Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 4 Issue 4 (2012), Pages 56-66

INTEGRAL OPERATORS CONTAINING SHEFFER POLYNOMIALS

(COMMUNICATED BY VIJAY GUPTA)

SEZGİN SUCU AND İBRAHİM BÜYÜKYAZICI

Abstract. The aim of the present paper is to introduce new type integral operators which involve Sheffer polynomials. We investigate approximation properties of the our operators with the help of the universal Korovkin-type property and also establish the rate of convergence by using modulus of continuity, second order modulus of smoothness and Petree's K- functional. Moreover, some examples which include Sheffer type sequence known as monomial, Bell, Toscano and Laguerre polynomials are given to compute error estimation by modulus of continuity.

1. Introduction

The main aim of an approximation theory is to present nonarithmetic quantities by arithmetic quantities so that the correctness can be determined to a desired degree. In 1953, Korovkin discovered the most powerful and simplest criterion in order to decide approximation process with positive linear operators on continuous functions space. After this year a considerable amount of research has been done by several mathematicians.

Mazhar and Totik [5] modified the Szász operator [6] and defined another class of positive linear operators

$$S_n^*(f;x) := ne^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\int_0^{\infty}e^{-nt}\frac{(nt)^k}{k!}f(t)\,dt \tag{1.1}$$

for function f which is of exponential type.

Jakimovski and Leviatan [4] introduced a generalization of Szász operators including Appell polynomials. Let us remind these operators. It is known that the Appell polynomials $p_k(x)$ can be defined by

$$g(u)\,e^{ux} = \sum_{k=0}^{\infty}p_k(x)\,u^k \tag{1.2}$$

[0]2010 Mathematics Subject Classification: 41A25, 41A35.

Keywords and phrases. Szász operators, Modulus of continuity, Korovkin's theorem, Sheffer polynomials, Toscano polynomials.

Submitted 23 April, 2012. Accepted 30 October, 2012.

where $g(z) = \sum_{n=0}^{\infty} a_n z^n$ is an analytic function in the disk $|z| < R$ $(R > 1)$ and $g(1) \neq 0$. From the generating functions (1.2),

$$P_n(f; x) := \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right)$$

is defined by Jakimovski and Leviatan. Ciupa [1] modified the operator P_n as follows

$$P_n^*(f; x) := \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} f(t)\, dt \tag{1.3}$$

where Γ is gamma function and $\lambda \geq 0$. For the special case $g(z) = 1$ and $\lambda = 0$, the operators defined by (1.3) become operators S_n^*.

Let $p_k(x)$ be Sheffer polynomials defined by

$$A(u)\, e^{xH(u)} = \sum_{k=0}^{\infty} p_k(x)\, u^k \tag{1.4}$$

where

$$\begin{aligned} A(z) &= \sum_{n=0}^{\infty} a_n z^n, \quad (a_0 \neq 0) \\ H(z) &= \sum_{n=1}^{\infty} h_n z^n, \quad (h_1 \neq 0) \end{aligned} \tag{1.5}$$

and suppose that

$$\begin{aligned} &(i) \quad \text{For } x \in [0, \infty) \text{ and } k \in \mathbb{N} \cup \{0\}, p_k(x) \geq 0, \\ &(ii) \quad A(1) \neq 0 \text{ and } H'(1) = 1, \\ &(iii) \quad (1.4) \text{ relation is valid for } |u| < R \text{ and the power series} \\ &\text{given by } (1.5) \text{ converge for } |z| < R, (R > 1). \end{aligned} \tag{1.6}$$

Under the assumption (1.6), Ismail [3] introduced and throughly investigated the positive linear operators

$$T_n(f; x) := \frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right) \tag{1.7}$$

whenever function f is an exponential type.

Now we will revise the operator T_n as follows

$$T_n^*(f; x) := \frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} f(t)\, dt \tag{1.8}$$

where the parameter $\lambda \geq 0$. For $H(t) = t$, the operators T_n^* reduce to P_n^* given by (1.3).

In the present paper, in order to get more general approximation operators we use Sheffer polynomials and gamma functions.

The structure of this paper as follows. In section 2, we study convergence of the operators T_n^* with the help of the universal Korovkin-type property, furthermore the order of convergence is obtained by means of the first and second modulus of continuity and Peetre's K-functional. Finally, in the last section we give some examples of these type operators (1.8) including monomial polynomials, Bell polynomials, Toscano polynomials and Laguerre polynomials and also obtain numerical error estimation by using Maple13 for getting new type operators.

2. APPROXIMATION PROPERTIES OF T_n^* OPERATORS

Now we are going to give some auxiliary definitions and lemmas before state our main theorems. Let us define the class E as follows

$$E := \left\{ f : x \in [0,\infty) , \ \frac{f(x)}{1+x^2} \text{ is convergent as } x \to \infty \right\}.$$

Lemma 2.1. *T_n^* operators satisfy the following equalities*

$$T_n^*(1;x) = 1 \tag{2.1}$$

$$T_n^*(\xi;x) = x + \frac{1}{n}\left(1+\lambda+\frac{A'(1)}{A(1)}\right) \tag{2.2}$$

$$\begin{aligned} T_n^*\left(\xi^2;x\right) &= x^2 + \frac{x}{n}\left(2\lambda+4+2\frac{A'(1)}{A(1)}+H''(1)\right) \\ &\quad + \frac{1}{n^2}\left((\lambda+1)(\lambda+2)+\frac{2(\lambda+2)A'(1)+A''(1)}{A(1)}\right) . \end{aligned} \tag{2.3}$$

Proof. Using the generating functions (1.4) and properties of gamma function, we get above results simply. □

Lemma 2.2. *For T_n^* operators, the below equality is verified*

$$T_n^*\left((\xi-x)^2;x\right) = x\frac{H''(1)+2}{n} + \frac{(\lambda+1)(\lambda+2)A(1)+2(\lambda+2)A'(1)+A''(1)}{n^2A(1)} . \tag{2.4}$$

Proof. From the linearity of T_n^* operators and applying Lemma 2.1, one can find (2.4). □

Definition 2.1. *The modulus of continuity of a function $f \in \tilde{C}[0,\infty)$ is a function $\omega(f;\delta)$ defined by the relation*

$$\omega(f;\delta) := \sup_{\substack{|x-y|\le\delta \\ x,y\in[0,\infty)}} |f(x)-f(y)|$$

where $\tilde{C}[0,\infty)$ is uniformly continuous functions space.

Definition 2.2. *The Peetre's K-functional is defined by*

$$\mathcal{K}(f;\delta) := \inf_{g\in W_\infty^2} \left\{ \|f-g\|_{\tilde{C}_B} + \delta\|g\|_{W_\infty^2} \right\}$$

where $W_\infty^2 := \left\{ g \in \tilde{C}_B[0,\infty) : g', g'' \in \tilde{C}_B[0,\infty) \right\}$, *with the norm*

$$\|f\|_{W_\infty^2} := \|f\|_{\tilde{C}_B} + \left\|f'\right\|_{\tilde{C}_B} + \left\|f''\right\|_{\tilde{C}_B}$$

and the second order modulus of smoothness is defined as

$$\omega_2(f;\delta) := \sup_{0\le h\le\delta} \sup_{x\in[0,\infty)} |f(x+2h)-2f(x+h)+f(x)| .$$

It is known that there is connection between the second order modulus of smoothness and Peetre's K-functional as follows [1]*:*

$$\mathcal{K}\left(f;\delta\right) \leq M\left\{\omega_2\left(f;\sqrt{\delta}\right)+\min\left(1,\delta\right)\|f\|_{\tilde{C}_B}\right\}$$

where M is absolute constant and $\tilde{C}_B\left[0,\infty\right)$ is the class of real valued functions defined on $\left[0,\infty\right)$ which are bounded and uniformly continuous with the norm $\|f\|_{\tilde{C}_B}:=\sup\limits_{x\in[0,\infty)}\left|f\left(x\right)\right|.$

Lemma 2.3. *(Gavrea and Rasa* [2]*) Let $g\in C^2\left[0,a\right]$ and $\left\{L_n\left(g;x\right)\right\}_{n\geq1}$ be a sequence of positive linear operators with the property $L_n\left(1;x\right)=1$. Then*

$$\left|L_n\left(g;x\right)-g\left(x\right)\right|\leq\left\|g^{'}\right\|\sqrt{L_n\left(\left(t-x\right)^2;x\right)}+\frac{1}{2}\left\|g^{''}\right\|L_n\left(\left(t-x\right)^2;x\right)\ .$$

Let f_h be the second order Steklov function attached to the function f. We will use the following result proved by Zhuk [7]: if $f\in C\left[a,b\right]$ and $h\in\left(0,\frac{b-a}{2}\right)$, then

$$\|f_h-f\| \leq \frac{3}{4}\omega_2\left(f;h\right) \tag{2.5}$$

$$\left\|f_h^{''}\right\| \leq \frac{3}{2}\frac{1}{h^2}\omega_2\left(f;h\right)\ . \tag{2.6}$$

We now deal with the approximation properties of our operators defined by (1.8). We begin by stating the following fundamental result.

Theorem 2.1. *For given $f\in C\left[0,\infty\right)\cap E$,*

$$\lim_{n\to\infty}T_n^*\left(f;x\right)=f\left(x\right)$$

the convergence being uniform in each compact subset of $\left[0,\infty\right)$.

Proof. By using (2.1), (2.2) and (2.3), we deduce that

$$\lim_{n\to\infty}T_n^*\left(\xi^i;x\right)=x^i\quad,\quad i=0,1,2$$

uniformly on compact subset of $\left[0,\infty\right)$. Hence, an application of the universal Korovkin-type property completes the proof. □

Usually, the error estimates in approximation theory are provided in terms of modulus of continuity, second order modulus of smoothness and Peetre's K-functional. So, let us state the order of approximation to function f by T_n^* with the help of above tools.

Theorem 2.2. *If $f\in\tilde{C}\left[0,\infty\right)\cap E$, then we have*

$$\left|T_n^*\left(f;x\right)-f\left(x\right)\right|\leq\left(1+\vartheta_n\left(x\right)\right)\omega\left(f;\frac{1}{\sqrt{n}}\right)$$

where

$$\vartheta_n\left(x\right)=\sqrt{x\left(H''\left(1\right)+2\right)+\frac{1}{n}\left(\left(\lambda+1\right)\left(\lambda+2\right)+\frac{2\left(\lambda+2\right)A'\left(1\right)+A''\left(1\right)}{A\left(1\right)}\right)}\ .$$

Proof. By using (2.1), property of the modulus of continuity and after some simple calculations, we can write

$$\begin{aligned} & \left|T_n^*(f;x)-f(x)\right| \\ \leq\ & \frac{e^{-nxH(1)}}{A(1)}\sum_{k=0}^{\infty}p_k(nx)\left(1+\frac{1}{\delta}\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-nt}t^{\lambda+k}\left|t-x\right|dt\right)\omega(f;\delta)\ . \end{aligned}$$

Applying the Cauchy-Schwarz inequality for the integral term on the right hand side of the above inequality, we conclude

$$\begin{aligned} \left|T_n^*(f;x)-f(x)\right| &\leq \frac{e^{-nxH(1)}}{A(1)}\sum_{k=0}^{\infty}p_k(nx) \\ &\times\left(1+\frac{1}{\delta}\sqrt{x^2-\frac{2(\lambda+k+1)}{n}x+\frac{(\lambda+k+1)(\lambda+k+2)}{n^2}}\right)\omega(f;\delta) \\ =\ & \left\{1+\frac{1}{\delta}\frac{e^{-nxH(1)}}{A(1)}\sum_{k=0}^{\infty}p_k(nx)\right. \\ &\left.\times\sqrt{x^2-\frac{2(\lambda+k+1)}{n}x+\frac{(\lambda+k+1)(\lambda+k+2)}{n^2}}\right\}\omega(f;\delta)\ . \end{aligned} \tag{2.7}$$

If we use again the Cauchy-Schwarz inequality in the above result (2.7), one can obtain the following

$$\begin{aligned} & \left|T_n^*(f;x)-f(x)\right| \\ \leq\ & \left\{1+\frac{1}{\delta}\sqrt{\frac{H''(1)+2}{n}x+\frac{1}{n^2}\left(\begin{array}{c}(\lambda+1)(\lambda+2)\\ +\frac{2(\lambda+2)A'(1)+A''(1)}{A(1)}\end{array}\right)}\right\}\omega(f;\delta) \\ =\ & \left\{1+\frac{1}{\delta}\frac{1}{\sqrt{n}}\sqrt{\left(H''(1)+2\right)x+\frac{1}{n}\left(\begin{array}{c}(\lambda+1)(\lambda+2)\\ +\frac{2(\lambda+2)A'(1)+A''(1)}{A(1)}\end{array}\right)}\right\}\omega(f;\delta)\ . \end{aligned}$$

In the previous inequality, choosing $\delta=\frac{1}{\sqrt{n}}$ one can get desired result. □

Now, we compute the rate of convergence of the operators T_n^* with the help of the second order modulus of smoothness.

Theorem 2.3. *Let f be defined on $[0,\infty)$ and $f\in C[0,a]$, then the rate of convergence the sequence of T_n^* is governed by*

$$\left|T_n^*(f;x)-f(x)\right|\leq\frac{2}{a}h^2\left\|f\right\|+\frac{3}{4}\left(2+a+h^2\right)\omega_2(f;h)$$

where $h:=\sqrt[4]{T_n^*\left((\xi-x)^2;x\right)}$.

Proof. Let us denote the second order Steklov function of f as f_h. Because of $T_n^*(1;x)=1$, one can write

$$\left|T_n^*(f;x)-f(x)\right|\leq 2\left\|f_h-f\right\|+\left|T_n^*(f_h;x)-f_h(x)\right|\ . \tag{2.8}$$

From the Landau inequality and applying (2.6), we may derive the following

$$\begin{aligned}\left\|f_h^{'}\right\| &\leq \frac{2}{a}\left\|f_h\right\|+\frac{a}{2}\left\|f_h^{''}\right\| \\ &\leq \frac{2}{a}\left\|f\right\|+\frac{3a}{4}\frac{1}{h^2}\omega_2\left(f;h\right) \ . \end{aligned} \tag{2.9}$$

By virtue of $f_h\in C^2\left[0,a\right]$, if we use the Lemma 2.3, (2.6) and (2.9) we obtain the estimate

$$\begin{aligned}\left|T_n^*\left(f_h;x\right)-f_h\left(x\right)\right| &\leq \left(\frac{2}{a}\left\|f\right\|+\frac{3a}{4}\frac{1}{h^2}\omega_2\left(f;h\right)\right)\sqrt{T_n^*\left(\left(\xi-x\right)^2;x\right)} \\ &\quad +\frac{3}{4}\frac{1}{h^2}T_n^*\left(\left(\xi-x\right)^2;x\right)\omega_2\left(f;h\right) \ . \end{aligned} \tag{2.10}$$

Choosing $h=\sqrt[4]{T_n^*\left(\left(\xi-x\right)^2;x\right)}$ in inequality (2.10) and then considering the last statement in (2.8), so the proof is completed. □

Furthermore, in the case f is smooth function the following theorem gives the estimation of approximation to function f.

Theorem 2.4. *For $f\in W_\infty^2$, we have*

$$\left|T_n^*\left(f;x\right)-f\left(x\right)\right|\leq\frac{1}{n}\mu\left(x\right)\left\|f\right\|_{W_\infty^2} \tag{2.11}$$

where

$$\mu\left(x\right):=x\left(1+\frac{H^{''}\left(1\right)}{2}\right)+\frac{1}{2}\left(\left(\lambda+1\right)\left(\lambda+4\right)+\frac{2\left(\lambda+3\right)A^{'}\left(1\right)+A^{''}\left(1\right)}{A\left(1\right)}\right) \ .$$

Proof. From the Taylor formula

$$f\left(\xi\right)=f\left(x\right)+f^{'}\left(x\right)\left(\xi-x\right)+\frac{f^{''}\left(\eta\right)}{2}\left(\xi-x\right)^2$$

where $\eta\in\left(x,\xi\right)$. Due to linearity property of operators T_n^*, one can write

$$T_n^*\left(f;x\right)-f\left(x\right)=f^{'}\left(x\right)T_n^*\left(\xi-x;x\right)+\frac{f^{''}\left(\eta\right)}{2}T_n^*\left(\left(\xi-x\right)^2;x\right) \ .$$

From this fact and using Lemma 2.1, we obtain

$$\begin{aligned}\left|T_n^*\left(f;x\right)-f\left(x\right)\right| &\leq \frac{1}{n}\left(1+\lambda+\frac{A^{'}\left(1\right)}{A\left(1\right)}\right)\left\|f^{'}\right\|_{C_B}+\frac{1}{2}\left\{\frac{H^{''}\left(1\right)+2}{n}x\right. \\ &\quad \left.+\frac{1}{n^2}\left(\left(\lambda+1\right)\left(\lambda+2\right)+\frac{2\left(\lambda+2\right)A^{'}\left(1\right)+A^{''}\left(1\right)}{A\left(1\right)}\right)\right\}\left\|f^{''}\right\|_{C_B} .\end{aligned}$$

By a simple calculation in the above inequality, we immediately derive (2.11). □

The next theorem contains quantitative estimate by means of Peetre's K-functional.

Theorem 2.5. *For every $f\in\tilde{C}_B\left[0,\infty\right)$, we have*

$$\left|T_n^*\left(f;x\right)-f\left(x\right)\right|\leq 2M\left(\omega_2\left(f;h\right)+\min\left(1,h^2\right)\left\|f\right\|_{\tilde{C}_B}\right)$$

where $h:=\sqrt{\frac{\mu(x)}{2n}}$.

Proof. Let be $g \in W_{\infty}^{2}$. From the previous theorem, it is clear that

$$\begin{aligned} \left|T_n^*(f;x)-f(x)\right| \leq\; & 2\left\{\|f-g\|_{\tilde{C}_B}+\frac{1}{2n}\left[\left(1+\frac{H''(1)}{2}\right)x\right.\right. \\ & \left.\left.+\frac{1}{2}\left((\lambda+1)(\lambda+4)+\frac{2(\lambda+3)A'(1)+A''(1)}{A(1)}\right)\right]\|g\|_{W_{\infty}^{2}}\right\} \end{aligned} \tag{2.12}$$

Because of the left hand side of inequality (2.12) does not depend on function g, the following result satisfies

$$\left|T_n^*(f;x)-f(x)\right| \leq 2\mathcal{K}\left(f;\frac{\mu(x)}{2n}\right) .$$

If we use connection between Peetre's K-functional and the second order modulus of smoothness, then by choosing $h:=\sqrt{\frac{\mu(x)}{2n}}$ we get

$$\left|T_n^*(f;x)-f(x)\right| \leq 2M\left(\omega_2(f;h)+\min\left(1,h^2\right)\|f\|_{\tilde{C}_B}\right) .$$

□

3. EXAMPLES OF THESE TYPE OPERATORS

Example 3.1. *The sequence* $\left\{x^k\right\}_{k=1}^{\infty}$ *which is the Sheffer sequence for* $A(t)=1$ *and* $H(t)=t$ *has the generating functions following type*

$$e^{xt}=\sum_{k=0}^{\infty}\frac{x^k}{k!}t^k .$$

Let us select $p_k(x)=\frac{x^k}{k!}$. *Considering these polynomials in* (1.8), *we find operators as follows*

$$T_n^*(f;x)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-nt}t^{\lambda+k}f(t)\,dt .$$

If we take $\lambda=0$ *in above operators, we get modified Szász operators* (1.1) *which are defined by Mazhar and Totik* [5].

n	*Error estimate by* T_n^* *operators including* $\left\{x^k\right\}_{k=1}^{\infty}$ *sequence*
10	*0.8085809766*
10^2	*0.2429249200*
10^3	*0.0763919187*
10^4	*0.0241436473*
10^5	*0.0076344614*
10^6	*0.0024142150*
10^7	*0.0007634414*
10^8	*0.0002414213*
10^9	*0.0000763441*

Table 1. The error bound of function $f(x)=\sin\left(x\sqrt{1+x^2}\right)$ *by using modulus of continuity*

Example 3.2. $\{B_k(x)\}$ *which are known as Bell polynomials forms the associated Sheffer sequence for* $A(t) = 1$ *and* $H(t) = e^t - 1$, *so the polynomials have generating functions*

$$\exp\left(x\left(e^t - 1\right)\right) = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} t^k \ . \tag{3.1}$$

Furthermore, $B_k(x)$ *polynomials are given by*

$$B_k(x) = e^{-x} \sum_{m=0}^{\infty} \frac{m^k}{m!} x^m \ .$$

Let us take $p_k(x) = \frac{B_k\left(\frac{x}{e}\right)}{k!}$. *By virtue of* (3.1), *we can get* $A(t) = 1$ *and* $H(t) = \frac{e^t - 1}{e}$. *Making use of above knowledge* $p_k(x) \geq 0$ *for* $x \in [0, \infty)$, $A(1) \neq 0$ *and* $H'(1) = 1$ *are provided. Hence, the following operators*

$$T_n^*(f; x) = e^{-nx\left(1 - \frac{1}{e}\right)} \sum_{k=0}^{\infty} \frac{B_k\left(\frac{nx}{e}\right)}{k!} \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} f(t)\, dt$$

are obtained.

n	*Error estimate by* T_n^* *operators including* $\{B_k(x)\}_{k=1}^{\infty}$ *sequence*
10	*0.9080938106*
10^2	*0.2746840438*
10^3	*0.0864420594*
10^4	*0.0273219960*
10^5	*0.0086395502*
10^6	*0.0027320522*
10^7	*0.0008639503*
10^8	*0.0002732050*
10^9	*0.0000863950*

Table 2. The error bound of function $f(x) = \sin\left(x\sqrt{1+x^2}\right)$ *by using modulus of continuity*

Example 3.3. *The polynomials* $g_k^{(\alpha)}(x)$ *which are Sheffer for* $A(t) = e^{\alpha t}$, $H(t) = 1 - e^t$ *have the generating functions of the form*

$$\exp(\alpha t + x(1 - e^t)) = \sum_{k=0}^{\infty} \frac{g_k^{(\alpha)}(x)}{k!} t^k \ . \tag{3.2}$$

These polynomials are known as Toscano polynomials or actuarial polynomials, since they were introduced in connection with problems of actuarial mathematics. The relation

$$g_k^{(\alpha)}(-x) = e^{-x} \sum_{m=0}^{\infty} \frac{(\alpha+m)^k}{m!} x^m$$

stated by Whittaker and Watson. It is clear that for $x \in [0, \infty)$ *and* $\alpha \geq 0$, *the polynomials* $g_k^{(\alpha)}(-x)$ *are positive. Let us choose* $p_k(x) := \frac{g_k^{(\alpha)}\left(-\frac{x}{e}\right)}{k!}$. *In view of the generating functions* (3.2), *one can find* $A(t) = e^{\alpha t}$ *and* $H(t) = (e^t - 1)/e$.

By using the above information, $p_k(x) \geq 0$ $(x \in [0,\infty), \alpha \geq 0)$, $A(1) \neq 0$ *and* $H'(1) = 1$ *are satisfied. So, we obtain the following special operators*

$$T_n^*(f;x) = \frac{e^{-nx\left(1-\frac{1}{e}\right)}}{e^{\alpha}} \sum_{k=0}^{\infty} \frac{g_k^{(\alpha)}\left(-\frac{nx}{e}\right)}{k!} \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} f(t)\, dt \ .$$

n	*Error estimate by* T_n^* *operators including* $\left\{g_k^{(0.001)}(x)\right\}_{k=1}^{\infty}$ *sequence*
10	*0.9081302237*
10^2	*0.2746851987*
10^3	*0.0864420959*
10^4	*0.0273219971*
10^5	*0.0086395503*
10^6	*0.0027320522*
10^7	*0.0008639503*
10^8	*0.0002732050*
10^9	*0.0000863950*

Table 3. The error bound of function $f(x) = \sin\left(x\sqrt{1+x^2}\right)$ *by using modulus of continuity*

Example 3.4. *Laguerre differential equation is*

$$xy'' + (1-x)y' + ky = 0$$

where k *is a positive integer. The standard solution of this equation called the Laguerre polynomial of order* k*, and is given by*

$$L_k(x) = \sum_{m=0}^{k} \frac{(-1)^m k!}{(k-m)!(m!)^2} x^m \ . \tag{3.3}$$

Laguerre polynomials satisfy the generating relation

$$\frac{1}{1-t} e^{-\frac{t}{1-t}x} = \sum_{k=0}^{\infty} L_k(x) t^k \ . \tag{3.4}$$

If we consider the above equality, then Laguerre polynomials are Sheffer type polynomials. Taking into account of formula (3.3), Laguerre polynomials $L_k(-x)$ *are positive for* $x \geq 0$*. Now, let be* $p_k(x) = \frac{L_k\left(-\frac{x}{2}\right)}{2^k}$*. Then by virtue of (3.4), one can get* $A(t) = \frac{2}{2-t}$ *and* $H(t) = \frac{t}{2(2-t)}$*. From these facts,* $p_k(x) \geq 0$ *for* $x \geq 0$*,* $A(1) \neq 0$ *and* $H'(1) = 1$ *are satisfied. After all, we obtain operators as follows*

$$T_n^*(f;x) = e^{-\frac{nx}{2}} \sum_{k=0}^{\infty} \frac{L_k\left(-\frac{nx}{2}\right)}{2^{k+1}} \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} f(t)\, dt \ .$$

n	*Error estimate by* T_n^* *operators including* $\{L_k(x)\}_{k=1}^{\infty}$ *sequence*
10	*1.0390028560*
10^2	*0.3029856656*
10^3	*0.0949631527*
10^4	*0.0300029998*
10^5	*0.0094869278*
10^6	*0.0030000029*
10^7	*0.0009486833*
10^8	*0.0003000000*
10^9	*0.0000948683*

Table 4. The error bound of function $f(x) = \sin\left(x\sqrt{1+x^2}\right)$ *by using modulus of continuity*

Algorithm 3.1. *The estimates found by following algorithm are given in Table 1. In the Table 1, we establish error estimates for the approximation with* T_n^* *operators including* $\{x^k\}_{k=1}^{\infty}$ *sequence.*

```
restart;
f:=x->sin(x*sqrt(1+x^2));
n:=1:
for i from 1 to 9 do
n:=10*n;
delta1:=evalf(1/sqrt(n));
omega1(f,delta1):=evalf(maximize(expand(abs(f(x+h)-f(x))),x=0..1-delta1,h=0..delta1)):
error1:=evalf((1+sqrt(2+2/n))*omega1(f,delta1));
end do;
```

Remark 3.1. *The algorithms for the numbers obtained in Table 2, Table 3 and Table 4 are pretty similar to the previous one.*

Remark 3.2. *Because of Bell polynomials, Toscano polynomials and Laguerre polynomials are not Appell polynomials, the operators constructed above aren't included Ciupa's article* [1].

References

[1] Ciupa, A., A class of integral Favard-Szász type operators, Studia Univ. Babeş-Bolyai Math., 40 (1) (1995), 39–47.

[2] Gavrea, I. and Rasa, I., Remarks on some quantitative Korovkin-type results, Rev. Anal. Numér. Théor. Approx., 22 (2) (1993), 173-176.

[3] Ismail, M.E.H., On a generalization of Szász operators, Mathematica (Cluj), 39 (1974), 259–267.

[4] Jakimovski, A. and Leviatan, D., Generalized Szász operators for the approximation in the infinite interval, Mathematica (Cluj), 11 (1969), 97-103.

[5] Mazhar, S.M. and Totik V., Approximation by modified Szász operators, Acta Sci. Math., 49 (1985), 257-269.

[6] Szász, O., Generalization of S. Bernstein's polynomials to the infinite interval, J. Research Nat. Bur. Standards, 45 (1950), 239–245.

[7] Zhuk, V.V., Functions of the *Lip*1 class and S. N. Bernstein's polynomials (Russian), Vestnik Leningrad. Univ. Mat. Mekh. Astronom., 1 (1989), 25–30.

SEZGİN SUCU
Ankara University Faculty of Science, Department of Mathematics, Tandoğan TR-06100, Ankara, Turkey.
E-mail address: ssucu@ankara.edu.tr

İBRAHİM BÜYÜKYAZICI
Ankara University Faculty of Science, Department of Mathematics, Tandoğan TR-06100, Ankara, Turkey.
E-mail address: ibuyukyazici@gmail.com