M(\alpha_{m,n};\sigma_1,\sigma_2)}{\lambda_m+\mu_n} \leq A_0$ for all $m+n \geq N$
or
$M(\alpha_{m,n};\sigma_1,\sigma_2) \leq \exp A_0(\lambda_m+\mu_n)$ for all $m+n \geq N$
Therefore, noticing that equation (3) is already valid for the coefficients $a_{m,n}$'s , we find that $\sum a_{m,n}\delta_{m,n}$ converges in Y and so it represents an elements of X.Hence there is natural transformation $T : X \to X$ such that:

$$T(\alpha) = \sum_{m,n=0}^{\infty} a_{m,n}\delta_{m,n}$$

with

$$\alpha = \sum_{m,n=0}^{\infty} a_{m,n}\delta_{m,n}.$$

Clearly $T(\delta_{m,n}) = \alpha_{m,n} \quad , m,n \geq 0$. We are now required to show that $T$is continuous on X with respect to the toplogy T.It is sufficient to show that T is continuous on (X,ρ). The norms $||...,\sigma_1,\sigma_2||$ are continuous on X , therefore given any two numbers $\sigma_1,\sigma_2 > 0$, we find

$$||T(\alpha),\sigma_1,\sigma_2|| \;=\; ||\lim_{p+q\to\infty}\sum_{m,n=0}^{p+q} a_{m,n}\ \alpha_{m,n},\sigma_1,\sigma_2||$$

$$\leq \lim_{p+q\to\infty}\sum_{m,n=0}^{p+q} ||a_{m,n}|| \quad ||\,\alpha_{m,n};\sigma_1,\sigma_2||$$

$$\leq \lim_{p+q\to\infty}\sum_{m,n=0}^{p+q} ||a_{m,n}|| \quad \exp(\lambda_m+\mu_n)\sigma = ||\alpha;\sigma,\sigma||,$$

where $\sigma = \sigma(\sigma_1,\sigma_2)$.Thus $T : (X, ||...,\sigma,\sigma||) \to (X, ||...,\sigma_1,\sigma_2||)$ is continuous and as σ_1,σ_2 are arbitrary, it follows that $T : X \to X$ is continuous.

Lastly we prove

Theorem 3.5. *If T is a linear operator from X into itself, such that T and T^{-1} are continuous then $\{T(\delta_{m,n});\ m,n \geq 0\}$ is a proper base in the closed subspace $T[X]$ of X. Conversely if $\{\alpha_{m,n}\ :\ m,n \geq 0\}$ is a proper base in a closed subspace Y of$X$, then there exists a continuous linear operator $T : X \to X$, such that $T(\delta_{m,n}) = \alpha_{m,n}$.*

Proof. Suppose that T is the linear operator mentioned in the hypothesis. Then $T[X]$ is a closed subspace of X.
Let $T(\delta_{m,n}) = \alpha_{m,n}$, $m,n \geq 0$ and let$f \in T[X]$, then

$$T^{-1}(f) = \sum_{m,n=0}^{\infty} a_{m,n}\delta_{m,n}$$

where $a_{m,n}$'s satisfy equation (3) . Now

$$\sum_{m,n=0}^{p+q} a_{m,n}\delta_{m,n} \to T^{-1}(f) \tag{14}$$

in X as $p+q \to \infty$.
Now T is continuous and linear, and equation (14) implies

$$f = \sum_{m,n=0}^{\infty} a_{m,n}\alpha_{m,n} \tag{15}$$

Since equation (13) holds, then $\sum M(a_{m,n}\alpha_{m,n}, \sigma_1, \sigma_2)$ converges for every real and positive$\sigma_1, \sigma_2$and the representation of f in (15) is unique. Since T^{-1} is continuous. We conclude that $\{\alpha_{m,n}\ :\ m,n \geq 0\}$ is a proper base for $T[X]$.
Conversely: let $\{\alpha_{m,n}\ :\ m,n \geq 0\}$ be a proper base for a closed subspace $Y$of X Hence equation (13) holds. Therefore by Theorem 3.4, there exists a continuous linear operator T on X into itself, such that $T(\delta_{m,n}) = \alpha_{m,n},\ m,n \geq 0$.
Now let $f \in X$, $\quad f \neq 0$, then f is represented by equation (1) whose coefficients $a_{m,n}$'s satisfy equation (3). Thus

$$T(f) = \sum_{m,n=0}^{\infty} a_{m,n}\alpha_{m,n} \neq 0$$

Therefore T is one-to-one. Hence T is a continuous algebraic isomorphism from X into X .Now apply the well known theorem of Banach ([1]) we find that T^{-1}exists and is continuous .Hence proof of the theorem is complete. □

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