Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 4 Issue 2 (2012), Pages 91-102.

OSCILLATION RESULTS FOR THIRD ORDER HALF-LINEAR NEUTRAL DIFFERENCE EQUATIONS

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Abstract. In this paper some new sufficient conditions for the oscillation of solutions of the third order half-linear difference equations

$$\Delta\left(a_n(\Delta^2(x_n+b_nh(x_{n-\delta})))^{\alpha}\right)+q_nf(x_{n+1-\tau})=0$$

and

$$\Delta\left(a_n(\Delta^2(x_n-b_nh(x_{n-\delta})))^{\alpha}\right)+q_nf(x_{n+1-\tau})=0$$

are established. Some examples are presented to illustrate the main results.

1. Introduction

In this paper we consider the following neutral type difference equations of the form

$$\Delta\left(a_n(\Delta^2(x_n+b_nh(x_{n-\delta})))^{\alpha}\right)+q_nf(x_{n+1-\tau})=0 \tag{1.1}$$

and

$$\Delta\left(a_n(\Delta^2(x_n-b_nh(x_{n-\delta})))^{\alpha}\right)+q_nf(x_{n+1-\tau})=0 \tag{1.2}$$

where $n\in\mathbb{N}(n_0)=\{n_0,n_0+1\dots\}$, n_0 is a nonnegative integer, subject to

i) f and h are real-valued continuous functions with $uh(u)>0$, and $uf(u)>0$ for all $u\neq 0$;

ii) there exist $M_1>0$ and $M_2>0$ such that $\dfrac{h(u)}{u}\leq M_1$ and $\dfrac{f(u)}{u^{\alpha}}\geq M_2$, where α is ratio of odd positive integers.

iii) $\{a_n\}$ is a positive nonincreasing real sequence with

$$A(n)=\sum_{s=n_0}^{n-1}\frac{1}{a_s^{\frac{1}{\alpha}}}\to\infty \text{ as } n\to\infty; \tag{1.3}$$

iv) $\{b_n\}$ is a real sequence with $0\leq b_n\leq M_1b<1$ for all $n\in\mathbb{N}(n_0)$ and $\{q_n\}$ is a positive real sequence.

2000 *Mathematics Subject Classification.* 39A10.
Key words and phrases. Half linear, Neutral difference equation, Third order oscillation.

Submitted January 04, 2012. Accepted May 10,2012.

Let $\theta = \max\{\delta, \tau\}$. By a solution of equation (1.1)((1.2)), we mean a real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$ and satisfies (1.1) ((1.2)) for all $n \geq n_0$. A nontrivial solution $\{x_n\}$ is said to be a nonoscillatory if it is either eventually positive or eventually negative and it is oscillatory otherwise.

The oscillation theory of difference equations and their applications have been receiving intensive attention in the last few decades, see for example [1, 2, 4], and the references cited therein. Especially the study of oscillatory behavior of second order equations of various types occupied a great deal of interest. However the study of third order difference equations have received considerably less attention even though such equations have wide applications. In [3, 8, 9, 10, 12, 13] the authors investigated the oscillatory properties of solutions of third order delay difference equations and in [4, 11, 14, 15, 16], the authors studied similar properties for neutral delay difference equations. Motivated by the above observations in this paper we investigate oscillatory behavior of solutions of equation (1.1) and (1.2).

The equations (1.1) and (1.2) can be considered as the discrete analogue of the equations

$$\left(a(t)\left((x(t)+b(t)x(t-\delta))'\right)^{\alpha}\right)' + q(t)x^{\alpha}(t-\tau) = 0$$

and

$$\left(a(t)\left((x(t)-b(t)x(t-\delta))'\right)^{\alpha}\right)' + q(t)x^{\alpha}(t-\tau) = 0$$

when $h(u) = u$ and $f(u) = u^{\alpha}$ respectively, whose oscillatory properties are discussed in [5, 6, 7].

In Section 2, we present sufficient conditions which ensure that all solutions of equation (1.1) are either oscillatory or converges to zero, and we present similar results for equation (1.2) in Section 3. Examples are provided to illustrate the main results.

2. Oscillation of Equation (1.1)

First, we state and prove some useful lemmas. For each solution $\{x_n\}$ of equation (1.1), we define the corresponding $\{z_n\}$ by

$$z_n = x_n + b_n h(x_{n-\delta}). \tag{2.1}$$

Lemma 2.1. *Let $\{x_n\}$ be a positive solution of equation* (1.1). *Then there are only the following two cases for $\{z_n\}$ defined in* (2.1)*;*

i) $z_n > 0$, $\Delta z_n > 0$, $\Delta^2 z_n > 0$, $\Delta(a_n \Delta^2 z_n) \leq 0$;
ii) $z_n > 0$, $\Delta z_n < 0$, $\Delta^2 z_n > 0$, $\Delta(a_n \Delta^2 z_n) \leq 0$

for $n \geq n_1 \in \mathbb{N}(n_0)$, where n_1 is sufficiently large.

Lemma 2.2. *Let $\{x_n\}$ be a positive solution of equation* (1.1)*, and the corresponding $\{z_n\}$ satisfies case(ii) of Lemma 2.1. If*

$$\sum_{n=n_0}^{\infty} \sum_{s=n}^{\infty} \left[\frac{1}{a_s} \sum_{t=s}^{\infty} q_t\right]^{\frac{1}{\alpha}} = \infty, \tag{2.2}$$

then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = 0$.

Proof. The proofs for the Lemmas 2.1 and 2.2 can be proved as in [15]. ∎

Lemma 2.3. *Assume that* $u_n > 0$, $\Delta u_n \geq 0$, $\Delta(a_n(\Delta u_n)^\alpha) \leq 0$ *for all* $n \geq n_0$. *Then, for each* $l \in (0,1)$, *there exists an integer* $N \geq n_0$ *such that* $\frac{u_{n-\tau}}{A(n-\tau)} \geq l\frac{u_n}{A(n)}$ *for* $n \geq N$.

Proof. Since $a_n(\Delta u_n)^\alpha$ is nonincreasing so is $a_n^{\frac{1}{\alpha}}\Delta u_n$. Then by the definition of $A(n)$, we have

$$u_n - u_{n-\tau} = \sum_{s=n-\tau}^{n-1} \Delta u_s \leq a_{n-\tau}^{\frac{1}{\alpha}}\Delta u_{n-\tau}(A(n) - A(n-\tau)). \tag{2.3}$$

Also

$$u_{n-\tau} \geq u_{n-\tau} - u_{n_0} \geq a_{n-\tau}^{\frac{1}{\alpha}}\Delta u_{n-\tau}(A(n-\tau) - A(n_0)).$$

Since $\lim_{n\to\infty} \frac{A(n-\tau) - A(n_0)}{A(n-\tau)} = 1$, for each $l \in (0,1)$ there exists an integer $N \geq n_0$ such that $A(n-\tau) - A(n_0) > lA(n-\tau)$ for $n \geq N$.

From the above inequality

$$\frac{u_{n-\tau}}{\Delta u_{n-\tau}} \geq l a_{n-\tau}^{\frac{1}{\alpha}} A(n-\tau), \ n \geq N. \tag{2.4}$$

Combining (2.3) and (2.4), we obtain

$$\frac{u_n}{u_{n-\tau}} \leq 1 + \frac{A(n) - A(n-\tau)}{lA(n-\tau)} \leq \frac{A(n)}{lA(n-\tau)},$$

and the proof is complete. ■

Lemma 2.4. *Assume that* $z_n > 0$, $\Delta z_n > 0$, $\Delta^2 z_n > 0$, *and* $\Delta\left(a_n(\Delta^2 z_n)^\alpha\right) \leq 0$ *for all* $n \geq N$. *Then*

$$\frac{z_{n+1}}{\Delta z_n} \geq \frac{a_n^{\frac{1}{\alpha}} A(n)}{2} \ \textit{for all} \ \ n \geq N.$$

Proof. Since $a_n(\Delta^2 z_n)^\alpha$ is positive and nonincreasing so is $a_n^{\frac{1}{\alpha}}\Delta^2 z_n$. From $\Delta z_n > 0, a_n > 0$, we have

$$\Delta z_n \geq \Delta z_n - \Delta z_N = \sum_{s=N}^{n-1} \frac{a_s^{\frac{1}{\alpha}}\Delta^2 z_s}{a_s^{\frac{1}{\alpha}}} \geq a_n^{\frac{1}{\alpha}} A(n)\Delta^2 z_n.$$

Since $\Delta A(n) = a_n^{\frac{-1}{\alpha}}$, we have

$$(\Delta A(n))(\Delta z_n) \geq A(n)\Delta^2 z_n \text{ for } n \geq N. \tag{2.5}$$

Summing the inequality (2.5) from N to $n-1$, we have

$$\sum_{s=N}^{n-1} (\Delta A(s))(\Delta z_s) \geq A(n)\Delta z_n - \sum_{s=N}^{n-1} \Delta z_{s+1}\Delta A(s)$$

or

$$\sum_{s=N}^{n-1} (\Delta z_{s+1})(\Delta A(s)) \geq \frac{A(n)}{2}\Delta z_n, \ n \geq N. \tag{2.6}$$

Since $\{a_n\}$ is nonincreasing, we have $A(n) > 0$, $\Delta A(n) > 0$, $\Delta^2 A(n) \geq 0$, and therefore,

$$\Delta(z_{n+1}\Delta A(n)) = (\Delta z_{n+1})(\Delta A(n)) + z_{n+2}\Delta^2 A(n). \tag{2.7}$$

From (2.6) and (2.7), we obtain

$$z_{n+1} \geq \frac{a_n^{\frac{1}{\alpha}} A(n)}{2} \Delta z_n, \ n \geq N.$$

This completes the proof. ■

Lemma 2.5. *Assume that* $\Delta z_n > 0$, $\Delta^2 z_n > 0$, *and* $\Delta(a_n(\Delta^2 z_n)^\alpha) \leq 0$ *for all* $n \geq N$. *Then* $a_n^{\frac{1}{\alpha}} A(n) \frac{\Delta^2 z_n}{\Delta z_n} \leq 1$ *for all* $n \geq N$.

Proof. The result follows from the inequality

$$\Delta z_n \geq \Delta z_n - \Delta z_N \geq \sum_{s=N}^{n-1} \frac{a_s^{\frac{1}{\alpha}} \Delta^2 z_s}{a_s^{\frac{1}{\alpha}}} \geq \left(a_n^{\frac{1}{\alpha}} \Delta^2 z_n \right) A(n).$$

■

Lemma 2.6. *If* $\lim_{n\to\infty} \frac{a_n^{\frac{-1}{\alpha}}}{A(n)} = 0$, *then*

$$\lim_{n\to\infty} \frac{1}{A(n+1)} \sum_{s=N}^{n} \left(1 + \frac{a_s^{\frac{-1}{\alpha}}}{A(s)} \right)^{\alpha(\alpha+1)} \frac{1}{a_{s+1}^{\frac{1}{\alpha}}} = 1.$$

Proof. By discrete L'Hospital rule [1], we have

$$\lim_{n\to\infty} \frac{1}{A(n+1)} \sum_{s=N}^{n} \left(1 + \frac{a_s^{\frac{-1}{\alpha}}}{A(s)} \right)^{\alpha(\alpha+1)} \frac{1}{a_{s+1}^{\frac{1}{\alpha}}}$$
$$= \lim_{n\to\infty} \left(1 + \frac{a_{n+1}^{\frac{-1}{\alpha}}}{A(n+1)} \right) = 1.$$

Next, we present oscillation results for the equation (1.1). For simplicity, we introduce the following notations;

$$P = \lim_{n\to\infty} \inf A^\alpha(n+1) \sum_{s=n+1}^{\infty} p_l(s),$$

$$Q = \lim_{n\to\infty} \sup \frac{1}{A(n+1)} \sum_{s=n_0}^{n} A^{\alpha+1}(s) p_l(s) \tag{2.8}$$

where $p_l(n) = l^\alpha (1 - M_1 b)^\alpha M_2 q_n A^{2\alpha}(n-\tau) \left(\frac{a_{n-\tau}^{\frac{1}{\alpha}}}{2A(n)} \right)^\alpha$ with $l \in (0, 1)$. Moreover, for $\{z_n\}$ satisfying the case (i) of Lemma 2.1, we define

$$w_n = a_n \left(\frac{\Delta^2 z_n}{\Delta z_n} \right)^\alpha, \ n \geq N, \tag{2.9}$$

$$r = \lim_{n\to\infty} \inf A^\alpha(n+1) w_{n+1}, \text{ and } R = \lim_{n\to\infty} \sup A^\alpha(n+1) w_{n+1}. \tag{2.10}$$

■

Lemma 2.7. *Assume that $\{a_n\}$ is nonincreasing. Let $\{z_n\}$ be a positive solution of equation* (1.1).

(I) Let $P < \infty$ and suppose that the corresponding $\{z_n\}$ satisfies case (i) of Lemma 2.1. Then

$$P \leq r - r^{1+\frac{1}{\alpha}}. \tag{2.11}$$

(II) If $Q < \infty$ and $\lim_{n\to\infty} \frac{a_n^{\frac{-1}{\alpha}}}{A(n)} = 0$ and $\{z_n\}$ satisfies case (i) of Lemma 2.1, then

$$P + Q \leq 1. \tag{2.12}$$

(III) If $P = \infty$ or $Q = \infty$ and $\lim_{n\to\infty} \frac{a_n^{\frac{-1}{\alpha}}}{A(n)} = 0$, then $\{z_n\}$ does not have the case (i) of Lemma 2.1.

Proof. **Part (I)**. Assume that $\{x_n\}$ is a positive solution of equation (1.1), and the corresponding $\{z_n\}$ satisfies case (i) of Lemma 2.1. First, note that

$$x_n = z_n - b_n h(x_{n-\delta}) \geq z_n - bM_1 z_{n-\delta} \geq (1 - bM_1) z_n.$$

Using the last inequality in equation (1.1), we obtain

$$\Delta\left(a_n(\Delta^2 z_n)^\alpha\right) \leq -(1 - M_1 b)^\alpha M_2 q_n z_{n+1-\tau}^\alpha \leq 0. \tag{2.13}$$

From the definition of w_n and (2.13), we see that $w_n > 0$ and satisfies

$$\Delta w_n \leq -M_2 q_n (1 - M_1 b)^\alpha \left(\frac{z_{n+1-\tau}}{\Delta z_n}\right)^\alpha - \frac{\alpha}{a_{n+1}^{\frac{1}{\alpha}}} w_{n+1}^{1+\frac{1}{\alpha}}. \tag{2.14}$$

From Lemma 2.3 with $u_n = \Delta z_n$, we have for l, the same as in $p_l(n)$

$$\frac{1}{\Delta z_n} \geq l \frac{A(n-\tau)}{A(n)} \frac{1}{\Delta z_{n-\tau}}, \ n \geq N$$

which with (2.14)gives

$$\Delta w_n \leq -l^\alpha M_2 q_n \left(\frac{A(n-\tau)}{A(n)}\right)^\alpha \left(\frac{z_{n+1-\tau}}{\Delta z_{n-\tau}}\right)^\alpha (1 - M_1 b)^\alpha - \frac{\alpha}{a_{n+1}^{\frac{1}{\alpha}}} w_{n+1}^{1+\frac{1}{\alpha}}.$$

Using the fact from Lemma 2.4 that $z_{n+1} \geq \frac{a_n^{\frac{1}{\alpha}} A(n)}{2} \Delta z_n$, we have

$$\Delta w_n + p_l(n) + \frac{\alpha}{a_{n+1}^{\frac{1}{\alpha}}} w_{n+1}^{1+\frac{1}{\alpha}} \leq 0, \text{ for } n \geq N. \tag{2.15}$$

Since $p_l(n) > 0$ and $w_n > 0$ for $n \geq N$, we have from (2.15) that $\Delta w_n \leq 0$ and

$$-\frac{\Delta w_n}{\alpha w_{n+1}^{1+\frac{1}{\alpha}}} \geq \frac{1}{a_{n+1}^{\frac{1}{\alpha}}} \text{ for } n \geq N.$$

Summing the last inequality from N to $n-1$, and using the decreasing property of w_n, we obtain

$$\frac{-w_n + w_N}{\alpha w_n^{\frac{\alpha+1}{\alpha}}} \geq \sum_{s=N}^{n-1} \frac{1}{a_{s+1}^{\frac{1}{\alpha}}}$$

or

$$w_n \leq \left(\frac{w_N}{\alpha \sum\limits_{s=N}^{n-1} \frac{1}{a_{s+1}^{\frac{1}{\alpha}}}} \right)^{\dfrac{\alpha}{\alpha+1}}$$

which inview of (1.3) implies that $\lim\limits_{n\to\infty} w_n = 0$. On the other hand from the definition of w_n and Lemma 2.5, we see that

$$0 \leq r \leq R \leq 1. \tag{2.16}$$

Let $\epsilon > 0$. Then from the definition of P and r, we can choose an integer $n_2 \geq N$ sufficiently large that

$$A^{\alpha}(n+1) \sum_{s=n+1}^{\infty} p_l(s) \geq P - \epsilon \text{ and } A^{\alpha}(n+1) w_{n+1} \geq r - \epsilon$$

for all $n \geq n_2$. Summing (2.15) from $n+1$ to ∞ and using $\lim\limits_{n\to\infty} w_n = 0$, we have

$$w_{n+1} \geq \sum_{s=n+1}^{\infty} p_l(s) + \alpha \sum_{s=n+1}^{\infty} \frac{w_{s+1}^{\frac{\alpha+1}{\alpha}}}{a_{s+1}^{\frac{1}{\alpha}}}, \; n \geq n_2. \tag{2.17}$$

Multiplying the last inequality by $A^{\alpha}(n+1)$, we have

$$\begin{aligned} A^{\alpha}(n+1) w_{n+1} \quad &\geq \quad A^{\alpha}(n+1) \sum_{s=n+1}^{\infty} p_l(s) \\ &\quad + \alpha A^{\alpha}(n+1) \sum_{s=n+1}^{\infty} \frac{w_{s+1}^{\frac{\alpha+1}{\alpha}}}{a_{s+1}^{\frac{1}{\alpha}}} \\ &\geq \quad (P - \epsilon) \\ &\quad + (r-\epsilon)^{\frac{\alpha+1}{\alpha}} A^{\alpha}(n+1) \sum_{s=n+1}^{\infty} \frac{\alpha \Delta(A(s+1))}{A^{\alpha}(s+1)}. \end{aligned} \tag{2.18}$$

From (2.18) and $\sum\limits_{s=n+1}^{\infty} \dfrac{\alpha \Delta A(s+1)}{A^{\alpha}(s+1)} \geq \alpha \displaystyle\int_{A(n+1)}^{\infty} \frac{ds}{s^{\alpha}}$, we have,

$$A^{\alpha}(n+1) w_{n+1} \geq (P - \epsilon) + (r - \epsilon)^{\frac{\alpha+1}{\alpha}}.$$

Taking $\liminf$ on both sides, we obtain that

$$r \geq (P - \epsilon) + (r - \epsilon)^{\frac{\alpha+1}{\alpha}}.$$

Since $\epsilon > 0$ is arbitrary, we obtain the desired result

$$P \leq r - r^{1+\frac{1}{\alpha}}. \tag{2.19}$$

Part(II). Multiplying (2.15) by $A^{\alpha+1}(n)$ and summing from N to n, and then using summation by parts formula, we obtain

$$\begin{aligned} A^{\alpha+1}(n+1)w_{n+1} \quad &\leq \quad A^{\alpha+1}(N+1)w_N - \sum_{s=N}^{n} A^{\alpha+1}(s)p_l(s) \\ &\quad + \sum_{s=N}^{n} w_{s+1}\Delta A^{\alpha+1}(s) \\ &\quad - \sum_{s=N}^{n} \alpha \frac{A^{\alpha+1}(s)w_{s+1}^{\frac{\alpha+1}{\alpha}}}{a_{s+1}^{\frac{1}{\alpha}}} \\ &\leq \quad A^{\alpha+1}(N+1)w_N - \sum_{s=N}^{n} A^{\alpha+1}(s) \\ &\quad + \sum_{s=N}^{n} (\alpha+1)A^{\alpha}(s+1)\Delta A(s)w_{s+1} \\ &\quad - \sum_{s=N}^{n} \alpha A^{\alpha+1}(s)\Delta A(s)w_{s+1}^{\frac{\alpha+1}{\alpha}}. \end{aligned}$$

Using the inequality $Bu - Au^{\frac{\alpha+1}{\alpha}} \leq \dfrac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\dfrac{B^{\alpha+1}}{A^{\alpha}}$ with $u = w_{n+1}$,

$$A = \alpha A^{\alpha+1}(s)\Delta A(s),\ B = (\alpha+1)A^{\alpha}(s+1)\Delta A(s),$$

we obtain

$$\begin{aligned} A^{\alpha+1}(n+1)w_{n+1} \quad &\leq \quad A^{\alpha+1}(N+1)w_N - \sum_{s=N}^{n} A^{\alpha+1}(s)p_l(s) \\ &\quad + \sum_{s=N}^{n} \left(\frac{(A(s+1))}{A(s)}\right)^{\alpha(\alpha+1)} \frac{1}{a_s^{\frac{1}{\alpha}}}. \end{aligned}$$

It follows that

$$\begin{aligned} A^{\alpha}(n+1)w_{n+1} \quad &\leq \quad \frac{A^{\alpha+1}(N+1)w_N}{A(n+1)} \\ &\quad - \frac{1}{A(n+1)} \sum_{s=N}^{n} A^{\alpha+1}(s)p_l(s) \\ &\quad + \frac{1}{A(n+1)} \sum_{s=N}^{n} \left(1 + \frac{a_s^{\frac{-1}{\alpha}}}{A(s)}\right)^{\alpha(\alpha+1)} \frac{1}{a_s^{\frac{1}{\alpha}}}. \end{aligned} \tag{2.20}$$

Taking $\limsup$ on both sides and using Lemma 2.6, we obtain

$$R \leq -Q + 1.$$

Combining this with the inequalities in (2.19) and (2.16) we obtain

$$P \leq r - r^{\frac{\alpha+1}{\alpha}} \leq r \leq R \leq -Q + 1$$

which proves the inequality (2.12).

Part(III). Assume that $\{x_n\}$ is a positive solution of equation (1.1). We shall show that $\{z_n\}$ does not have case (i) of Lemma 2.1. Assume the contrary. First, we assume $P = \infty$. Then exactly as in the proof of Part (I), we obtain (2.17). Then

$$A^{\alpha}(n+1)w_{n+1} \geq A^{\alpha}(n+1) \sum_{s=n+1}^{\infty} p_l(s).$$

Taking lim inf on both sides, we obtain in view of (2.16) that

$$1 \geq r \geq \infty.$$

This is a contradiction. Next, we assume that $Q = \infty$ and $\lim_{n\to\infty} \frac{a_n^{\frac{-1}{\alpha}}}{A(n)} = 0$. Then taking lim inf and lim sup on the left and right hand side of (2.20) respectively, we obtain

$$0 \leq R \leq -\infty.$$

This contradiction completes the proof.

Now we are ready to present the following oscillation criteria for the equation (1.1). ■

Theorem 2.8. *Assume that condition* (2.2) *holds and* $\{a_n\}$ *is non-increasing. If*

$$P > \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}, \tag{2.21}$$

then every solution $\{x_n\}$ *of equation* (1.1) *is either oscillatory or tends to zero as* $n \to \infty$.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that $\{x_n\}$ is a positive solution (since the proof for the opposite case is similar) of equation (1.1). If $P = \infty$, then by Lemma 2.7, $\{z_n\}$ does not have case (i) of Lemma 2.1. That is, $\{z_n\}$ has to satisfy case (ii) of Lemma 2.1 and from Lemma 2.2, we see that $\lim_{n\to\infty} x_n = 0$.

Next, we assume that $P < \infty$. We shall discuss two possibilities. If for $\{z_n\}$, case (ii) of Lemma 2.1 holds, then exactly as above we are led by Lemma 2.2 to $\lim_{n\to\infty} x_n = 0$.

Now, we assume that for $\{z_n\}$, case (i) of Lemma 2.1 holds. Let w_n and r be defined by (2.9) and (2.10), respectively. Then from Lemma 2.7 we see that r satisfies the inequality

$$P \leq r - r^{\frac{\alpha+1}{\alpha}}.$$

Using the inequality $Bu - Au^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}$ with $A = B = 1$ and $u = r$, we obtain that

$$P \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}$$

which contradicts (2.21). This completes the proof. ■

Theorem 2.9. *Assume that condition* (2.4) *holds and* $\{a_n\}$ *is nonincreasing with* $\lim_{n\to\infty} \frac{a_n^{\frac{-1}{\alpha}}}{A(n)} = 0$. *If*

$$P + Q > 1, \tag{2.22}$$

then every solution $\{x_n\}$ *of equation* (1.1) *is either oscillatory or tends to zero as* $n \to \infty$.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that $\{x_n\}$ is a positive solution of equation (1.1). If $P = \infty$ or $Q = \infty$, then by Lemma 2.7, $\{z_n\}$ does not have case (i) of Lemma 2.1.

That is $\{z_n\}$ has to satisfy case (ii) of Lemma 2.1. From Lemma 2.2, we see that $\lim_{n\to\infty} x_n = 0$.

Next, we assume that $P < \infty$ and $Q < \infty$. We shall discuss two possibilities. If, for $\{z_n\}$, case (ii) of Lemma 2.1 holds, then exactly as above we led by Lemma 2.2, to $\lim_{n\to\infty} x_n = 0$. Now, we assume that for $\{z_n\}$, case (i) of Lemma 2.1 holds. Let w_n and r be defined by (2.9) and (2.10) respectively. Then from Lemma 2.7, we see that P and Q satisfy the inequality $P + Q \leq 1$ which contradicts (2.22). This completes the proof.

As a consequence of Theorem 2.9 we have the following result. ■

Corollary 2.10. *Assume that condition* (2.4) *holds and* $\{a_n\}$ *is non-increasing with* $\lim_{n\to\infty} \frac{a_n^{\frac{-1}{\alpha}}}{A(n)} = 0$. *If*

$$Q = \lim_{x\to\infty} \sup \frac{1}{A(n+1)} \sum_{s=n_0}^{n} A^{\alpha+1}(s) p_l(s) > 1, \tag{2.23}$$

then every solution $\{x_n\}$ *of equation* (1.1) *is either oscillatory or tends to zero as* $n \to \infty$.

We conclude this section with the following example.

Example 2.1. *Consider the third order nonlinear difference equation*

$$\Delta\left(\frac{1}{n^3}(\Delta^2(x_n + \frac{1}{3}x_{n-1}(1 - x_{n-1}^2)))^3\right) + \frac{\lambda}{n^6} x_{n-1}^3 (1 + x_{n-1}^2) = 0, \ n \geq 1. \tag{2.24}$$

Here $a_n = \frac{1}{q}$, $b_n = \frac{1}{3}$, $h(u) = u(1-u^2)$, $q_n = \frac{\lambda}{n^6}$, $f(u) = u^3(1+u^2)$ *and* $\alpha = 3$. *Then we find* $M_1 = 1$ *and* $M_2 = 1$ *and it is easy to see that conditions* (2.2) *and* (2.21) *are hold for* $\lambda > 0$.. *Hence by Theorem 2.8, we see that every solution of equation* (2.24) *is either oscillatory or converges to zero as* $n \to \infty$.

3. Oscillation of Equation(1.2)

In this section, we present oscillatory criteria for equation (1.2). We define

$$z_n = x_n - b_n h(x_{n-\delta}). \tag{3.1}$$

Lemma 3.1. *Let* $\{x_n\}$ *be a positive solution of equation* (1.2). *Then the corresponding function* $\{z_n\}$ *defined in* (3.1) *satisfies the following cases.*

(iii) $z_n > 0$, $\Delta z_n > 0$, $\Delta^2 z_n > 0$, $\Delta(a_n \Delta^2 z_n) \leq 0$;
(iv) $z_n > 0$, $\Delta z_n < 0$, $\Delta^2 z_n > 0$, $\Delta(a_n \Delta^2 z_n) \leq 0$;
(v) $z_n < 0$, $\Delta z_n < 0$ $\Delta^2 z_n > 0$, $\Delta(a_n \Delta^2 z_n) \leq 0$;
(vi) $z_n < 0$, $\Delta z_n < 0$ $\Delta^2 z_n < 0$, $\Delta(a_n \Delta^2 z_n) \leq 0$

for $n \geq n_1$, *where* n_1 *is sufficiently large.*

Lemma 3.2. *Let* $\{x_n\}$ *be a positive solution of equation* (1.2), *and the corresponding* z_n *satisfies the case of Lemma 3.1 (iv). If* (2.4) *holds, then* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = 0$.

The proofs of Lemma 3.1 and Lemma 3.2 can be proved as in [15]. *For simplicity, we introduce the following notations;*

$$\overline{p}_l(n) = l^\alpha M_2 q_n A^{2\alpha}(n-\tau) \left(\frac{a_{n-\tau}^{\frac{1}{\alpha}}}{2A(n)}\right)^\alpha \quad \textit{with } l \in (0,1).$$

$$\overline{P} = \lim_{n\to\infty}\inf A^{\alpha}(n+1)\sum_{s=n+1}^{\infty}\overline{p}_l(s)$$

$$\text{and } \overline{Q} = \lim_{n\to\infty}\sup \frac{1}{A(n+1)}\sum_{s=n_0}^{n}A^{\alpha}(s,N)\overline{p}_l(s) \tag{3.2}$$

where w_n, r and R are as defined in Section 2.

Lemma 3.3. *Assume that $\{a_n\}$ is nonincreasing. Let $\{x_n\}$ be a positive solution of equation (1.2).*
(I) Let $\overline{P}<\infty$ and suppose that the corresponding $\{z_n\}$ satisfied case (iii) of Lemma 3.1. Then

$$\overline{P} \le r - r^{1+\frac{1}{\alpha}}. \tag{3.3}$$

(II) If $\overline{Q}<\infty$ and $\lim_{n\to\infty}\frac{a_n^{\frac{-1}{\alpha}}}{A(n)}=0$ and $\{z_n\}$ satisfies case (iii) of Lemma 3.1, then

$$\overline{P}+\overline{Q}\le 1. \tag{3.4}$$

(III) If $\overline{P}=\infty$ or $\overline{Q}=\infty$ and $\lim_{n\to\infty}\frac{a_n^{\frac{-1}{\alpha}}}{A(n)}=0$ and $\{z_n\}$ does not satisfy case (iii) of Lemma 3.1.

Proof. Let $\{x_n\}$ be a positive solution of equation (1.2) and $\{z_n\}$ satisfies case (iii) of Lemma 3.1. Since $0<z_n<x_n$, equation (1.2) can be written in the form

$$\Delta(a_n(\Delta^2 z_n)^{\alpha}) + M_2 q_n z_{n+1-\tau}^{\alpha} \le 0.$$

The rest of the proof for the parts (I), (II) and (III) are similar to that of Lemma 2.7, and hence the details are omitted. ■

Theorem 3.4. *Assume that $\{a_n\}$ is nonincreasing and condition (2.4) holds. If*

$$\overline{P} > \frac{\alpha^{\alpha}}{(\alpha+1)^{1+\alpha}} \tag{3.5}$$

then every solution $\{x_n\}$ of equation (1.2) is either oscillatory or tends to zero as $n\to\infty$.

Proof. Let $\{x_n\}$ be a positive solution of equation (1.2). Then

$$\Delta(a_n(\Delta^2 z_n)^{\alpha}) + M_2 q_n x_{n+1-\tau}^{\alpha} \le 0.$$

We claim that $\{x_n\}$ is bounded. If not, then there exists a sequence $\{n_j\}$ such that $\lim_{j\to\infty} n_j=\infty$ and $\lim_{j\to\infty} x_{n_j}=\infty$ and,

$$x_{n_j} = \max[x_s;\ n_0\le s\le n_j].$$

Since $n-\delta\to\infty$ as $n\to\infty$, we can choose $n_j-\delta>n_0$. As $n-\delta\le n$, we have

$$x_{n_j-\delta}\le \max[x_s : n_0\le s\le n_j-\delta].$$

Therefore, for all large j

$$z_{n_j} = x_{n_j} - b_{n_j}h(x_{n_j-\delta}) \ge (1-M_1 b)x_{n_j}.$$

Thus $z_{n_j}\to\infty$ as $j\to\infty$, so $\{z_n\}$ is positive and unbounded. It follows from Lemma 3.1 that case (iii) has to hold. Part(I) of Lemma 3.3 provides

$$\overline{P}\le r - r^{1+\frac{1}{\alpha}}.$$

Using the inequality $Bu - Au^{1+\frac{1}{\alpha}} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}$, with $A = B = 1$ and $u = r$, we obtain

$$\overline{P} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}$$

which contradicts (3.5). So, we conclude that both $\{x_n\}$ and $\{z_n\}$ are bounded. Lemma 3.1 now implies that for z_n either case (iv) or case (v) holds.

If case (iv) holds, then Lemma 3.2 ensures that $\lim_{n\to\infty} x_n = 0$. On the other hand if the case (v) holds, then there exists a finite limit $\lim_{n\to\infty} z_n = -d < 0$. We know that $0 < x_n$ is bounded, so

$$\lim_{n\to\infty} \sup x_n = c, \; 0 \leq c < \infty.$$

We claim that $c = 0$. If not, then there exists a sequence $\{n_j\}$ such that $\lim_{j\to\infty} n_j = \infty$ and $\lim_{j\to\infty} x_{n_j} = c$. It is easy to see that for $\epsilon = \frac{c(1 - M_1 b)}{2bM_1} > 0$, we have $x_{n_j - \delta} < c + \epsilon$. Moreover,

$$0 > -\delta = \lim_{j\to\infty} z_{n_j} \geq \lim_{j\to\infty} (x_{n_j} - M_1 b(c+\epsilon)) = \frac{c}{2}(1 - M_1 b) > 0$$

which is a contradiction. Thus $c = 0$ and $\lim_{n\to\infty} x_n = 0$. This completes the proof.

The proof of the next result is similar to that of Theorem 2.9. so it is omitted. ■

Theorem 3.5. *Assume that condition* (2.4) *holds and* $\{a_n\}$ *is nonincreasing with* $\lim_{n\to\infty} \frac{a_n^{\frac{-1}{\alpha}}}{A(n)} = 0$. *If*

$$\overline{P} + \overline{Q} > 1,$$

then every solution $\{x_n\}$ *of equation* (1.2) *is either oscillatory or tends to zero as* $n \to \infty$.

Corollary 3.6. *Assume that condition* (2.4) *holds and* $\{a_n\}$ *is nonincreasing with* $\lim_{n\to\infty} \sup \frac{a_n^{\frac{-1}{\alpha}}}{A(n)} = 0$. *If*

$$\overline{Q} = \lim_{n\to\infty} \sup \frac{1}{A(n+1)} \sum_{s=n_0}^{n} A^{\alpha+1}(s) \overline{p}_l(s) > 1,$$

then every solution $\{x_n\}$ *of equation* (1.2) *is either oscillatory or tends to zero as* $n \to \infty$.

We conclude this section with the following example.

Example 3.1. *Consider the third order difference equation*

$$\Delta\left(\frac{1}{n}\left(\Delta^2\left(x_n - \frac{1}{3}\left(\frac{x_{n-1}}{1 + x_{n-1}^2}\right)\right)\right)^3\right) + \frac{\lambda}{n^6} x_{n-1}^3 (1 + x_{n-1}^2) = 0, \; n \geq 1. \quad (3.6)$$

Corollary 3.6 implies that every solution of equation (3.6) *is either oscillatory or converges to zero as* $n \to \infty$, *provided that* $\lambda > 0$.

4. Conclusion

In this paper we establish sufficient conditions which ensure that all solutions of equations (1.1) and (1.2) are either oscillatory or tend to zero as $n \to \infty$, under the condition $\{a_n\}$ is nonincreasing. Therefore our results complement to those obtained in [15] for the case $\{a_n\}$ is nondecreasing, $f(u) = u^\alpha$ and $h(u) = u$. It would be interesting to obtain similar results to equations (1.1) and (1.2) when $\sum\limits_{n=n_0}^{\infty} \frac{1}{a_n^{\frac{1}{\alpha}}} < \infty$.

Acknowledgement

The authors thank the referee for his/her valuable suggestions which improve the contents of the paper.

References

[1] R. P. Agarwal, *Difference Equations and Inequalities,* 2nd Edition, Marcel Dekker, New York, 2000.

[2] R. P. Agarwal, S. R. Grace and D. O. Regan, *Oscillation Theory for Difference and Functional Differential Equations,* Kluwer, Dordrecht, 2000.

[3] R. P. Agarwal, S. R. Grace and D. O. Regan, *On the oscillation of certain third order difference equations,* Adv. Differ. Equ., 3(2005), 345-367.

[4] R. P. Agarwal, M. Bohner, S. R. Grace and D. O. Regan, *Discrete Oscillation Theory,* Hindawi, New York, 2005.

[5] B. Baculíková and J. Dźurina, *Oscillation of third order netural differential eqautions,* Math. Comp. Modelling, 52(2010), 215-226.

[6] B. Baculíková and J. Dźurina, *On the asymptotic behavior of a class of third nonlinear neutral differential equations,* Cent. Eur. J. Math., 8(6)(2010), 1091-1103.

[7] J. Džirina, E. Thandapani and S. Tamilvanan, *Oscillatory solutions to third order half-linear netural differential equations,* Elec. J. Diff. Eqns., Vol 2012(2012), No.29, pp. 1-9.

[8] S. R. Grace, R. P. Agarwal and J. R. Graef, *Oscillation criteria for certain third order non-linear difference equations,* Appl. Anal. Discrete Math., 3(2009), 27-38.

[9] J. R. Graef and E. Thandapani, *Oscillatory and asymptoic behavior of solutions of third order delay difference equations,* Funkcial. Ekvac.42(1999), 355-369.

[10] S. H. Saker, *Oscillation of third order difference equations,* Port. Math., 61(2004), 249-257.

[11] S. H. Saker, *Oscillation and asymptoic behavior of third order nonlinear neutral delay difference equations,* Dyna. Sys. Appl., 15(2006) 549-568.

[12] S. H. Saker, J. O. Alzabut and A. Mukheimer, *On the oscillatory behavior for a certain class of third order nonlinear delay difference equations,* Elec. J. Qual. Theory. Diff. Eqns., Vol 2010 (2010) No.67, pp.1-16.

[13] E. Schmeidel, *Oscillatory and asymptotically zero solutions of third order difference equations with quasidifferences,* Opuscula Math., 26(2006) 361-369.

[14] E. Thandapani and K. Mahalingam, *Oscillatory properties of third order neutral delay difference equations,* Demons. Math 35(2002), 325-336.

[15] E. Thandapani and S. Selvarangam, *Oscillation of third order halflinear neutral difference equation,* Math. Bohemica (to appear).

[16] E. Thandapani and M. Vijaya, *On the oscillation of third order half linear netural type difference equations*, Elec. J. Qual. Theory. Diff. Eqns., Vol.2011 (2011) No.76, pp.1-13.

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