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DIFFERENCE OF WEIGHTED COMPOSITION OPERATOR FROM WEIGHTED BERGMAN SPACES TO WEIGHTED-TYPE SPACES

(COMMUNICATED BY SONGXIAO LI)

XIAOHONG FU

Abstract. The boundedness of the difference of two weighted composition operators from weighted Bergman spaces to weighted-type spaces in the unit ball are investigated in this paper.

1. Introduction

Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in $\mathbb{C}^n$, we write

$$|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2},\ \langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n.$$

Let B be the unit ball of $\mathbb{C}^n$. For $a, z \in B$, $a \neq 0$, let σ_a denote the Möbius transformation of B taking 0 to a, i.e.,

$$\sigma_a(z) = \frac{a - P_a(z) - \sqrt{1-|z|^2}Q_a(z)}{1 - \langle z, a\rangle},$$

where $P_a(z)$ is the orthogonal projection of z onto the one dimensional subspace of $\mathbb{C}^n$ spanned by a, and $Q_a(z) = z - P_a(z)$. When $a = 0$, we define $\sigma_a(z) = -z$. The pseudo-hyperbolic metric is defined by

$$\rho(z, w) = |\sigma_z(w)|.$$

Let $H(B)$ be the space of all holomorphic functions on B. For $p \in (0, \infty)$ and $\alpha > -1$, the weighted Bergman space , denoted by $A^p_\alpha(B) = A^p_\alpha$, is defined to be the space of all $f \in H(B)$ such that

$$\|f\|^p_{A^p_\alpha} = \int_B |f(z)|^p d\nu_\alpha(z) = c_\alpha \int_B |f(z)|^p (1-|z|^2)^\alpha d\nu(z) < \infty.$$

Here $c_\alpha = \Gamma(n+\alpha+1)/(\Gamma(n+1)\Gamma(\alpha+1))$, $d\nu$ is the normalized Lebesgue measure of B. When $p \geq 1$, A^p_α is a Banach space. When $\alpha = 0$, $A^p_0(B) = A^p(B)$ is the standard Bergman space. See [20] for some basic facts on weighted Bergman spaces on B.

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A positive continuous function μ on $[0,1)$ is called normal, if there exist positive numbers s and t, $0<s<t$, and $\delta \in [0,1)$ such that (see [14])

$$\frac{\mu(r)}{(1-r)^s} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\mu(r)}{(1-r)^s}=0;$$

$$\frac{\mu(r)}{(1-r)^t} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\mu(r)}{(1-r)^t}=\infty.$$

Let μ be a normal function on $[0,1)$. An $f\in H(B)$ is said to belong to the weighted-type space $H_\mu^\infty = H_\mu^\infty(B)$, if

$$\|f\|_{H_\mu^\infty} = \sup_{z\in B}\mu(|z|)\,|f(z)|<\infty.$$

H_μ^∞ is a Banach space with the norm $\|\cdot\|_{H_\mu^\infty}$.

Let $u\in H(B)$ and φ be a holomorphic self-map of B. For $f\in H(B)$, the weighted composition operator, denoted by uC_φ, is defined as follows:

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad z\in B.$$

The weighted composition operator can be regarded as a generalization of a multiplication operator and a composition operator, which is defined by

$$(C_\varphi f)(z) = f(\varphi(z)), \quad z\in B.$$

See [2] for more information on this topic.

Recently, there are many researchers studied the difference of two composition operators or two weighted composition operators. The motivation for study the difference of composition operators is to understand the topological structure of the set of composition operators acting on a given function space. For the study of difference of composition operators and difference of weighted composition operators, see, for example, [1,3,5–7,10–13,17–19] and the references therein. See [4,9,15,16,21] for the study of the weighted composition operators from some function spaces to weighted-type spaces in the unit ball.

In [7], the authors investigated the compactness of the difference of two weighted composition operators acting from the weighted Bergman space A^p_α to the weighted-type space H_μ^∞ in the unit disk. In [17], the authors generalized the results in [7] to the unit ball. Here we consider the boundedness. We give some necessary and sufficient conditions for the boundedness of the difference of two weighted composition operators from weighted Bergman spaces to weighted-type spaces in the unit ball by using the method of [8].

Constants are denoted by C in this paper, they are positive and not necessary the same in each occurrence.

2. Main results and proofs

In this section we give our main results and proofs. For this purpose, we need a auxiliary result. The following lemma can be found in [17].

Lemma 2.1. *Let $p\ge 1, \alpha>-1$. There exists a constant $C>0$ such that*

$$\left|(1-|z|^2)^{\frac{n+1+\alpha}{p}}f(z)-(1-|w|^2)^{\frac{n+1+\alpha}{p}}f(w)\right| \le C\|f\|_{A^p_\alpha}\rho(z,w).$$

for all $f\in A^p_\alpha$ and for all $z,w\in B$.

To be convenient for the statements of our main results, we define

$$I_1(z) = \frac{\mu(|z|)u(z)}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}},\ I_2(z) = \frac{\mu(|z|)v(z)}{(1-|\psi(z)|^2)^{\frac{n+1+\alpha}{p}}}. \tag{2.1}$$

Now we are in a position to state and prove our main results in this paper.

Theorem 2.1. *Assume that φ, ψ are holomorphic self-maps of B, $u, v \in H(B)$, $p \geq 1, \alpha > -1$. Then $uC_\varphi - vC_\psi : A^p_\alpha \to H^\infty_\mu$ is bounded if and only if*

$$\sup_{z\in B} |I_1(z) - I_2(z)| < \infty \tag{2.2}$$

and

$$\max\Big\{\sup_{z\in B} |I_1(z)|\rho(\varphi(z),\psi(z)),\ \sup_{z\in B} |I_2(z)|\rho(\varphi(z),\psi(z))\Big\} < \infty. \tag{2.3}$$

Proof. First we assume (2.2) and (2.3) hold. Let $f \in A^p_\alpha$. It is well known that there exists a constant C such that (see [20])

$$|f(z)| \leq C\frac{\|f\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{n+1+\alpha}{p}}}. \tag{2.4}$$

By Lemma 2.1 and (2.4), we have

$$\begin{aligned}
&\sup_{z\in B}\mu(|z|)|(uC_\varphi - vC_\psi)f(z)|\\
=&\sup_{z\in B}\mu(|z|)|f(\varphi(z))u(z) - f(\psi(z))v(z)|\\
=&\sup_{z\in B}\bigg|(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}} f(\varphi(z))\Big[\frac{\mu(|z|)u(z)}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}} - \frac{\mu(|z|)v(z)}{(1-|\psi(z)|^2)^{\frac{n+1+\alpha}{p}}}\Big]\\
&+\frac{\mu(|z|)v(z)}{(1-|\psi(z)|^2)^{\frac{n+1+\alpha}{p}}}[(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}} f(\varphi(z)) - (1-|\psi(z)|^2)^{\frac{n+1+\alpha}{p}} f(\psi(z))]\bigg|\\
\leq& C\|f\|_{A^p_\alpha}\sup_{z\in B}\big|I_1(z) - I_2(z)\big| + C\|f\|_{A^p_\alpha}\sup_{z\in B}|I_2(z)|\rho(\varphi(z),\psi(z))\\
<&\infty,
\end{aligned}$$

which implies that $uC_\varphi - vC_\psi : A^p_\alpha \to H^\infty_\mu$ is bounded.

Conversely, assume that $uC_\varphi - vC_\psi : A^p_\alpha \to H^\infty_\mu$ is bounded. Taking $f = 1$, by the boundedness of $uC_\varphi - vC_\psi$, we get

$$\sup_{z\in B}\mu(|z|)|u(z) - v(z)| < \infty. \tag{2.5}$$

For $a \in B$ with $\varphi(a) \neq 0$, we define

$$f_a(z) = \frac{\sigma_{\psi(a)}(z)}{(1-\langle z, \varphi(a)\rangle)^{\frac{n+1+\alpha}{p}}},\ z \in B.$$

It is easy to see that $f_a \in A^p_\alpha$ with $\|f_a\|_{A^p_\alpha} \leq C$. Note that

$$f_a(\varphi(a)) = \frac{\sigma_{\psi(a)}(\varphi(a))}{(1-|\varphi(a)|^2)^{\frac{n+1+\alpha}{p}}},\ f_a(\psi(a)) = 0.$$

By the boundedness of $uC_\varphi - vC_\psi$, we have

$$\begin{aligned}
\infty &> \|(uC_\varphi - vC_\psi)f_a\|_{H^\infty_\mu} \\
&= \sup_{z\in B} \mu(|z|)|f_a(\varphi(z))u(z) - f_a(\psi(z))v(z)| \\
&\geq \mu(|a|)|f_a(\varphi(a))u(a) - f_a(\psi(a))v(a)| \\
&= \frac{\mu(|a|)|u(a)|}{(1-|\varphi(a)|^2)^{\frac{n+1+\alpha}{p}}}\rho(\varphi(a),\psi(a)) \\
&= |I_1(a)|\rho(\varphi(a),\psi(a))
\end{aligned}$$

for any $a \in B$ with $\varphi(a) \neq 0$. From which we see that

$$\sup_{a\in B\setminus B_1} |I_1(a)|\rho(\varphi(a),\psi(a)) < \infty, \tag{2.6}$$

where $B_1 = \{a \in B : \varphi(a) = 0\}$.

Similar process implies

$$\sup_{a\in B\setminus B_2} |I_2(a)|\rho(\varphi(a),\psi(a)) < \infty, \tag{2.7}$$

where $B_2 = \{a \in B : \psi(a) = 0\}$.

If $\varphi(a) \neq 0, \psi(a) \neq 0$, let

$$g_a(z) = \frac{1}{(1-\langle z,\psi(a)\rangle)^{\frac{n+1+\alpha}{p}}},\ z \in B.$$

Then $g_a \in A^p_\alpha$ with $\|g_a\|_{A^p_\alpha} \leq C$. Then, by the boundedness of $uC_\varphi - vC_\psi$, we obtain

$$\begin{aligned}
\infty &> \|(uC_\varphi - vC_\psi)g_a\|_{H^\infty_\mu} \\
&\geq \mu(|a|)|g_a(\varphi(a))u(a) - g_a(\psi(a))v(a)| \\
&= |I_1(a) - I_2(a) + J(a)|,
\end{aligned} \tag{2.8}$$

where

$$J(a) = I_1(a)\Big[(1-|\varphi(a)|^2)^{\frac{n+1+\alpha}{p}} g_a(\varphi(a)) - (1-|\psi(a)|^2)^{\frac{n+1+\alpha}{p}} g_a(\psi(a))\Big].$$

Therefore, by Lemma 2.1 and (2.6),

$$\sup_{a\in B\setminus B_1\cup B_2} |J(a)| \leq C\|g_a\|_{A^p_\alpha} \sup_{a\in B\setminus B_1\cup B_2} |I_1(a)|\rho(\varphi(a),\psi(a)) < \infty,$$

which together with (2.8) implies

$$\sup_{a\in B\setminus B_1\cup B_2} |I_1(a) - I_2(a)| < \infty. \tag{2.9}$$

If $\varphi(a) \neq 0, \psi(a) = 0$, then $\rho(\varphi(a),\psi(a)) = |\varphi(a)|$. Taking

$$h_a(z) = \frac{\langle z,\varphi(a)\rangle}{|\varphi(a)|^2}$$

and using the boundedness of $uC_\varphi - vC_\psi$, we obtain

$$\sup_{a\in B_2\setminus B_1} \mu(|a|)|u(a)| < \infty, \tag{2.10}$$

which together with (2.5) implies

$$\sup_{a\in B_2\setminus B_1} \mu(|a|)|v(a)| < \infty. \tag{2.11}$$

Therefore

$$\begin{aligned}\sup_{a\in B_2\setminus B_1}|I_2(a)|\rho(\varphi(a),\psi(a)) &= \sup_{a\in B_2\setminus B_1}\mu(|a|)|v(a)||\varphi(a)| \\ &\le \sup_{a\in B_2\setminus B_1}\mu(|a|)|v(a)|<\infty. \qquad (2.12)\end{aligned}$$

Next, set

$$p_a(z)=\frac{1}{(1-\langle z,\varphi(z)\rangle)^{\frac{n+1+\alpha}{p}}}\frac{\langle z,\varphi(a)\rangle}{|\varphi(a)|^2}.$$

Then $p_a\in A^p_\alpha$. By the boundedness of $uC_\varphi-vC_\psi$, we obtain

$$\sup_{a\in B_2\setminus B_1}|I_1(a)|=\sup_{a\in B_2\setminus B_1}\frac{\mu(|a|)|u(a)|}{(1-|\varphi(a)|^2)^{\frac{n+1+\alpha}{p}}}<\infty,$$

which together with (2.11) we obtain

$$\begin{aligned}&\sup_{a\in B_2\setminus B_1}|I_1(a)-I_2(a)| \\ =&\sup_{a\in B_2\setminus B_1}\Big|I_1(a)-\mu(|a|)v(a)\Big| \\ \le&\sup_{a\in B_2\setminus B_1}|I_1(a)|+\sup_{a\in B_2\setminus B_1}\mu(|a|)|v(a)|<\infty. \qquad (2.13)\end{aligned}$$

If $\psi(a)\neq 0,\varphi(a)=0$, then similarly to the above proof we get

$$\sup_{a\in B_1\setminus B_2}|I_1(a)|\rho(\varphi(a),\psi(a))<\infty \qquad (2.14)$$

and

$$\sup_{a\in B_1\setminus B_2}|I_1(a)-I_2(a)|<\infty. \qquad (2.15)$$

If $\varphi(a)=\psi(a)=0$, then $\rho(\varphi(a),\psi(a))=0$. It is clear that

$$\sup_{a\in B_1\cap B_2}|I_1(a)-I_2(a)|=\sup_{a\in B_1\cap B_2}\mu(|a|)|u(a)-v(a)|<\infty, \qquad (2.16)$$

$$\sup_{a\in B_1\cap B_2}|I_1(a)|\rho(\varphi(a),\psi(a))<\infty, \qquad (2.17)$$

$$\sup_{a\in B_1\cap B_2}|I_2(a)|\rho(\varphi(a),\psi(a))<\infty. \qquad (2.18)$$

By (2.9), (2.13), (2.15) and (2.16) we get (2.2). By (2.6), (2.14) and (2.17) we obtain

$$\sup_{a\in B}|I_1(a)|\rho(\varphi(a),\psi(a))<\infty.$$

By (2.7), (2.12) and (2.18) we have

$$\sup_{a\in B}|I_2(a)|\rho(\varphi(a),\psi(a))<\infty,$$

as desired. The proof is completed. □

From Theorem 2.1, we get the following result. See also [4, 15].

Corollary 2.1. *Assume that φ is holomorphic self-maps of B, $u\in H(B)$, $p\ge 1,\alpha>-1$. Then $uC_\varphi: A^p_\alpha\to H^\infty_\mu$ is bounded if and only if*

$$\sup_{z\in B}|I_1(z)|<\infty.$$

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Department of Mathematics,
JiaYing University,
514015, Meizhou, GuangDong, China.
E-mail address: jyufxh@163.com